Dual Systems for Minor and Principal Component Computation

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Abstract: Converting principal component dynamical system to a minor component dynamical system and vice versa sometimes leads to unstable systems. In this paper, classes of globally stable dynamical systems that can be converted between PCA and MCA systems by merely switching the signs of some terms of a given system are developed. These systems are shown to be applicable to symmetric and nonsymmetric matrices. These systems are then modified to be asymptotically stable by adding a penalty term. The proposed systems may apply to both the standard and the generalized eigenvalue problems. Lyapunov stability theory and LaSalle invariance principle are used to derive invariant sets for these systems.

Keywords: Principal components, minor components, generalized eigenvalue problem, Lyapunov stability, global convergence, Oja's learning rule, Rayleigh quotient, dual-purpose MCA/PCA systems.

1 Introduction

Let $g(x) : \mathbb{R}^{n\times p} \to \mathbb{R}^{n\times p}$, $p \leq n$, be a continuously differentiable function and consider the dynamical systems

$$x' = g(x), \tag{1a}$$

$$x' = -g(x). \tag{1b}$$

The symbol $\mathbb{R}$ denote the set of real numbers. A set $S \in \mathbb{R}^{n\times p}$ is an *invariant set* for the system (1) if every trajectory $x(t)$ which starts from a point in S remains in S for all time. For example, any equilibrium point is an invariant set. The domain of attraction of an equilibrium point is also an invariant set.

From the Lyapunov's indirect method (Theorem 6, see Appendix), an equilibrium point, $x = \hat{x}$, for the nonlinear system (1), i.e., $g(\hat{x}) = 0$, is asymptotically stable if all eigenvalues of $G = \frac{\partial g}{\partial x}|_{x=\hat{x}}$ have negative real parts, i.e., $Re(\lambda_i) < 0$ for each eigenvalue λ_i of G, where $Re(\lambda)$ denotes the real part of λ. The point $\hat{x}$ is unstable if $Re(\lambda_i) > 0$ for some eigenvalue of G.

Thus if $Re(\lambda_i) \neq 0$ for each i then the systems $\dot{x} = g(x)$, and $\dot{x} = -g(x)$, can not be both stable. In other words, if both systems are stable, then $Re(\lambda_i) = 0$. This is particularly true if $\frac{\partial g}{\partial x}|_{x=\hat{x}}$ is skew symmetric.

Using the perspective of the Lyapunov's direct method (Theorem 7, see Appendix), one may consider the Lyapunov function $V = \frac{1}{2}tr(x^T x)$. Here x^T denotes the transpose of a real matrix x, and $tr(C)$ denotes the trace of a square matrix C, which is the sume of the diagonal elements of C. The time derivative of V along the trajectories of (1a) is $\dot{V} = tr(x^T g(x))$. Thus if $x^T g(x)$ is skew symmetric, then $tr(x^T g(x)) = 0$, and consequently $V(x(t))$ is constant, i.e., $V(x(t)) = V(x(0))$ for $t \geq 0$. This shows that if $x^T g(x)$ is skew symmetric, then the systems (1a) and (1b) are stable and that $||x(t)|| = ||x_0||$. It should be noted that most of the results of this paper are based on the assumption that $x^T g(x)$ is skew symmetric.

Now assuming that $x^T g(x)$ is skew symmetric, one may modify the systems (1a) and (1b) so that these are asymptotically stable. Thus consider

$$\begin{aligned} \dot{x} &= g(x) - \alpha x(x^T x - D), \\ \dot{x} &= -g(x) - \alpha x(x^T x - D), \end{aligned} \tag{2}$$

where D is some positive definite matrix, and $\alpha \geq 0$.

Let $V = tr(x^T x - D)^2$, then the time derivative of V along any solution of (2) is

$$\begin{aligned} \dot{V} = &\pm tr\{(x^T x - D)x^T g(x)\} - \alpha tr\{x^T x(x^T x - D)^2\} \\ &- \alpha tr\{x^T x(x^T x - D)^2\} \leq 0. \end{aligned} \tag{3}$$

The last inequality follows from the observation that $x^T x - D$ is symmetric, $x^T g(x)$ is skew symmetric, and Proposition 8. Theorem 7 implies that both systems of (2) are asymptotically stable provided that $x^T g(x)$ is skew symmetric and $\alpha > 0$.

For the special case where $x \in \mathbb{R}^{n\times 1}$, it follows that $\dot{V} = -\alpha V(V - D)^2$. It can be shown that the exact solution $V(t)$ satisfies the equation

$$\frac{V}{V - D} e^{\frac{D}{V - D}} = ce^{-\alpha D^2 t}, \tag{4}$$

where c is a constant given by $c = \frac{V(0)}{V(0)-D} e^{\frac{D}{V(0)-D}}$. Clearly as $t \to \infty$, we have $V \to D$.

Asymptotic stablility can also be obtained by replacing the axtra terms in (2) with $-\alpha x(x^T x)^{-1}(x^T x - D)^2$, $\alpha > 0$ and D is positive definite matrix:

$$\begin{aligned} \dot{x} &= g(x) - \alpha x(x^T x)^{-1}(x^T x - D)^2, \\ \dot{x} &= -g(x) - \alpha x(x^T x)^{-1}(x^T x - D)^2. \end{aligned} \tag{5}$$

This case has the additional advantage of having $V(x) = \frac{1}{2}tr(x^T x)$ as a Lyapunov function for the systems. Clearly $\dot{V} = -\alpha(x^T x - D)^2 \leq 0$. Theorem 7 implies that the systems of (5) are asymptotically stable. Note that if $x \in \mathbb{R}^{n\times 1}$, then $\dot{V} = -\alpha(V - D)^2$. The exact solution satisfies $\frac{1}{V-D} = \alpha t + \frac{1}{V(0)-D}$, or $V(t) = D + \frac{1}{\alpha t + \frac{1}{V(0)-D}}$ from which it follows that as $t \to \infty$, we have $V(t) \to D$. Also, if $V(0) = D$, then $V(t) = D$ for $t \geq 0$.

The next result provides a generalization of (5).

Proposition 1. *Consider the autonomous systems defined by*

$$x' = g(x)P - \alpha x(x^Tx)^{-1}(x^Tx - D)^2, \qquad (6a)$$

$$x' = -g(x)P - \alpha x(x^Tx)^{-1}(x^Tx - D)^2, \qquad (6b)$$

where $x^Tg(x)$ is skew symmetric for each $x \in \mathbb{R}^{n\times p}$, D is any positive definite matrix, and $\alpha > 0$. Then for each positive definite matrix $P(x)$, the systems (6a) and (6b) are asymptotically stable. In particular, if $x(t)$ is any full rank solution of (6a) or (6b), then $x(t)^Tx(t) \to D$ as $t \to \infty$.

Proof: Let $V(x) = \frac{1}{2}tr(x^Tx)$, then $\dot{V} = -\alpha(x^Tx - D)^2 \leq 0$ according to Proposition 8. Theorem 7 implies that the systems of (6) are asymptotically stable.

2 Applications to The Eigenvalue problem

Let $A \in \mathbb{R}^{n\times n}$ and assume that all eigenvalues of A are of distinct magnitude. It is known that the columns of $x \in \mathbb{R}^{n\times p}$ span a p-dimensional eigenspace of a matrix A if and only if x is a rank p solution of the equation $Axf = xh$ for some full rank matrices $f(x)$ and $h(x)$ of appropriate dimensions.

In this paper, principal (PCA) and minor (MCA) component analyzers of a real matrix are developed. These analyzers are matrix differential equations that converge to the eigenvectors associated with the largest and smallest eigenvalues of a given matrix. Similarly, principal (PSA) and minor (MSA) subspace analyzers of a symmetric matrix are matrix differential equations that converge to a matrix whose columns's span is the subspace spanned by the eigenvectors corresponding to the largest and smallest eigenvalues, respectively.

The interest in PCA and MCA stems from the fact that they are useful tools in adaptive antenna arrays in signal processing, multiuser detection in wireless communication, and truncated model reduction tasks.

Minor and principal subspace/component analyzers have been investigated by many authors. Early works include those of Oja [1], Chen and Amari [2], Sanger [3], Xu [4], and others [5,6]. A common method for converting a PCA/PSA flow into an MCA/MSA one is by changing the sign of the given matrix, or by using the inverse of the original matrix. However, inverting a large matrix is a costly task, and changing the sign of the original matrix does not always generate a stable system unless frequent orthonormalization is employed during the numerical implementation. A stabilization approach to principal and minor components algorithms is given in [2]. The main objective of this paper is to introduce a framework for developing classes of stable dynamical systems that can be easily converted from PCA flow into MCA flow and vice versa.

Throughout this paper, the following notation will be used. The identity matrix of appropriate dimension is expressed with the symbol I. Also, the derivative of $V(x)$ with respect to time along a trajectory $x' = g(x)$ is denoted by $\dot{V}$. Finally, it will be assumed that the matrix A has distinct eigenvalues unless otherwise stated.

2.1 The proposed MSA/PSA Dynamical Systems

Let Ω_1 be defined so that $\Omega_1 = \{x \in \mathbb{R}^{n\times p} : x^Tx \text{ is positive definite}\}$. In this section, dynamical systems for implementing dual-purpose MCA/MSA and PCA/PSA algorithms are developed. In what follows, it will be assumed that the initial condition x_0 for dynamical systems satisfies $x_0 = x(0) \in \Omega_1$. A special case of the system (2) is given in the next result.

Proposition 2. *Let $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{p\times p}$, and $x, x(0) \in \Omega_1 \subset R^{n\times p}$ and consider the following systems:*

$$x' = Axf_1(x) - x(x^Tx)^{-1}f_1(x)^Tx^TA^Tx - \alpha x(x^Tx - D) \quad (7a)$$

$$\begin{aligned} x' = {} & Ax - x(x^Tx)^{-1}f_2(x)^Tx^TA^Txf_2(x)^{-1} \\ & - \alpha x(x^Tx)^{-1}(x^Tx - D)^2, \end{aligned} \qquad (7b)$$

where D is positive definite matrix, $f_1(x)$ and $f_2(x)$ are full rank matrices for every full rank x. Assume also that $f_2(x)^T = f_2(x)$ is positive definite and that all eigenvalues of A are of distinct magnitude. Then the systems (7a) and (7b) are asymptotically stable. Moreover, $x(t)^Tx(t) \to D$ as $t \to \infty$. Additionally, if $f_1 = D$, where D is diagonal with positive distinct eigenvalues, and $A^T = A$, then both x^Tx and x^TAx converge to diagonal matrices, i.e., depending on the initial condition $x(0)$, both (7a) and (7b) are asymptotically stable dynamical systems for computing principal subspaces of the matrix A.

Outline of a Proof: Assume that $\alpha = 1$ for convenience. Asymptotic stability of the system of (7a) follows from Proposition 8 and Theorem 7. Assume that $x(t)$ is a full rank solution of (7a) such that $x(t) \to \hat{x}$ as $t \to \infty$. To show that $\hat{x}^T\hat{x}$ and $\hat{x}^TA\hat{x}$ converge to diagonal matrices, let $P = \hat{x}^T\hat{x}$ and $B = \hat{x}^TA\hat{x}$, then

$$BP - PB^T = \alpha P(P - D).$$

Thus $P(P - D)$ is skew symmetric and therefore $P(P - D) + P(P - D) = 0$. Let z be an eigenvector of $P - D$ with corresponding eigenvalue λ, then $(P - D)z = \lambda z$ and therefore $2\lambda(z^TPz) = 0$. Since P is positive definite, then $\lambda = 0$. This shows that each eigenvalue of the symmetric matrix $P - D$ is zero and hence $P = D$. Consequently, $BD = DB^T$. If it is assumed that $A^T = A$, then $BD = DB$ and from Proposition 9, it follows that B is diagonal.

Assume that $x(t)$ is a full rank solution for (7b) and let $P_1 = f_2(\hat{x})$. Then $B - P_1B^TP_1^{-1} = (P - D)^2$, or equivalently, $BP_1 - P_1B^T = (P - D)^2P_1$. Thus $(P - D)^2P_1$ is skew symmetric and hence $(P - D)^2P_1 + P_1(P - D)^2 = 0$. Since $(P - D)^2$ is symmetric and P_1 is positive definite, it follows that $P = D$. Consequently, if $f_2(x) = q(x^Tx)$ where $q(z)$ is a polynomial of degree $m \geq 1$, then $P_1 = q(D)$ is diagonal. It follows from Proposition 9 that B is diagonal provided that A is symmetric and all eigenvalues of P_1 are distinct.

Remark 1: If it is assumed in the system (7a) that $f_1(x) = (x^Tx)^r$, where r is any integer, the following systems will be obtained:

$$x' = \pm(Ax(x^Tx)^r - x(x^Tx)^{r-1}x^TA^Tx) - \alpha x(x^Tx - D).$$

As in the previous analysis, if $A^T = A$ and if $x(t)$ is a full rank solution of any of the above systems, then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^T\hat{x}$ and $\hat{x}^TA\hat{x}$ are diagonal.

The derivation of the systems in Remark 1 can be obtained by optimizing the Rayleigh quotient $tr\{(x^TAx)(x^Tx)^{-1}\}$ over the set Ω_1. In particular, if $f(x) = x^Tx$, i.e., $r = 1$, then (7a) reduces to

$$x' = \pm(Axx^Tx - xx^TA^Tx) - \alpha x(x^Tx - D). \qquad (8)$$

When A is positive definite and $x \in \mathbb{R}^{n\times 1}$, these can be viewed as generalized gradient systems since

$$Axx^Tx - xx^TAx - \alpha x(x^Tx - D) = \{\nabla f(x)\}(x^Tx)^2 + \nabla h(x), \quad (9a)$$

where $f(x) = \frac{x^TAx}{x^Tx}$ and $h(x) = -\frac{\alpha}{4}(x^Tx - D)^2$. Similarly,

$$xx^TAx - Axx^Tx - \alpha x(x^Tx - D) = \{\nabla \frac{1}{f(x)}\}(x^TAx)^2 + \nabla h(x). \quad (9b)$$

Stable dynamical systems can also be derived using Proposition 1 in which case for any symmetric positive definite $f(x)$, the following systems

$$x' = (Ax - x(x^Tx)^{-1}x^TA^Tx)f - \alpha x(x^Tx)^{-1}(x^Tx - D)^2, \quad (10a)$$

$$x' = (x(x^Tx)^{-1}x^TA^Tx - Ax)f - \alpha x(x^Tx)^{-1}(x^Tx - D)^2, \quad (10b)$$

can be shown to be asymptotically stable.

Proposition 3. *In the systems defined in (10) let $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{p\times p}$, be symmetric matrices and all their eigenvalues are distinct and positive. Assume that $x(t)$ is a full rank solution of (7a) such that $x(t) \to \hat{x}$ as $t \to \infty$, then $\hat{x}^T\hat{x}$ and $\hat{x}^TA\hat{x}$ are diagonal.*

Outline of a Proof: Assume that $\alpha = 1$ for convenience. Let $P = \hat{x}^T\hat{x}$, $F = f(\hat{x})$, and $B = \hat{x}^TA\hat{x}$, then

$$(BP - PB^T)F = (P - D)^2.$$

Thus $(P - D)^2F^{-1}$ is skew symmetric and therefore $F(P - D)^2 = -(P - D)^2F$. Let z be and eigenvector of $P - D$ with corresponding eigenvalue λ, then $(P - D)z = \lambda z$ and therefore $2\lambda(z^TFz) = 0$. Since F is positive definite, then $\lambda = 0$. This shows that each eigenvalue of the symmetric matrix $P - D$ is zero and hence $P = D$. Consequently, $BD = DB$. From Proposition 9, it follows that B is diagonal.

In the next result we show that a slight modification of (8) leads to upper-triangulization of x^TAx.

Proposition 4. *Let $A \in \mathbb{R}^{n\times n}$, and consider the following systems:*

$$x' = \pm(Axx^Tx - xUT\{x^TA^Tx\}). \quad (11)$$

If $x(t)$ be a full rank solution of (11), then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^TA\hat{x}$ is upper triangular. In particular, if $A^T = A$, then both $\hat{x}^T\hat{x}$ and $\hat{x}^TA\hat{x}$ are diagonal. Here $UT\{y\}$ denotes the matrix whose diagonal and the upper triangular part are the same as that of y while the entries of the lower triangular part are zeros.

Outline of a Proof: Let $P = \hat{x}^T\hat{x}$ and $B = \hat{x}^TA\hat{x}$, then

$$BP = PU,$$

where $B^T = U + L$. Here U and L are upper triangular and lower triangular matrices so that all diagonal elements of L are zeros. Hence $BP = P(B^T - L)$ or $BP - PB^T = -PL$ is skew symmetric. Consequently, $PL + L^TP = 0$. Since $L^r = 0$ for some positive integer r, one can show that $L^{r-1} = 0$ as follows: Post- and pre-multiplying the last equation by $(L^{r-2})^T$ and L^{r-1}, respectively, yield $(L^{r-2})^TPL^r + (L^{r-1})^TPL^{r-1} = 0$. This shows that $L^{r-1} = 0$ since P is positive definite. Hence $L = 0$ by induction. This implies that $B = U$. If $B^T = B = U$, then B must be diagonal, i.e., $B = D_1$ for some diagonal matrix D_1, and

$$D_1P = PD_1.$$

From Proposition 9, if all eigenvalues of D_1 are distinct, then P is diagonal.

Remark 2: One can also consider analogous systems to those of (11) so that

$$x' = +(AxUT(x^Tx) - xx^TA^Tx), \quad (12a)$$

$$x' = -(AxUT(x^Tx) - xx^TA^Tx). \quad (12b)$$

Let $x(t)$ be a full rank solution of (12a) or (12b). Then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^T\hat{x}$ is diagonal and $B = D_1B^TD_1^{-1}$, where $D_1 = \hat{x}^T\hat{x}$. Hence if A is symmetric, then $B = \hat{x}^TA\hat{x}$ is diagonal provided that all eigenvalues of D_1 are distinct.

3 Generalized Eigenvalue Problem

The analysis of PCA/PSA and MCA/MSA learning systems of the previous section may be applied to the generalized eigenvalue problem involving two matrices $A, B \in \mathbb{R}^{n\times n}$, where B is positive definite. Thus we consider the following systems:

$$x' = +(Axx^TB^Tx - Bxx^TA^Tx) - \alpha Bx(x^TBx - D), \quad (13a)$$

$$x' = -(Axx^TB^Tx - Bxx^TA^Tx) - \alpha Bx(x^TBx - D), \quad (13a)$$

where D is positive definite diagonal matrix and $\alpha \geq 0$.

It can be shown that these systems are stable and that if $x(t)$ is a full rank solution of (13a), then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^TB\hat{x} = D$. If it is also assumed that $A^T = A$, then $\hat{x}^TA\hat{x}$ is diagonal. Additionally, the column space of $\hat{x}$ is a p-dimensional eigen subspace of the pencil (A, B).

Proposition 5. *Let $A, B \in \mathbb{R}^{n\times n}$, and assume that B is invertible so that all eigenvalues of the matrix $B^{-1}A$ are of distinct magnitude. Consider the following systems:*

$$x' = (Axx^TB^Tx - BxUT\{x^TA^Tx\}), \quad (14a)$$

$$x' = -(Axx^TB^Tx - BxUT\{x^TA^Tx\}). \quad (14b)$$

Assume that $x(t)$ is a full rank solution of (14a). Then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^TA\hat{x}$ is diagonal. Additionally, the column space of $\hat{x}$ is a p-dimensional eigen subspace of the pencil (A, B). Similarly, if $x(t)$ is a full rank solution of (14b), then $x(t)$ converges to a matrix $\hat{x}$ such that $\hat{x}^TA\hat{x}$ is diagonal. Moreover if $A^T = A$, and B is positive definite, then both $\hat{x}^TB\hat{x}$ and $\hat{x}^TA\hat{x}$ are diagonal.

Remark 3: It should noted that in Propositions 2, 3, 4, and 5, a solution $x(t)$ may or may not converge to principal or minor subspaces depending on the initial condition $x(0)$. Simulation results have been conducted to examine the convergence behavior of these systems using matrices or pencils having eigenvalues of distinct magnitude. These indicated that starting from a random full rank initial condition, the proposed systems converge to PSA or MSA all the time. A reasonable conjecture is that if $x(0)^TU$ is non-singular where U is an $n \times p$ matrix consists of p principal eigenvectors or p minor eigenvectors, then $x(t)$ converges to PSA or MSA, respectively.

4 Conclusions

Many dual-purpose learning systems for computing minor and principal components of general matrices are derived. The stability of these systems is based on properties of skew symmetric matrices. The most important features are that: (1) a PCA/PSA system may be switched into a learning rule for extracting MCA/MSA by merely multiplying few terms of the learning rule by -1, (2) these systems apply to more general matrices such as definite and non-definite matrices or symmetric and non-symmetric matrices, and (3) many of these systems can be easily extended to the generalized eigenvalue problem. There are many issues that still need to be investigated. These include the dependence of these systems on the rank of initial conditions and the behavior of these systems when they are applied to complex matrices. These will be detailed in a forthcoming paper.

5 Appendix

In this section, we introduce several known results from Lyapunov stability theory of dynamical systems. Let $g(x) : \mathbb{R}^{n\times p} \to \mathbb{R}^{n\times p}$, $p \le n$, be continuously differentiable function and consider the dynamical system

$$x' = g(x). \qquad (A_1)$$

A few stability results for nonlinear autonomous systems will be stated here.

The first result in this appendix gives conditions under which we can draw conclusions about the local stability of an equilibrium point of a nonlinear system by investigating the stability of a linearized system.

Theorem 6 (Lyapunov's Indirect Method). *Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = g(x)$, where $g : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let the Jacobian matrix A at $x = 0$ be:*

$$A = \frac{\partial g}{\partial x}|_{x=0}. \qquad (A_2)$$

Let λ_i, $i = 1, \cdots, n$ be the eigenvalues of A. Then,

1. *The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A.*
2. *The origin is unstable if $Re(\lambda_i) > 0$ for any of the eigenvalues of A.*

Here $Re(\lambda)$ denotes the real part of λ.

Proof. The proof of this theorem can be found in Khalil (2002) [7].

If some of the eigenvalues of $\frac{\partial g}{\partial x}|_{x=0}$ lie on the imaginary axis, further analysis using the central manifold theory is needed to establish stability.

Theorem 7 (Local Invariant Set Theorem) [8]. *Consider the autonomous system (1) with g continuous and let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a scalar function with continuous first partial derivatives. Assume that*

1. *for some $l > 0$, the set Ω_l defined by $V(x) \le l$ is bounded.*
2. *$V'(x) \le 0$ for all x in Ω_l.*

Let R be the set of all points within Ω_l where $V'(x) = 0$ and M be the largest invariant set in R. Then, every solution $x(t)$ originating in Ω_l tends to M as $t \to \infty$.

In Theorem 7, the word largest means that M is the union of all invariant sets within R. Notice that R is not necessarily connected, nor is the set M.

The invariant set theorems reflect the intuition that the decrease of a Lyapunov function V has to gradually vanish. In other words $\dot{V}$ has to converge to zero because V is lower bounded.

To prove stability for some of the proposed systems in this paper, the following theorem is needed. It examines the trace of a product of symmetric and anti-symmetric matrices.

Proposition 8. *Let $C, S \in \mathbb{R}^{n\times n}$ and assume that S is antisymmetric matrix. Then $tr((A + A^T)S) = 0$. Hence if P is symmetric, then $tr(PS) = 0$.*

Proof. Clearly, $tr(AS) = tr(A^T S^T) = -tr(A^T S)$. Hence $tr((A + A^T)S) = 0$.

Proposition 9 [9]. *Let $D, B \in \mathbb{R}^{n\times n}$ be positive definite matrices and assume that D is diagonal having distinct eigenvalues. If $BD = DB$, then B is diagonal.*

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