OPPORTUNISTIC SAMPLING BY LEVEL-CROSSING

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ABSTRACT

Level-crossing A/D converters (LCA/D) have been considered in the literature and have been shown to efficiently sample certain classes of signals. In this paper we provide a stable algorithm to perfectly reconstruct signals of finite rate of innovation using level-crossing samples. Furthermore, we also apply level-crossing sampling to detection of event-arrival signals.

***Index Terms*—** non-uniform sampling, level-crossing, FRI, point processes,

1. INTRODUCTION

There is an extensive body of literature on sampling of analog signals [1]. The celebrated Shannon's sampling theorem established a uniform sampling method for bandlimited signals. For non-bandlimited signals, however, no one single scheme has emerged. An alternative method to uniform sampling, called level-crossing (LC) has been proposed in the literature in a variety of context [2]-[7]. In this scheme, signals are level-crossed with a fixed set of thresholds and samples are taken on the time axis. It is a form of nonuniform sampling that lets the signal dictate the frequency of data collection and quantization: more samples are taken when the signal is bursty, and less when otherwise. Intuitively, for certain signals, it is opportunistic to sample this way. As such, we want to further explore this efficiency in this paper.

An immediate question follows: What types of signals are suited for LC sampling? We focus our attention on event-arrival signals. A typical such signal is composed of scaled and delayed copies of a known pulse, where information is carried entirely by the arrival instances. Event-arrival signals exist in a diverse range of settings from neural spiking activities, communication signal transmissions, to queueing networks. They can be modelled as (deterministic) signals of finite rate of innovation (FRI) and (stochastic) point processes. We will use both frameworks to study the capabilities of LC sampling.

One such capability is stable reconstruction. Deterministic event-arrival signals are well modelled by the class of FRI signals, which is established by recent work of Vetterli *et al.*[8] to have finite number of free parameters per unit interval. A class of non-bandlimited FRI signals has been shown to be recoverable from uniform sampling with the aid of an annihilating filter. This innovative method however comes with limits, such that the reconstruction algorithm is unstable and physically non-realizable. This is addressed in [9], where a local reconstruction algorithm is proposed. In this paper we will show that with a properly chosen pre-filter, we can reconstruct a class of such signals perfectly and stably with LC samples.

Another application that showcases the opportunistic nature of LC sampling is detection of random parameters from a finite-interval observation. The signals are modelled as point processes with time-varying and history-dependent conditional intensity. We will show that for a nontrivial class of event-arrival signals, level-crossing sampling can guarantee a certain threshold of performance in less time than uniform sampling.

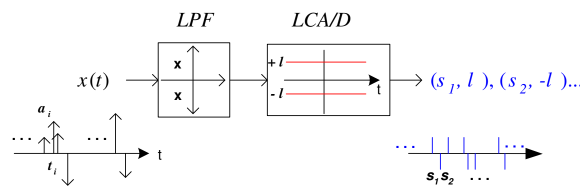

Fig. 1. The level-crossing sampling scheme of a FRI signal.

2. PERFECT RECONSTRUCTION FROM LEVEL-CROSSING SAMPLES

We consider a typical FRI signal model as in [8]:

$$x(t) = \sum_{i=1}^{K} a_i \delta(t - t_i), \ 0 \leq t \leq T. \tag{1}$$

$x(t)$ is a stream of K Diracs, where the coefficients $\{a_i\}_{i=1}^K$ and the time delays $\{t_i\}_{i=1}^K$ are free parameters. Together there are $2K$ unknowns in an interval of T. As such, $x(t)$ has a rate of innovation $r = \frac{2K}{T}$. For the ease of analysis, let $x(t)$ also have the following properties:

Property 1: The amplitude coefficients a_i, $1 \leq i \leq K$, are bounded, i.e., $0 < |a_i| < \infty$.

Property 2: The Diracs are ϵ-distinct: $\inf_{i\neq j}|t_i - t_j| > \epsilon,\ \epsilon > 0$.

2.1. Low-pass filter h

Instead of sampling $x(t)$ directly, we sample the output of $x(t)$ through a low-pass filter h, as shown in Figure 1, where h is a causal two-pole system, characterized by amplitude $C > 0$, decay constant α, and oscillating frequency ω_o. Its impulse response is of the form

$$h(C, \alpha, \omega_o, t) = C\, e^{-\alpha t} \cos(\omega_o t)\, u(t). \tag{2}$$

The output of the filter is

$$y(t) = \sum_{i=1}^{K} a_i\, C e^{-\alpha(t-t_i)} \cos \omega_o(t - t_i) u(t - t_i), \quad t \geq 0. \tag{3}$$

We will reconstruct $x(t)$ from LC samples of $y(t)$.

2.2. 2-Level Sampling

The level-crossing A/D (LCA/D) is defined in two parts, a level sampler $\mathcal{L}$ followed by a quantizer q [7]. Here the $\mathcal{L}$ has two symmetric levels, $+l$ and $-l$, hence it is an *2-level* sampler. It outputs a sequence of samples, $\{(s_j, \pm l)\}_{j=1}^{N}$, where $y(s_j) = \pm l$. For now, we assume the crossing instants $\{s_j\}_{j=1}^{N}$ are not quantized, so information obtained by sampling has perfect resolution.

Since LCA/D lets the signal dictate when and where to sample, samples arrive non-uniformly. Each one establishes an equation that can be used later to solve for the unknown parameters and reconstruct $x(t)$:

$$y(s_j) = \sum_{i=1}^{k} a_i h(s_j - t_i) = \pm l, \ \ j = 1, 2, \cdots, N. \tag{4}$$

2.3. Criterion for stable reconstruction

Ideally, as long as there are as many equations as unknowns, namely $N \geq 2K$, a solution can be found for (4). Unfortunately, a unique solution is not always forthcoming. First, the set of equations (4) is neither linear nor can it be decoupled and transformed into a set of bilinear equations, thus making it difficult to obtain closed-form solutions. Second, the equations are not convex in $\{(a_i, t_i)\}_{i=1}^{N}$. Since the number of local minima grows with the dimension of the unknowns, finding the optimal solution is nontrivial. These two issues can easily lead to instability during reconstruction. We will address this problem in the following theorem.

Theorem 1: For every signal of finite rate of innovation (1), there exists a filter $h(C, \alpha, \omega_o)$ and a 2-level $\mathcal{L}$, such that the level-crossing samples of $x * h(t)$ with $\mathcal{L}$ can be used recursively to reconstruct $x(t)$ perfectly and stably. In particular, the filter's rate of decay has a lower bound

$$\alpha > \frac{\ln \frac{AC}{l}}{\epsilon}, \tag{5}$$

where A is an upperbound on the amplitudes. In addition, the threshold l needs to satisfy the following inequality:

$$l < a_i C, \ \ 1 \leq i \leq K. \tag{6}$$

Proof: $y(t)$ is a (sequential) stream of K decaying sinusoids (cosines), where the i-th term is scaled by a_i and delayed by t_i seconds. The i-th term alone is only a function of two unknowns, namely a_i and t_i. When K terms are superimposed together, however, a particular sample $y(s_j)$ has global dependence, i.e., $y(s_j)$ is a function of all the unknown parameters, hence the difficulty in finding a solution. Fortunately, this can be avoided by a careful selection of filter h.

Without loss of generality, let's assume C, the amplitude of the filter, is normalized to 1. Let Δ denote a time duration that guarantees the amplitude of every decaying cosine in (3) drops below l after Δ seconds. In other words,

$$\Delta = \{t \,|\, t > 0\,;\, |a_i| e^{-\alpha t} < l, \ 1 \leq i \leq K\}. \tag{7}$$

As such, the signal crosses l and produces samples. In order to localize the information carried by the samples, we want the samples to be taken before the next cosine arrives, i.e., $\Delta < \epsilon$. This is accomplished by picking a value for α that satisfies the following constraint:

$$\alpha = \{v \,|\, v > 0\,;\, \frac{1}{v} \ln \frac{|a_i|}{l} > \epsilon, \ 1 \leq i \leq K\}. \tag{8}$$

Since the input is amplitude bounded, then there exists an A, such that $a_i \leq A, \ 1 \leq i \leq K$. Combining this with (7) and (8), we can derive the inequality (5).

Due to the oscillating nature of the filter impulse response h, each decaying cosine will cross $\pm l$ at least twice and produce, sequentially, at least two equations for each set of unknowns (a_i, t_i). This enables stable reconstruction, as we will show next.

2.4. Reconstruction algorithm

Let $\mathbf{a} = [a_1 a_2 \cdots a_K]^T$ and $\mathbf{t} = [t_1 t_2 \cdots t_K]^T$. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{t}}$ be estimates of $\mathbf{a}$ and $\mathbf{t}$, respectively. The estimated waveform, using these parameters, is:

$$\hat{y}(t) = \sum_{i=1}^{K} \hat{a}_i h(t - \hat{t}_i). \tag{9}$$

The mean-square-error (MSE) between $y(t)$ and $\hat{y}(t)$ is defined as $r(\hat{\mathbf{a}}, \hat{\mathbf{t}}) \triangleq \| \, y(t) - \hat{y}(t) \, \|_2^2$. Note that the MSE is a function of $y(t)$, the dependency on x being implicit. We will use the notation

$$\begin{aligned} &\text{minimize} \quad r(\hat{\mathbf{a}}, \hat{\mathbf{t}}) \\ &\text{subject to} \quad y(s_j) - \hat{y}(s_j) = 0, \ \ j = 1, 2, \cdots, \end{aligned} \tag{10}$$

to describe the optimization problem of finding a set of $(\hat{\mathbf{a}}, \hat{\mathbf{t}})$ that minimizes the objective $r(\hat{\mathbf{a}}, \hat{\mathbf{t}})$ among all that satisfy the constraints. As mentioned previously, the number of local minima grows with the dimension of unknowns, so finding the optimal solution is nontrivial. As such, let us formulate the solution set S^* [10].

The set of points on which the objective and constraint functions are defined is $S = \mathbb{R}^K \times \mathbb{T}$, where $\mathbb{T}$ is the time interval $[0, T]$.

Let the optimal value p^* be

$$p^* = \inf\{r(\hat{\mathbf{a}}, \hat{\mathbf{t}}) \mid y(s_j) - \hat{y}(s_j) = 0, \;\; j = 1, \ldots, N\}.$$

A pair $(\mathbf{a}^*, \mathbf{t}^*)$ is an optimal solution, if $(\mathbf{a}^*, \mathbf{t}^*) \in S$ and $r(\mathbf{a}^*, \mathbf{t}^*) = p^*$. The set of all optimal solutions is the optimal set, denoted by

$$\begin{aligned} S^* = \{(\hat{\mathbf{a}}, \hat{\mathbf{t}}) \mid y(s_j) - \hat{y}(s_j) &= 0, j = 1, \ldots, N; \\ r(\hat{\mathbf{a}}; \hat{\mathbf{t}}) &= p^*\}. \end{aligned} \tag{11}$$

The set S^* is nonempty, and when it contains only one solution $(\hat{\mathbf{a}}^*, \hat{\mathbf{t}}^*)$, $x(t)$ is reconstructed uniquely. When $p^* = 0$, then $x(t)$ is also reconstructed perfectly.

The key is to realize that the k-th pair of unknowns (a_k, t_k) can be solved independently from $\{a_i\}_{i=k+1}^K$ and $\{t_i\}_{i=k+1}^K$. For example, the first two samples s_1 and s_2 occur after the first cosine is triggered and before the second cosine arrives, i.e. $s_1, s_2 \in \; [t_1, t_1 + \Delta]$. As such,

$$(\hat{a}_1, \hat{t}_1) = \{(u, v) \,|(u, v) \in S, \; u\, h(t - v)|_{t=s_1,s_2} = \pm l\}. \tag{12}$$

A unique pair of solutions to a_1 and t_1 is found by evaluating (12) with (10), and it then is used to solve the other unknowns recursively,

$$\begin{aligned} (\hat{a}_k, \hat{t}_k) \;\;=\;\; & \{(u, v) \,|\sum_{i=1}^{k-1} \hat{a}_i h(t - \hat{t}_i) + uh(t - v) = \pm l, \\ & t = s_j, \, s_{j+1}, \;\, j \geq 2k\}, \\ & 2 \leq k \leq K. \end{aligned} \tag{13}$$

3. OPPORTUNISTIC DETECTION

In the previous section, we provided an algorithm for perfection reconstruction of FRI signals by LC. In addition, LC finds itself another important application in parameter detection and estimation, which will be focus of this section. As mentioned in the Introduction, we are particularly interested in signals that can be casted as event-arrival processes, where the pulse shape (event) is known, but arrivals are unknown. In a deterministic framework, such signals can be modelled as FRI, and its counterpart in a stochastic framework can be modelled as point processes, e.g., poisson processes.

Traditionally, when the pulse shape is known, the optimal detector employed in AWGN setting will be a continuous time (CT) matched filter. However the filter requires large bandwidth, and so does the discrete time matched filter obtained from uniform sampling of the CT filter. As such, they are impractical to implement. This leads us to consider LC sampling, which offers high instantaneous bandwidth and samples only when information is present. It is intuitively an opportunistic, as well as efficient, allocation of resources. Let us illustrate with a binary detection example.

Example: The received signal under one hypothesis is the noise N_t alone; the received signal under the other hypothesis is the signal X_t corrupted by noise N_t. Both X_t and N_t are poisson random processes with rate λ_0 and λ_1 over an interval T respectively:

$$\begin{aligned} H_0: \quad & N_t \qquad\;\; \sim \mathcal{P}(\lambda_0), \\ H_1: \quad & X_t + N_t \sim \mathcal{P}(\lambda_0 + \lambda_1). \end{aligned} \tag{14}$$

Assuming both hypotheses were equally likely, given n arrivals in T seconds, the likelihood ratio test (LRT) for maximum-likelihood (ML) detection is given by

$$\frac{p(n \text{ in } T|H_1)}{p(n \text{ in } T|H_0)} \gtrless^{'H_1'}_{'H_0'} 1. \tag{15}$$

Eq.(16) easily translates to a LRT on n,

$$n \gtrless \gamma(T), \tag{16}$$

where the threshold $\gamma(T) = \frac{\lambda_1}{\ln(1+\lambda_1/\lambda_0)} T$. It follows that the probability of detection and the probability of false alarm are, respectively, $P_D(T) = \sum_{n=\lceil\gamma\rceil}^{\infty} p(n|H_1)$ and $P_F(T) = \sum_{n=\lfloor\gamma\rfloor}^{\infty} p(n|H_0)$. We are interested to know how fast a detection can be made. In order to quantify this, we first need the minimum observation interval.

Question 1: In order to realize a pair (P_D, P_F), what is the minimum observation interval T?

For any realizable pair $(1 - \alpha, \beta)$, $0 < \alpha, \, \beta < 1$, i.e. any point on the receiver operating characteristic (ROC) curve of the ML decision rule, we can fix P_F and find a T that satisfies $P_D \geq 1 - \alpha$. The problem can be formulated as such:

$$\begin{aligned} \text{minimize} \quad & T \\ \text{subject to} \quad & T > 0, \; P_F(T) \leq \beta, \; P_D(T) \geq 1 - \alpha. \end{aligned} \tag{17}$$

Combining (16) and (17), we can find T by solving the following optimization problem:

$$\begin{aligned} \min \quad & T \\ \text{s.t.} \quad & T > 0, \qquad\qquad\qquad\qquad (18) \\ & \sum_{n=0}^{\lfloor\gamma(T)\rfloor - 1} \frac{(\lambda_0 T)^n}{n!} \geq (1 - \beta) e^{\lambda_0 T}, \qquad (19) \\ & \sum_{n=0}^{\lceil\gamma(T)\rceil - 1} \frac{((\lambda_0 + \lambda_1)T)^n}{n!} \leq \alpha\, e^{(\lambda_0+\lambda_1)T}. \quad (20) \end{aligned}$$

Note that both left-hand side summations of (19) and (20) monotonically increase from the value 1. Similarly, the terms on the right-hand side of (19) and (20) start at $1-\beta$ and α respectively, and monotonically increase with T as well. When $(1-\alpha, \beta)$ is realizable, then there exists a T'', $T''>0$, that satisfies (19) with equality. It follows that the open interval $(0, T'']$ satisfies both (18) and (19). When there exists a T', $T' \in (0, T'']$, that satisfies (20) with equality, then the closed interval $[T', T'']$ is the solution set to (18)-(20). Minimizing over $[T', T'']$, we arrive at a solution to the objective function (17), $T^* = T'$. In addition, a necessary (but not sufficient) lower bound on the observation interval is

$$T \geq \frac{1}{\lambda_0 + \lambda_1} \ln \frac{1}{\alpha}. \tag{21}$$

Question 2: With the observation interval T chosen to satisfy $(1-\alpha, \beta)$, how long does it take to make a decision?

The threshold test (15) indicates that we have to wait for $\gamma(T)$ event arrivals in order to make a detection. Let t_d be the waiting time until a detection is made. Its distribution $f(t;\gamma)$ is Erlang, therefore on average, we have to wait for an interval of $\bar{t}_d = E[t_d] = \frac{1}{2}(T + \frac{\gamma(T)}{\lambda_0+\lambda_1})$.

We say it is opportunistic when a detection can be made before the expected waiting time, i.e., the time it takes to make a decision t is less than t_d,

$$t < \min(\bar{t}_d, T). \tag{22}$$

As such, opportunistic detection can be made with probability p, where after some grooming, it can be shown to have a closed form of

$$p = \int_0^{\bar{t}_d} f(t;\gamma)dt = 1 - e^{-\lambda \bar{t}_d} \sum_{i=0}^{\gamma-1} \frac{(\lambda \bar{t}_d)^i}{i!}. \tag{23}$$

Here we identified a class of opportunistic Poisson processes that can be efficiently sampled by level-crossing.

Furthermore, a broader class of point processes can be mapped into a unit-rate Poisson process, and analyzed as above. A point process X_t is a sequence of event arrival times in $[0, T]$, with the conditional intensity function $\lambda(t|X_t)$ that can be both time-varying and history dependent. Let $t_1, t_2, \cdots, t_K$ be event arrivals in $[0, T]$, and define $t_0 = 0$ and $t_{K+1} = T$. With the mapping $z_i = \int_{t_{i-1}}^{t_i} \lambda(t|X_t)dt$. it can be shown that the times $\{z_i\}_{i=1}^{K+1}$ correspond to the inter-arrival times of a unit-rate Poisson process [11]. In other words, the class of opportunistic signals obtained above not only includes Poisson processes , but all point processes for which a detection can be made before the average wait. As such, this class of signals is particularly suited to level-crossing sampling.

4. CONCLUSION

We have shown that LC samples can be used to perfectly reconstruct FRI signals. The reconstruction algorithm is also recursive and stable. In addition, LC can also be used in parameter detection in event-arrival processes, where we identified a particular sub-class of signals for which detection can be made faster using LC sampling.

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