ROOT-MUSIC BASED DIRECTION-OF-ARRIVAL ESTIMATION METHODS FOR ARBITRARY NON-UNIFORM ARRAYS

Michael Rübsamen *Alex B. Gershman*

Communication Systems Group
Technische Universität Darmstadt
Merckstr. 25, 64283 Darmstadt, Germany

ABSTRACT

Two computationally efficient high-resolution methods are proposed for direction-of-arrival (DOA) estimation in arbitrary non-uniform sensor arrays. Our first algorithm is based on the fact that the spectral MUSIC function is periodic in angle. Expanding this function using Fourier series, we reformulate the DOA estimation problem as an equivalent polynomial rooting problem. Our second approach applies the inverse Fourier transform to the so-obtained root-MUSIC polynomial to compute the null-spectrum without any polynomial rooting, using a simple line search. The proposed techniques are shown to offer substantially improved performance-to-complexity tradeoffs as compared to the existing root-MUSIC-type methods applicable to non-uniform arrays.

Index Terms— Direction-of-arrival estimation, non-uniform sensor arrays, root-MUSIC

1. INTRODUCTION

The multiple signal classification (MUSIC) algorithm [1], [2] is one of the most popular and widely used subspace-based techniques for estimating the DOAs of multiple signal sources. The conventional (spectral) MUSIC algorithm involves, however, a computationally demanding spectral search over the angle and, therefore, its implementation can be prohibitively expensive in real-world applications. To reduce the computational complexity of MUSIC, a numerically efficient search-free modification of this approach has been proposed in [3]. The latter algorithm is commonly referred to as root-MUSIC because it exploits polynomial rooting instead of spectral search. Although the root-MUSIC technique enjoys a substantially reduced computational complexity and an improved threshold estimation performance as compared to the spectral MUSIC approach [4], it is only applicable to uniform linear arrays (ULAs) or non-uniform arrays whose sensors are restricted to lie on a uniform grid. Several extensions of root-MUSIC to a more general class of subarray-based array geometries have been proposed in [5] and [6], but these geometries still remain quite specific as they require the sensors of each subarray to belong to a uniform grid.

There have been several promising attempts to extend the concept of root-MUSIC to arbitrary non-uniform array geometries. For example, the approach of [7] uses the idea of interpolating a virtual ULA and applying the standard root-MUSIC technique to the virtual array observations. However, the performance of this interpolated root-MUSIC technique can be substantially limited by the array mapping errors [7], [8].

An interesting approach to extend root-MUSIC to non-uniform arrays of arbitrary geometry has been recently reported in [9]. This approach uses the idea of [10] to model the non-uniform array steering vector as a product of a matrix that depends only on the array parameters and a Vandermonde vector depending only on the angle. The Vandermonde structure of the latter vector is exploited in [9] to obtain a polynomial whose roots can be used to estimate the source DOAs.

In this paper, we propose an alternative approach referred to as Fourier-domain (FD) root-MUSIC that extends the concept of root-MUSIC to the case of non-uniform arrays of arbitrary geometry. Our technique exploits the fact that the null-spectrum MUSIC function is periodic in angle and uses the truncated Fourier series expansion of this function to reformulate the DOA estimation problem as a polynomial rooting problem.

As the order of the FD root-MUSIC polynomial is entirely determined by the number of terms used in the truncated Fourier series, and since the resulting DOA estimation performance can suffer from truncation errors, the order of the FD root-MUSIC polynomial should be chosen rather high. Therefore, to avoid the polynomial rooting step and further reduce the computational complexity, we use the idea of [11] and apply the inverse Fourier transform to the FD root-MUSIC polynomial to compute the null-spectrum and obtain the source DOAs by means of a simple line search.

It is demonstrated that the proposed techniques offer computationally more attractive implementations and/or an improved DOA estimation performance as compared to the methods of [7] and [9].

2. BACKGROUND

Let an array of N omnidirectional sensors receive signals from L ($L < N$) narrowband far-field sources with the unknown DOAs $\{\theta_1, \dots, \theta_L\}$. The $N \times 1$ array snapshot vector at time k can be modeled as [1]-[4]

$$\boldsymbol{x}(k) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(k) + \boldsymbol{n}(k) \tag{1}$$

where $\boldsymbol{\theta} = [\theta_1, \dots, \theta_L]^T$ is the $L \times 1$ vector of signal DOAs,

$$\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), \dots, \boldsymbol{a}(\theta_L)] \tag{2}$$

is the $N \times L$ signal steering matrix, $\boldsymbol{s}(k)$ is the $L \times 1$ vector of signal waveforms, $\boldsymbol{n}(k)$ is the $N \times 1$ vector of sensor noise, and $(\cdot)^T$ denotes the transpose.

Assuming a non-uniform array of arbitrary geometry, the $N \times 1$ steering vector can be expressed as

$$\boldsymbol{a}(\theta) = \left[e^{j\frac{2\pi}{\lambda}(x_1 \sin\theta + y_1 \cos\theta)}, \dots, e^{j\frac{2\pi}{\lambda}(x_N \sin\theta + y_N \cos\theta)}\right]^T \tag{3}$$

where λ is the signal wavelength, $j = \sqrt{-1}$, and $\{x_i, y_i\}$ are the coordinates of the ith array sensor.

The $N \times N$ array covariance matrix can be written as [1]-[4]

$$\boldsymbol{R_x} = \mathrm{E}\{\boldsymbol{x}(k)\boldsymbol{x}^H(k)\} = \boldsymbol{A}\boldsymbol{R_s}\boldsymbol{A}^H + \sigma^2\boldsymbol{I} \tag{4}$$

where $\boldsymbol{R_s} = \mathrm{E}\{\boldsymbol{s}(k)\boldsymbol{s}^H(k)\}$ is the source covariance matrix, σ^2 is the sensor noise variance, $\boldsymbol{I}$ is the identity matrix, $\mathrm{E}(\cdot)$ denotes the statistical expectation, and $(\cdot)^H$ stands for the Hermitian transpose.

In practice, the exact array covariance matrix $\boldsymbol{R_x}$ is unavailable, and its sample estimate

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \frac{1}{K}\sum_{k=1}^{K}\boldsymbol{x}(k)\boldsymbol{x}^H(k) \tag{5}$$

is used, where K is the number of snapshots.

The eigendecomposition of the sample covariance matrix (5) can be written as [1]-[4]

$$\hat{\boldsymbol{R}}_{\boldsymbol{x}} = \hat{\boldsymbol{E}}_S\hat{\boldsymbol{\Lambda}}_S\hat{\boldsymbol{E}}_S^H + \hat{\boldsymbol{E}}_N\hat{\boldsymbol{\Lambda}}_N\hat{\boldsymbol{E}}_N^H \tag{6}$$

where the orthonormal columns of $\hat{\boldsymbol{E}}_S$ and $\hat{\boldsymbol{E}}_N$ contain the signal- and noise-subspace eigenvectors of $\hat{\boldsymbol{R}}_{\boldsymbol{x}}$, respectively, and the diagonal matrices $\hat{\boldsymbol{\Lambda}}_S$ and $\hat{\boldsymbol{\Lambda}}_N$ are built from the signal- and noise-subspace eigenvalues of $\hat{\boldsymbol{R}}_{\boldsymbol{x}}$, respectively.

The conventional MUSIC null-spectrum function can be expressed as [1]

$$f(\theta) = \boldsymbol{a}^H(\theta)\hat{\boldsymbol{E}}_N\hat{\boldsymbol{E}}_N^H\boldsymbol{a}(\theta) = \|\hat{\boldsymbol{E}}_N^H\boldsymbol{a}(\theta)\|^2 \tag{7}$$

where $\|\cdot\|$ is the vector 2-norm. The spectral MUSIC technique estimates the signal DOAs from the minima of this function by means of a search over θ.

The key idea of the array interpolation method [7] is to interpolate a virtual ULA for preliminary specified angular sectors using actual non-uniform array data. Then, the root-MUSIC technique can be applied to the interpolated array observations.

Another root-MUSIC method for non-uniform arrays has been proposed in [9]. Using the results of [10], it can be shown that for any arbitrary array, the steering vector can be approximated as

$$\boldsymbol{a}(\theta) \approx \boldsymbol{G}\boldsymbol{b}(\theta) \tag{8}$$

where $\boldsymbol{G}$ is an $N \times M$ matrix that depends only on the array parameters, and

$$\boldsymbol{b}(\theta) = \left[e^{-j\frac{M-1}{2}\theta}, \ldots, e^{j\frac{M-1}{2}\theta}\right]^T$$

is an $M \times 1$ Vandermonde vector which depends only on θ and M. The truncation parameter M characterizes the accuracy of the approximation used in (8). Specifically, equation (8) is exact only for $M \to \infty$, and the approximation in (8) becomes more accurate when increasing M.

The authors of [9] have proposed to use the approximation (8) for some finite M to rewrite the MUSIC null-spectrum (7) in the form

$$\begin{aligned} f(\theta) &= \boldsymbol{a}^H(\theta)\hat{\boldsymbol{E}}_N\hat{\boldsymbol{E}}_N^H\boldsymbol{a}(\theta) \\ &\approx \boldsymbol{b}^H(\theta)\boldsymbol{G}^H\hat{\boldsymbol{E}}_N\hat{\boldsymbol{E}}_N^H\boldsymbol{G}\boldsymbol{b}(\theta) \\ &= \boldsymbol{b}^T(1/z)\boldsymbol{G}^H\hat{\boldsymbol{E}}_N\hat{\boldsymbol{E}}_N^H\boldsymbol{G}\boldsymbol{b}(z) \triangleq g(z) \end{aligned} \tag{9}$$

where $z \triangleq e^{j\theta}$, and the degree of the polynomial $g(z)$ is $2M-2$. It is suggested in [9] to obtain the signal DOAs from the closest to the unit circle roots of $g(z)$ in the way similar to that used in the conventional root-MUSIC technique.

Several ways to compute the matrix $\boldsymbol{G}$ have been discussed in [10] and [9], such as the least squares (LS) approach and an approach that determines each element of $\boldsymbol{G}$ via the inverse discrete Fourier transform (IDFT) of different components of the steering vector $\boldsymbol{a}(\theta)$ taken at different angles. The authors of [9] mention that, irrespectively of the chosen way to compute $\boldsymbol{G}$, the parameter M should be taken large enough (typically $M > N$ and often $M \gg N$) to achieve an acceptable DOA estimation performance.

3. FOURIER-DOMAIN ROOT-MUSIC

In this section, we will develop an alternative polynomial rooting based approach to DOA estimation in arbitrary non-uniform arrays. Let us make use of the fact that, according to (3), the MUSIC null-spectrum (7) is a periodic function of θ with the period 2π. Then, its Fourier series expansion yields

$$f(\theta) = \sum_{m=-\infty}^{\infty} F_m\, e^{jm\theta} \tag{10}$$

where the Fourier coefficients are given by

$$F_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-jm\theta}\, d\theta. \tag{11}$$

Truncating the Fourier series in (10) to $2M-1$ points[1] and using the notation $z = e^{j\theta}$, we can approximate $f(\theta)$ as

$$f(\theta) \simeq \sum_{m=-M+1}^{M-1} F_m z^m \triangleq p(z) \tag{12}$$

The Fourier coefficients F_m, $m = -M+1, \ldots, M-1$ can be computed using DFT in a standard way. Although in this case a close approximation of the Fourier series coefficients can be obtained in a computationally efficient way, the DFT coefficients are not exactly equal to the Fourier series coefficients due to aliasing introduced by sampling the null-spectrum $f(\theta)$. Hence, we use the following DFT-based approximation of $p(z)$:

$$\tilde{p}(z) \triangleq \sum_{m=-M+1}^{M-1} \tilde{F}_m z^m \tag{13}$$

where $\tilde{F}_m$, $m = -M+1, \ldots, M-1$ are the DFT coefficients.

According to (13), the degree of $\tilde{p}(z)$ is $2M-2$, and the source DOAs can be obtained by means of rooting this polynomial. Let us now prove that the roots of (13) satisfy the conjugate reciprocity property, that is, if z_0 is a root of $\tilde{p}(z)$, then $z_0' \triangleq 1/z_0^*$ is also a root of this polynomial, where $(\cdot)^*$ stands for the complex conjugate. Assuming that z_0 is a root of $\tilde{p}(z)$ and taking into account that $\tilde{F}_m^* = \tilde{F}_{-m}$, we have

$$\begin{aligned} 0 &= \sum_{m=-M+1}^{M-1} \tilde{F}_m z_0^m \\ &= \sum_{m=-M+1}^{M-1} \tilde{F}_{-m} z_0^{*m} \\ &= \sum_{m=-M+1}^{M-1} \tilde{F}_m {z_0'}^m, \end{aligned} \tag{14}$$

[1] To define the truncation parameter M in a consistent way throughout this paper, we require that the resulting polynomials for both the proposed method and the technique of [9] are of the same order $2M-2$.

Algorithm	Computational Complexity
Spectral MUSIC	$O(N^3 + JNL)$
Interpolated root-MUSIC [7]	$O(N^3 + IN^2L$ $+ I \cdot$ degree-N rooting)
Nonuniform root-MUSIC [9]	$O(N^3 + MNL + M^2L$ $+$ degree-M rooting)
FD root-MUSIC	$O(N^3 + MNL + M\log_2 M$ $+$ degree-M rooting)
FD line-search MUSIC	$O(N^3 + MNL + J\log J)$

Table 1. The orders of computational complexities of spectral MUSIC, the algorithms of [7] and [9], and the proposed techniques.

that is, $z_0' = 1/z_0^*$ is also a root of $\tilde{p}(z)$.

Using the established conjugate reciprocity property, the signal DOAs can be estimated from the L closest to the unit circle roots of $\tilde{p}(z)$ lying inside this circle.

4. FOURIER-DOMAIN LINE-SEARCH MUSIC

Both the polynomials $g(z)$ and $\tilde{p}(z)$ are of degree $2M-2$. Therefore, as the value of M should be sufficiently large to warrant small truncation errors, the complexity of finding the roots of these polynomials may be rather high. Motivated by this fact, we will now consider a modification of the proposed FD root-MUSIC algorithm that avoids the polynomial rooting step by replacing it with a simple line search. The key idea of this modified approach follows the work in [11].

Let us use in (13) zero-padding to $2J-1$ ($J > M$) values where $2J-1$ is the required number of points of the null-spectrum. Then, we obtain

$$\tilde{p}(z) = \sum_{m=-J+1}^{J-1} \tilde{F}_m z^m = \sum_{m=-J+1}^{J-1} \tilde{F}_m e^{jm\theta} \triangleq \tilde{p}(\theta) \qquad (15)$$

where $\tilde{F}_m = 0$ for all values of m that satisfy $M-1 < |m| \leq J-1$. Hence, a total of $2J-1$ uniform in the interval $-\pi \leq \theta \leq \pi$ samples of the null-spectrum $\tilde{p}(\theta)$ can be found by applying IDFT to the zero-padded sequence of the coefficients $\tilde{F}_m$, $m = -J+1, \ldots, J-1$. As a result, no polynomial rooting is needed anymore, and the polynomial rooting step is replaced by a line search over θ.

There is an important difference between the spectral search involved in the conventional spectral MUSIC technique and in the proposed FD line-search MUSIC. The spectral search in (7) requires to compute the matrix-vector product $\hat{\boldsymbol{E}}_N^H \boldsymbol{a}(\theta)$ for each sample value of θ, while the proposed technique uses IDFT (that is computed in a numerically efficient way using FFT) to obtain the null-spectrum $\tilde{p}(\theta)$ for *all* samples of θ.

5. COMPARISON OF COMPUTATIONAL COMPLEXITIES

The computational complexities of spectral MUSIC, the techniques of [7] and [9], and the proposed two algorithms are compared in Table 1. In this table, I denotes the number of angular sectors in the interpolated root-MUSIC technique, and the number of virtual sensors in the latter technique is assumed to be of the same order as the number of actual sensors. As there is a variety of polynomial rooting algorithms with the cubic and lower complexity [12]-[13], the orders of complexity of the rooting steps in the second, third, and fourth techniques of Table 1 are not explicitly shown.

As can be observed from Table 1, the proposed techniques may offer substantial computational advantages with respect to some of the existing algorithms developed for non-uniform arrays. In particular, in the pragmatic case $M \gg N$, the computational complexity of the proposed methods can be substantially smaller than that of the approach of [9].

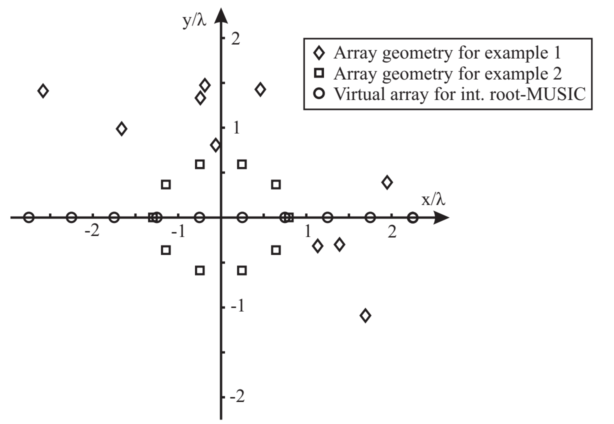

Fig. 1. Array geometries used in the first and second examples, and virtual array geometry used in interpolated root-MUSIC in both examples.

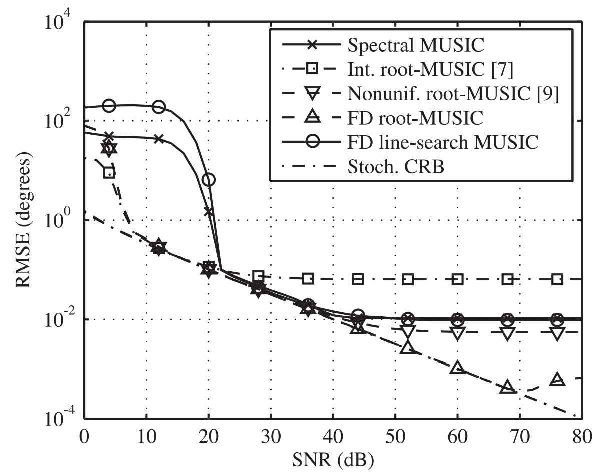

Fig. 2. DOA estimation RMSEs versus SNR; first example.

6. SIMULATION RESULTS

In all figures, we compare the performances of FD root-MUSIC and FD line-search MUSIC with that of spectral MUSIC, the techniques of [7] and [9], and the stochastic Cramér-Rao bound. Throughout simulations, we assume an array of $N = 10$ sensors and $L = 2$ equally powered signal sources with the DOAs $\theta_1 = 13^\circ$ and $\theta_2 = 15^\circ$ relative to the north direction of Fig. 1 where all the array geometries used in simulations are displayed.

In the first example, all sensor locations have been generated

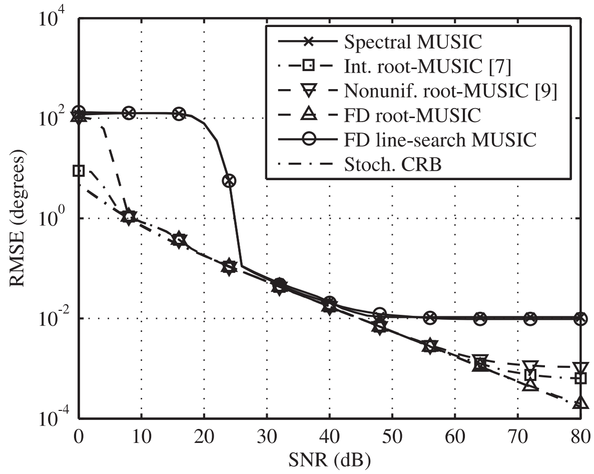

Fig. 3. DOA estimation RMSEs versus SNR; second example.

randomly. The geometry of this array is shown in Fig. 1 using diamond marks. Fig. 2 displays the DOA estimation root-mean-square errors (RMSEs) of all methods tested versus the signal-to-noise ratio (SNR) for $K = 100$ snapshots. In FD root-MUSIC, FD line-search MUSIC, and the technique of [9], $M = 50$ is taken. In interpolated root-MUSIC, the interpolation sector $[0°, 45°]$ is chosen, and the geometry of the interpolated ULA is shown in Fig. 1 by circle marks.

In the second example, the circular array is assumed whose geometry is displayed in Fig. 1 using square marks. Fig. 3 displays the DOA estimation RMSEs of all methods tested versus SNR for $K = 2000$ snapshots. In this figure, $M = 24$ is chosen for the proposed FD techniques and the method of [9]. The geometry of the interpolated ULA as well as the interpolation sector are the same as in the previous example.

From Figs. 2 and 3, it can be observed that in both examples, the proposed FD root-MUSIC method consistently outperforms the other methods tested. These performance improvements are achieved at high values of SNR. The fact that FD root-MUSIC outperforms the algorithm of [9] can be explained by substantially lower truncation errors of the FD root-MUSIC approach as compared to the method of [9]. It also follows from Figs. 2 and 3 that the performance of FD line-search MUSIC is quite close to that of spectral MUSIC, but is worse than that of FD root-MUSIC. Such a performance loss of FD line-search MUSIC with respect to FD root-MUSIC can be viewed as a price for avoiding the polynomial rooting step in the line-search MUSIC algorithm.

7. CONCLUSIONS

In this paper, we have developed a novel root-MUSIC-type approach to DOA estimation in sensor arrays of arbitrary geometry. Our approach is referred to as Fourier-domain root-MUSIC and exploits the fact that the null-spectrum MUSIC function is periodic in angle. It uses the truncated Fourier series expansion of this function to reformulate the DOA estimation problem as a polynomial rooting problem.

To avoid the polynomial rooting step and further reduce the computational complexity, the inverse Fourier transform has been applied to the Fourier-domain root-MUSIC polynomial to compute the null-spectrum and estimate the source DOAs by means of a simple line search.

It has been demonstrated through simulations with different array configurations that the proposed techniques offer attractive alternatives to the existing DOA estimation methods applicable to arrays with arbitrary geometry.

8. REFERENCES

[1] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," in *Proc. RADC Spectral Estimation Workshop*, Rome, NY, 1979, pp. 243-258.

[2] G. Bienvenu and L. Kopp, "Adaptivity to background noise spatial coherence for high resolution passive methods," in *Proc. ICASSP'80*, Denver, CO, USA, April 1980, pp. 307-310.

[3] A. Barabell, "Improving the resolution performance of eigenstructure-based direction-finding algorithms," in *Proc. ICASSP'83*, Boston, MA, USA, April 1983, pp. 336-339.

[4] B. D. Rao and K. V. S. Hari, "Performance analysis of root-MUSIC," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 37, pp. 1939-1949, Dec. 1989.

[5] M. Pesavento, A. B. Gershman, and K. M. Wong, "Direction finding using partly calibrated sensor arrays composed of multiple subarrays," *IEEE Trans. Signal Processing*, vol. 50, pp. 2103-2115, Sept. 2002.

[6] S. Abd Elkader, A. B. Gershman, and K. M. Wong, "Rank reduction direction-of-arrival estimators with improved robustness against subarray orientation errors," *IEEE Trans. Signal Processing*, vol. 54, pp. 1951-1955, May 2006.

[7] B. Friedlander, "The root-MUSIC algorithm for direction finding with interpolated arrays," *Signal Processing*, vol. 30, pp. 15-25, 1993.

[8] P. Hyberg, M. Jansson, and B. Ottersten, "Array interpolation and bias reduction," *IEEE Trans. Signal Processing*, vol. 52, pp. 2711-2720, Oct. 2004.

[9] F. Belloni, A. Richter and V. Koivunen, "Extension of root-MUSIC to non-ULA array configurations," in *Proc. ICASSP'06*, Toulouse, France, May 2006, vol. 4, pp. 897-900.

[10] M. A. Doron, E. Doron, and A. J. Weiss, "Coherent wide-band processing for arbitrary array geometry," *IEEE Trans. Signal Processing*, vol. 413-417, Jan. 1993.

[11] K. V. S. Babu, "A fast algorithm for adaptive estimation of root-MUSIC polynomial coefficients," in *Proc. ICASSP'91*, Toronto, Ontario, Canada, May 1991, pp. 2229-2232.

[12] C. A. Neff and J. H. Reif, "An efficient algorithm for the complex roots problem," *Journal of Complexity*, vol. 12, no. 2, pp. 81-115, 1996.

[13] M. Lang, B.-C. Frenzel, "Polynomial root finding," *IEEE Signal Processing Letters*, vol. 1, pp. 141-143, Oct. 1994.