# ROBUST ADAPTIVE BEAMFORMING FOR GENERAL-RANK SIGNAL MODELS USING POSITIVE SEMI-DEFINITE COVARIANCE CONSTRAINT

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**ABSTRACT**

In this paper, we develop an improved approach to the worst-case robust adaptive beamforming for general-rank signal models by means of taking into account the positive semi-definite constraint for the mismatched signal covariance matrix. The resulting robust adaptive beamforming problem is solved in an iterative way using semi-definite programming (SDP) at each iteration. Simulation results show that the proposed technique achieves a substantially improved performance as compared to the current robust adaptive beamforming techniques developed for the general-rank signal environments.

***Index Terms*—** Robust adaptive beamforming, semi-definite programming, worst-case performance optimization

## 1. INTRODUCTION

Robust adaptive beamforming has recently gained a significant interest in the literature; see [1]-[7] and references therein. Although in most of papers on this subject rank-one signal models are assumed, general-rank signal models are of significant interest as they are often encountered in radio communications and sonar where the signal sources can be dispersed in angle because of propagation effects [1], [3], [8]-[11].

In [3], a worst-case optimization based minimum variance (MV) robust adaptive beamformer has been proposed for the general-rank signal case. This beamformer is based on the explicit modelling of uncertainties in the desired signal covariance matrix and in the sample data covariance matrix, and subsequent worst-case performance optimization. Although the beamformer of [3] offers a computationally simple closed-form solution and excellent robustness capability [5], it ignores the positive semi-definiteness (PSD) constraint for the mismatched signal covariance matrix. As a result, the beamformer of [3] may be overly conservative in some cases.

In this paper, we propose a new robust MV beamformer that follows the idea of [3], but also takes into account the aforementioned PSD constraint. The resulting robust adaptive beamforming problem is solved iteratively using semi-definite programming (SDP) in each iteration. Simulation results validate substantial improvements offered by the proposed robust beamformer as compared to the technique of [3] and other popular general-rank beamformers.

## 2. BACKGROUND

The output of a narrowband beamformer can be written as

$$\boldsymbol{y}(k) = \boldsymbol{w}^H \boldsymbol{x}(k)$$

where $\boldsymbol{x}(k) = [x_1(k), \cdots, x_M(k)]^T$ is the complex array snapshot vector at time $k$, $\boldsymbol{w} = [w_1, \cdots, w_M]^T$ is the complex weight vector, $M$ is the number of sensors, and $(\cdot)^T$ and $(\cdot)^H$ denote the transpose and Hermitian transpose, respectively. The array snapshot vector can be expressed as

$$\boldsymbol{x}(k) = \boldsymbol{s}(k) + \boldsymbol{i}(k) + \boldsymbol{n}(k)$$

where $\boldsymbol{s}(k)$, $\boldsymbol{i}(k)$ and $\boldsymbol{n}(k)$ are the desired signal, interference, and noise components, respectively. The optimal weight vector $\boldsymbol{w}_{\rm opt}$ can be obtained by maximizing the signal-to-interference-plus-noise-ratio (SINR)

$$\mathrm{SINR} = \frac{\boldsymbol{w}^H \boldsymbol{R}_s \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \tag{1}$$

where $\boldsymbol{R}_s \triangleq \mathrm{E}\{\boldsymbol{s}(k)\boldsymbol{s}^H(k)\}$ and $\boldsymbol{R}_{i+n} \triangleq \mathrm{E}\{(\boldsymbol{i}(k) + \boldsymbol{n}(k))$ $(\boldsymbol{i}(k)+\boldsymbol{n}(k))^H\}$ are the $M \times M$ signal and interference-plus-noise covariance matrices, respectively, and $\mathrm{E}\{\cdot\}$ denotes the statistical expectation. Generally, $\boldsymbol{R}_s$ can be of arbitrary rank, that is, $1 \leq \mathrm{rank}\{\boldsymbol{R}_s\} \leq M$. In the particular case of a point signal source, $\boldsymbol{s}(k) = s(k)\boldsymbol{a}_s$ and $\boldsymbol{R}_s = \sigma_s^2 \boldsymbol{a}_s \boldsymbol{a}_s^H$ is rank-one, where $s(k)$ is the zero-mean signal waveform, $\sigma_s^2 = \mathrm{E}\{|s(k)|^2\}$ is the variance of $s(k)$, and $\boldsymbol{a}_s$ is the signal steering vector. In this particular case, the SINR in (1) reduces to

$$\mathrm{SINR} = \frac{\sigma_s^2 \left|\boldsymbol{w}^H \boldsymbol{a}_s\right|^2}{\boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w}} \,. \tag{2}$$

However, in many practical scenarios $\mathrm{rank}\{\boldsymbol{R}_s\} > 1$, for example, in scenarios with incoherently scattered signal sources

[9]-[10], or with signals propagating through random inhomogeneous media [8], [11].

In both the rank-one and general-rank signal cases, the optimal weight vector can be obtained by minimizing the output interference-plus-noise power under the distortionless response constraint [3]:

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \boldsymbol{R}_{i+n} \boldsymbol{w} \quad \text{s.t.} \quad \boldsymbol{w}^H \boldsymbol{R}_s \boldsymbol{w} = 1 \tag{3}$$

which is equivalent to maximizing the SINR in (1). In practical scenarios, the exact knowledge of $\boldsymbol{R}_{i+n}$ is unavailable because of the presence of the signal component in the training data snapshots and/or finite observation time. Therefore, the sample covariance matrix

$$\hat{\boldsymbol{R}} = \frac{1}{K} \sum_{k=1}^{K} \boldsymbol{x}(k) \boldsymbol{x}^H(k) \tag{4}$$

is usually used instead of $\boldsymbol{R}_{i+n}$, where $\boldsymbol{R} \triangleq \mathrm{E}\{\boldsymbol{x}(k)\boldsymbol{x}^H(k)\}$ is the data covariance matrix, and $K$ is the training sample size [12].

Replacing $\boldsymbol{R}_{i+n}$ by $\hat{\boldsymbol{R}}$ in (3) and solving the latter problem yields the following weight vector [3]

$$\boldsymbol{w}_{\mathrm{SMI}} = \mathcal{P}\{\hat{\boldsymbol{R}}^{-1} \boldsymbol{R}_s\} \tag{5}$$

where $\mathcal{P}\{\cdot\}$ denotes the principal eigenvector of a matrix. The beamformer (5) is commonly referred to as the sample matrix inversion (SMI) beamformer.

To provide robustness against possible norm-bounded matrix mismatches $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$ in the matrices $\boldsymbol{R}_s$ and $\boldsymbol{R}_{i+n}$, respectively, the authors of [3] suggested to obtain the beamformer weight vector via maximizing the worst-case output SINR. This corresponds to the following optimization problem:

$$\begin{aligned} &\max_{\boldsymbol{w}} \min_{\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2} \frac{\boldsymbol{w}^H(\boldsymbol{R}_{\mathrm{s}} + \boldsymbol{\Delta}_1)\boldsymbol{w}}{\boldsymbol{w}^H(\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_2)\boldsymbol{w}} \\ &\text{s.t.} \quad \|\boldsymbol{\Delta}_1\| \leq \varepsilon\,, \ \|\boldsymbol{\Delta}_2\|_{\leq}\gamma \end{aligned} \tag{6}$$

where $\|\cdot\|$ is the Frobenius norm of a matrix or the 2-norm of a vector, and $\varepsilon$ and $\gamma$ are some known bounds on the covariance matrix errors $\boldsymbol{\Delta}_1$ and $\boldsymbol{\Delta}_2$.

The solution to the robust MV beamforming problem (6) can be written as [3]

$$\boldsymbol{w}_{\mathrm{rob}} = \mathcal{P}\{(\hat{\boldsymbol{R}} + \gamma \boldsymbol{I})^{-1}(\boldsymbol{R}_s - \varepsilon \boldsymbol{I})\} \tag{7}$$

where $\boldsymbol{I}$ is the identity matrix.

Unfortunately, the problem (6) ignores the PSD constraint

$$\boldsymbol{R}_{\mathrm{s}} + \boldsymbol{\Delta}_1 \succeq 0 \tag{8}$$

which is violated in (7).

## 3. ROBUST BEAMFORMING WITH PSD CONSTRAINTS

In this section, we develop an improved, less conservative variant of the robust worst-case MV beamformer (7) that takes into account the PSD constraint (8). With this constraint, the problem (6) can be rewritten as

$$\begin{aligned} &\min_{\boldsymbol{w}} \max_{\|\boldsymbol{\Delta}_2\| \leq \gamma} \boldsymbol{w}^H(\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_2)\boldsymbol{w} \\ &\text{s.t.} \ \boldsymbol{w}^H(\boldsymbol{R}_s + \boldsymbol{\Delta}_1)\boldsymbol{w} \geq 1, \ \boldsymbol{R}_s + \boldsymbol{\Delta}_1 \succeq 0 \ \forall \ \|\boldsymbol{\Delta}_1\| \leq \varepsilon \end{aligned} \tag{9}$$

To enforce the PSD constraint, let us model uncertainty in the "square root" of the signal covariance matrix, $\boldsymbol{Q}$, rather than in the signal covariance matrix itself. The matrix $\boldsymbol{Q}$ is defined through the equation $\boldsymbol{R}_s = \boldsymbol{Q}^H \boldsymbol{Q}$. Then, defining $\boldsymbol{\Delta}$ as a norm-bounded mismatch in $\boldsymbol{Q}$, we can rewrite (9) as

$$\begin{aligned} &\min_{\boldsymbol{w}} \max_{\|\boldsymbol{\Delta}_2\| \leq \gamma} \boldsymbol{w}^H(\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_2)\boldsymbol{w} \\ &\text{s.t.} \ \min_{\|\boldsymbol{\Delta}\| \leq \eta} \boldsymbol{w}^H(\boldsymbol{Q} + \boldsymbol{\Delta})^H(\boldsymbol{Q} + \boldsymbol{\Delta})\boldsymbol{w} \geq 1 \end{aligned} \tag{10}$$

where $\eta$ is some known bound on $\boldsymbol{\Delta}$.

To simplify the problem (10), let us first find the solutions to the following problems:

$$\max_{\boldsymbol{\Delta}} \boldsymbol{w}^H(\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_2)\boldsymbol{w} \quad \text{s.t.} \quad \|\boldsymbol{\Delta}_2\| \leq \gamma \tag{11}$$

$$\min_{\boldsymbol{\Delta}} \|(\boldsymbol{Q} + \boldsymbol{\Delta})\boldsymbol{w}\| \quad \text{s.t.} \quad \|\boldsymbol{\Delta}\| \leq \eta \tag{12}$$

*Lemma 1:* The solution to (11) is

$$\boldsymbol{\Delta}_{2*} = \gamma \frac{\boldsymbol{w}\boldsymbol{w}^H}{\|\boldsymbol{w}\|^2}$$

and the maximum of the objective function in (11) is

$$\boldsymbol{w}^H(\hat{\boldsymbol{R}} + \gamma \boldsymbol{I})\boldsymbol{w}. \tag{13}$$

*Proof:* See [3]. □

*Lemma 2:* If the mismatch is small enough, that is,

$$\eta \|\boldsymbol{w}\| \leq \|\boldsymbol{Q}\boldsymbol{w}\|$$

then the solution to (12) is

$$\boldsymbol{\Delta}_* = -\frac{\eta \boldsymbol{Q}\boldsymbol{w}\boldsymbol{w}^H}{\|\boldsymbol{w}\|\|\boldsymbol{Q}\boldsymbol{w}\|}$$

and the minimum of the objective function in (12) is

$$\|\boldsymbol{Q}\boldsymbol{w}\| - \eta\|\boldsymbol{w}\|.$$

*Proof:* Using the triangle and Cauchy-Schwarz inequalities along with the constraint $\|\boldsymbol{\Delta}\| \leq \eta$, we have

$$\begin{aligned} \|(\boldsymbol{Q} + \boldsymbol{\Delta})\boldsymbol{w}\| &\geq \|\boldsymbol{Q}\boldsymbol{w}\| - \|\boldsymbol{\Delta}\boldsymbol{w}\| \\ &\geq \|\boldsymbol{Q}\boldsymbol{w}\| - \|\boldsymbol{\Delta}\|\|\boldsymbol{w}\| \\ &\geq \|\boldsymbol{Q}\boldsymbol{w}\| - \eta\|\boldsymbol{w}\|. \end{aligned} \tag{14}$$

It is straightforward to verify that if $\eta\|\boldsymbol{w}\| \leq \|\boldsymbol{Q}\boldsymbol{w}\|$, then

$$\|(\boldsymbol{Q} + \boldsymbol{\Delta}_*)\boldsymbol{w}\| = \|\boldsymbol{Q}\boldsymbol{w}\| - \eta\|\boldsymbol{w}\|. \quad (15)$$

Comparing (14) and (15), we prove Lemma 2. $\square$

Considering the small mismatch case (13) and using the results of Lemmas 1 and 2, the problem (10) can be simplified as

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H(\hat{\boldsymbol{R}} + \gamma\boldsymbol{I})\boldsymbol{w} \quad \text{s.t.} \quad \|\boldsymbol{Q}\boldsymbol{w}\| - \eta\|\boldsymbol{w}\| \geq 1. \quad (16)$$

The problem in (16) has a non-convex constraint and, therefore, is difficult to solve directly. To approximate this problem by a convex optimization problem, let us rewrite its constraint as

$$\|\boldsymbol{Q}\boldsymbol{w}\|^2 \geq (\eta\|\boldsymbol{w}\| + 1)^2.$$

The latter constraint can be expressed in the following form:

$$\boldsymbol{w}^H\boldsymbol{R}_s\boldsymbol{w} - \eta^2\boldsymbol{w}^H\boldsymbol{w} - 1 \quad \geq \quad 2\eta\|\boldsymbol{w}\|. \quad (17)$$

Defining a new variable

$$\boldsymbol{W} \triangleq \boldsymbol{w}\boldsymbol{w}^H$$

we can equivalently rewrite (17) as the following three constraints:

$$\begin{aligned} &\mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{W}\} - \eta^2\mathrm{tr}\{\boldsymbol{W}\} - 1 \geq 2\eta\sqrt{\mathrm{tr}\{\boldsymbol{W}\}} \\ &\boldsymbol{W} \succeq 0, \quad \mathrm{rank}\{\boldsymbol{W}\} = 1 \end{aligned} \quad (18)$$

where we have used the identities $\boldsymbol{w}^H\boldsymbol{R}_s\boldsymbol{w} = \mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{W}\}$ and $\boldsymbol{w}^H\boldsymbol{w} = \mathrm{tr}\{\boldsymbol{W}\}$. The constraint $\boldsymbol{X} \succeq 0$ means that $\boldsymbol{X}$ is symmetric PSD. A standard yet straightforward approach to get rid of non-convex rank constraints is to drop them from the optimization problem. This approach is commonly referred to as *semi-definite relaxation*. Dropping the rank-one constraint $\mathrm{rank}\{\boldsymbol{W}\} = 1$ in (18) and reformulating the objective function of (16) in terms of $\boldsymbol{W}$, we obtain the following problem:

$$\begin{aligned} &\min_{\boldsymbol{W}} \mathrm{tr}\{(\hat{\boldsymbol{R}} + \gamma\boldsymbol{I})\boldsymbol{W}\} \\ &\text{s.t.} \;\; \mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{W}\} - \eta^2\mathrm{tr}\{\boldsymbol{W}\} - 1 \geq 2\eta\sqrt{\mathrm{tr}\{\boldsymbol{W}\}}, \quad (19) \\ &\qquad \boldsymbol{W} \succeq 0. \end{aligned}$$

However, the first constraint in (19) still remains non-convex because of the term $2\eta\sqrt{\mathrm{tr}\{\boldsymbol{W}\}}$ in the right hand side. Therefore, to solve (19), we resort to an iterative procedure. In the $k$th iteration, we find $\boldsymbol{W}_k$ by means of solving the following problem:

$$\begin{aligned} &\min_{\boldsymbol{W}} \mathrm{tr}\{(\hat{\boldsymbol{R}} + \gamma\boldsymbol{I})\boldsymbol{W}\} \\ &\text{s.t.} \;\; \mathrm{tr}\{\boldsymbol{R}_s\boldsymbol{W}\} - \eta^2\mathrm{tr}\{\boldsymbol{W}\} - 1 \geq 2\eta\sqrt{\mathrm{tr}\{\boldsymbol{W}_{k-1}\}}, \quad (20) \\ &\qquad \boldsymbol{W} \succeq 0 \end{aligned}$$

where $\boldsymbol{W}_{k-1}$ is the solution obtained in the previous $(k-1)$th iteration. The problem (20) belongs to the class of SDP problems and, therefore, is convex for each particular iteration.

The problem (20) can be solved using currently available highly efficient convex optimization tools such as CVX [15] or SeDuMi [16]. However, the rank of the solution $\boldsymbol{W}_*$ is usually higher than one and, therefore, the optimal weight vector cannot be straightforwardly recovered from $\boldsymbol{W}_*$. In such a case, a common approach is to use *randomization techniques* whose essence is to draw multiple Gaussian random vectors from $\mathcal{N}_C(\boldsymbol{0}, \boldsymbol{W}_*)$ where $\mathcal{N}_C(\cdot,\cdot)$ stands for a complex circular multivariate Gaussian distribution. Then, the "best" solution is selected among such randomly generated candidates. Because of the randomization procedure, some of the weight vector candidates may violate the constraint in (16) and, therefore, they have to be re-scaled to satisfy this constraint. Finally, the best candidate that satisfies this constraint and minimizes the objective function is selected as an approximate solution to (16).

## 4. SIMULATION RESULTS

In our simulations, we assume a uniform linear array of $M = 20$ omnidirectional sensors spaced half a wavelength apart. The desired signal is always present in the training data cell. There is a single point-source interferer whose interference-to-noise-ratio (INR) is equal to 20 dB. The interferer is modelled as a moving source with the time-varying direction-of-arrival (DOA) $\theta(k) = -30° + 10°\sin(k/15)$. The desired signal is assumed to be an incoherently scattered source with the Gaussian angular power density whose central angle and the angular spread are equal to $30°$ and $4°$, respectively. The presumed shape of the signal angular power density is also Gaussian, but the presumed central angle and the angular spread are equal to $32°$ and $6°$, respectively. A rectangular sliding window of $K = 50$ snapshots is used and a total of 500 sliding windows are used to average the results.

The following general-rank beamformers are compared: the proposed robust beamformer based on (20), the robust beamformer of [3], the SMI beamformer (5), and the diagonally loaded SMI (LSMI) beamformer [13], [14]. The diagonal loading parameter $\gamma = 30$ is chosen in our beamformer, the beamformer of [3], and the LSMI beamformer. The value of $\varepsilon = 9\sigma_s^2$ is chosen for the beamformer of [3], as suggested in [3], where $\sigma_s^2$ is the variance of the desired signal. In the proposed beamformer, $\eta$ is chosen to be $0.75\sqrt{\mathrm{tr}\{\boldsymbol{R}_s\}}$, which appears to be a nearly optimal choice of this parameter. The CVX Matlab software [15] has been used to implement the proposed beamformer and about 10 iterations have been performed to solve (20).

Fig. 1 displays the output SINR versus the signal-to-noise ratio (SNR) for all the beamformers tested. As can be seen from this figure, the proposed beamformer has the best performance among the compared techniques. In particular, it

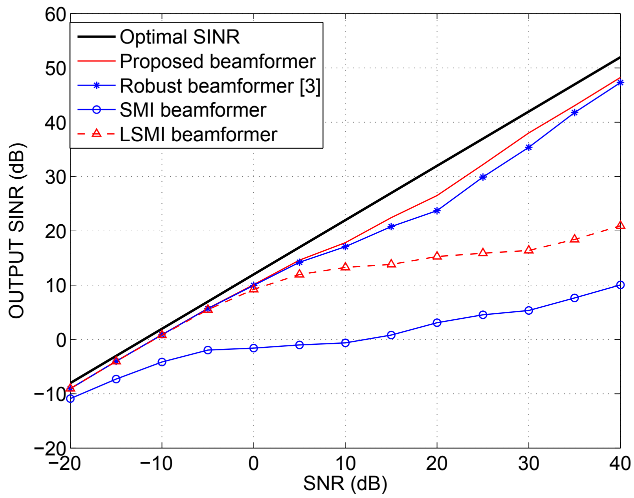


**Fig. 1**. Output SINR versus SNR.

outperforms the technique of [3] at high SNR values, and the achieved SINR improvements are up to 2.5 dB.

## 5. CONCLUSIONS

A new robust MV beamformer is proposed for general-rank signal models. Our approach is based on worst-case performance optimization and takes into account the PSD constraint on the mismatched signal covariance matrix. The resulting robust beamforming problem has been solved iteratively using SDP in each iteration. The proposed technique is shown to outperform the approach of [3] that ignores the PSD constraint.

## 6. REFERENCES


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