OPTIMAL AND BIDIRECTIONAL OPTIMAL EMPIRICAL MODE DECOMPOSITION

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ABSTRACT

The empirical mode decomposition (EMD) was recently proposed as a new time-frequency analysis tool for nonstationary and nonlinear signals. Although the EMD is able to find the intrinsic modes of the signal and is completely self-adaptive, it does not have any implication on optimality. In some situation, when certain optimality is considered, we need a more flexible signal decomposition and reconstruction scheme. We propose a modified version of the EMD, which enhances the capability of the EMD. The proposed modified EMD algorithm gives the best estimate to a given signal in the minimum mean square error sense. Two different formulations are proposed. The first one utilizes a linear weighting for the intrinsic mode functions (IMF). The second algorithm adopts a bidirectional weighting, namely, it not only uses weighting for IMF modes, but also exploits the correlations between samples in a specific window and carries out filtering in the window. These two new EMD methods extend the capability of the traditional EMD and is well suited for optimal signal recovery. Simulation studies are performed to show the application of the proposed optimal EMD algorithms to denoising problem.

***Index Terms*—** Optimal, empirical mode decomposition, signal reconstruction, denoising

1. INTRODUCTION

The empirical mode decomposition (EMD) is proposed by Huang *et al*. as a new signal decomposition method for nonlinear and nonstationary signals [1]. The EMD decomposes a signal into a collection of oscillatory modes, called intrinsic mode functions (IMF), which represent fast to slow oscillations in the signal. Each IMF can be viewed as a certain scale. Traditional signal analysis tools, like Fourier or wavelet-based methods, require some predefined basis functions to represent a signal. The EMD relies on a fully data-driven mechanism that does not require any *a priori* known basis. It has been used to solve many science and engineering problems [2, 3, 4].

The EMD depends only on the data itself and is completely unsupervised. In addition, it satisfies the perfect reconstruction (PR) property because the sum of all the IMFs yields the original signal. However, in some situations, when dealing with reconstructing signal from the IMFs, we do not need all the IMFs so that certain desired characteristics can be achieved. For instance, when the EMD is used for denoising a signal, partial reconstruction EMD based on the IMF energy eliminates the noise components [5]. The partial reconstruction utilizes a binary decision on IMFs, i.e., either discarding them or keeping them in the partial summation. Such reconstruction is not based on any optimality conditions. If we are given a signal and want to approximate the signal by the IMFs obtained from another signal which has some relationship with the given signal, then an optimal criterion could be set up. In order to approximate the given signal, we have many choices for operations on the IMFs. A direct approach is using linear weighting of IMFs. This in turn leads to our first proposed optimal EMD algorithm. A second approach is using weighting coefficients along both vertical IMF index direction and horizontal temporal index direction. Because of this, the second approach is named as the bidirectional optimal EMD algorithm.

2. EMPIRICAL MODE DECOMPOSITION

The aim of the EMD is to decompose the signal into a sum of Intrinsic Mode Functions (IMF). An IMF is defined as a function with equal number of extrema and zero crossings (or at most differed by one) with its envelopes, as defined by all the local maxima and minima, being symmetric with respect to zero. An IMF represents a simple oscillatory mode as a counterpart to the simple harmonic function used in Fourier analysis.

Given a signal $x(n)$, the starting point of the EMD is the identification of all the local maxima and minima. All the local maxima are then connected by a cubic spline curve as the upper envelop $e_u(n)$. Similarly, all the local minima are connected by a spline curve as the lower envelop $e_l(n)$. The mean of the two envelops is denoted as $m_1(n) = [e_u(n) + e_l(n)]/2$ and is subtracted from the signal. Thus the first proto-IMF $h_1(n)$ is obtained as

$$h_1(n) = x(n) - m_1(n). \tag{1}$$

The above procedure to extract the IMF is referred to as the sifting process. Since $h_1(n)$ still contains multiple extrema in between zero crossings, the sifting process is performed again on $h_1(n)$. This process is applied repetitively to the proto-IMF $h_k(n)$ until the first IMF $c_1(n)$, which satisfies the IMF condition, is obtained. Some stopping criteria are used to terminate the sifting process. A commonly used criterion is the Sum of Difference (SD)

$$SD = \sum_{n=0}^{T} \frac{|h_{k-1}(n) - h_k(n)|^2}{h_{k-1}^2(n)}. \tag{2}$$

When the SD is smaller than a threshold, the first IMF $c_1(n)$ is obtained, which is written as

$$r_1(n) = x(n) - c_1(n). \tag{3}$$

Note that the residue $r_1(n)$ still contains some useful information. We can therefore treat the residue as a new signal and apply the above procedure to obtain

$$r_{i-1}(n) - c_i(n) = r_i(n), \quad i = 2, \ldots, N. \tag{4}$$

The whole procedure terminates when the residue $r_N(n)$ is either a constant, a monotonic slope, or a function with only one extremum. Combining the equations in (3) and (4) yields the EMD of the original signal,

$$x(n) = \sum_{i=1}^{N} c_i(n) + r_N(n). \tag{5}$$

The result of the EMD produces N IMFs and a residue signal. For convenience, we refer to $c_i(n)$ as the ith-order IMF. By this convention, lower order IMFs capture fast oscillation modes while higher order IMFs typically represent slow oscillation modes. If we interpret the EMD as a time-scale analysis method, lower order IMFs and higher order IMFs correspond to the fine and coarse scales, respectively. The residue itself can also be regarded as the last IMF.

3. OPTIMAL EMPIRICAL MODE DECOMPOSITION

The traditional empirical mode decomposition given in the last section is a perfect reconstruction (PR) decomposition because the sum of all IMFs yields the original signal. Now given a signal $d(n)$, we want to approximate the signal by some operations on the IMFs. We could do this in various ways depending on the operators used. A linear combination of IMFs is one option. By linear weighting IMFs, we obtain the following estimated signal

$$\hat{x}(n) = \sum_{i=1}^{N} a_i c_i(n), \tag{6}$$

where the coefficients a_i is the weight assigned to the i-th IMF and can take any real value. Note that here for convenience, the residue term is absorbed in the summation as the last term $c_N(n)$. Since our objective is to approximate the desired signal $d(n)$ by the estimated signal $\hat{x}(n)$, we utilize the mean square error as the optimization criterion to find the coefficients a_i's:

$$J = E\{[d(n) - \hat{x}(n)]^2\} = E\left\{\left[d(n) - \sum_{i=1}^{N} a_i c_i(n)\right]^2\right\}. \tag{7}$$

The optimal coefficients can be determined by taking the derivative of (7) with respect to a_i and setting it to zero. Therefore, we obtain

$$\frac{\partial J}{\partial a_i} = -E\left\{2\left[d(n) - \sum_{i=1}^{N} a_i c_i(n)\right] c_i(n)\right\} = 0$$
$$\Longrightarrow \quad \sum_{j=1}^{N} a_j E\{c_i(n) c_j(n)\} = E\{d(n) c_i(n)\}. \tag{8}$$

Now define

$$p_i = E\{d(n) c_i(n)\} \tag{9}$$
$$R_{ij} = E\{c_i(n) c_j(n)\}. \tag{10}$$

Therefore, Eq. (8) becomes the following equation

$$\sum_{i=1}^{N} R_{ij} a_j = p_i, \quad i = 1, \ldots, N. \tag{11}$$

The above N equations can be written in a matrix form

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}, \tag{12}$$

which can be compactly written as

$$\mathbf{R}\mathbf{a} = \mathbf{p}. \tag{13}$$

The optimal coefficients are thus found to be

$$\mathbf{a}^* = \mathbf{R}^{-1}\mathbf{p}. \tag{14}$$

In practice, p_i and R_{ij} should be estimated by sample average. The dimension of the matrix $\mathbf{R}$ is $N \times N$. Since the number of IMFs N is usually a small integer number, the matrix inversion does not incur any numerical difficulties. The minimum MSE can also be found by substituting (14) into (7).

$$J_{\text{min}} = E\left\{\left[d(n) - \sum_{i=1}^{N} a_i^* c_i(n)\right]^2\right\} = \sigma_d^2 - \mathbf{p}^T \mathbf{R}^{-1} \mathbf{p}, \tag{15}$$

where $\sigma_d^2 = E\{d^2(n)\}$ is the variance of the desired signal.

From the above formulation, we see that the optimal EMD is very similar to the optimal filtering (Wiener filtering) which aims to estimate a desired signal by passing the input through a linear filter. The difference here is that the OEMD is a signal decomposition and reconstruction method rather than a filtering method. Two special cases of the OEMD are remarked as follows. If all the coefficients $a_i = 1$, then it is equivalent to the original perfect reconstruction EMD (PR-EMD). If some of the coefficients are set to zero while others are set to one, it reduces to the partial reconstruction EMD (PAR-EMD) used in [5, 4]. Therefore, the OEMD generalizes the traditional EMD and more importantly, yields the optimal estimate of a given signal in the mean square error sense.

4. BIDIRECTIONAL OPTIMAL EMPIRICAL MODE DECOMPOSITION

In the EMD, there are two directions in the resulting IMFs. The first direction is the vertical direction denoted by the IMF index i in (5). The vertical direction corresponds to different scales. The other direction is the horizontal direction represented by the time index n in (5). This direction captures the time evolution of the signal. The OEMD proposed in the last section only uses the weighting of the different IMFs, i.e, the weighting is performed in the vertical direction. Therefore, it lacks degree of freedom in the horizontal temporal direction. In some circumstances, adjacent signal samples are correlated and this factor must be considered when doing reconstruction. A more flexible EMD algorithm that incorporates the signal correlation among samples in a temporal window is defined as follows. For a specific time n, a temporal window of size $2M + 1$ is chosen with the current sample being the center of the window. At the same time, a weighting is also employed to account for the interaction between IMFs. Consequently, 2D weighting coefficients b_{ij} are utilized to yield the estimated signal

$$\hat{x}(n) = \sum_{i=1}^{N} \sum_{j=-M}^{M} b_{ij} c_i(n - j), \tag{16}$$

Table 1. Optimal coefficients of the OEMD algorithm

IMF order i	1	2	3	4	5	6	7	8
a_i^*	0.0652	0.5083	0.8972	0.9852	1.0131	1.0590	0.9249	1.0203

where M is the half window length. This formulation takes both vertical and horizontal directions into consideration and thus is called bidirectional optimal EMD (BOEMD). From (16), the bidirectional weighting of BOEMD can be interpreted as follows. The ith IMF $c_i(n)$ is passed through a FIR filter b_{ij} of length $2M+1$. So we have a filterbank consisting of N FIR filters, each of which is applied to an individual IMF. The output is the summation of all filter outputs. Compared to the OEMD, the BOEMD makes use of the correlation between the samples. We thus have more degree of freedom in choosing these coefficients to achieve certain desired property of the estimated signal than the OEMD algorithm. However, the cost paid for this advantage is the increased computational complexity.

Similar to the OEMD, the criterion chosen here is the mean square error which is

$$J_2 = E\left\{\left[d(n) - \sum_{i=1}^{N}\sum_{j=-M}^{M} b_{ij}c_i(n-j)\right]^2\right\}. \tag{17}$$

Differentiation with respect to the coefficient b_{ij} yields

$$\begin{aligned}\frac{\partial J_2}{\partial b_{ij}} &= -E\{2e(n)c_i(n-j)\}\\ &= -2E\left\{\left[d(n) - \sum_{k=1}^{N}\sum_{l=-M}^{M} b_{kl}c_k(n-l)\right]c_i(n-j)\right\} = 0.\end{aligned} \tag{18}$$

It follows from (18) that

$$\sum_{k=1}^{N}\sum_{l=-M}^{M} b_{kl}R_2(k,i;l,j) = p_2(i,j),\ i=1,\dots,N,\ j=-M,\dots,M, \tag{19}$$

where we define

$$R_2(k,i;l,j) = E\{c_k(n-l)c_i(n-j)\} \tag{20}$$

$$p_2(i,j) = E\{d(n)c_i(n-j)\}. \tag{21}$$

It can be seen that the correlation in (20) is bidirectional with a quadruple index representing both IMF order and temporal directions. There are altogether $(2M+1)N$ equations in (19) and if we rearrange the $R_2(k,i;l,j)$ and $p_2(i,j)$ according to the lexicographic order, we can put (19) into the matrix equation (22). Eq. (22) can be compactly written as

$$\mathbf{R}_2\mathbf{b} = \mathbf{p}_2, \tag{23}$$

from which the optimal solution $\mathbf{b}^*$ is given by

$$\mathbf{b}^* = \mathbf{R}_2^{-1}\mathbf{p}_2. \tag{24}$$

The dimension of the matrix $\mathbf{R}_2$ is $(2M+1)N \times (2M+1)N$, so the computational complexity is increased from $\mathcal{O}(N^3)$ of the OEMD algorithm to $\mathcal{O}((2M+1)^3N^3)$. However, since the BOEMD performs weighting in two directions, it can better capture signal correlation. As in the OEMD case, the elements of the matrix $\mathbf{R}_2$ and the vector $\mathbf{p}$ can be estimated by the sample average.

Table 2. Optimal coefficients of the BOEMD algorithm ($M=1$)

b_{ij}^*	IMF order i							
	1	2	3	4	5	6	7	8
-1	0.0070	0.3194	1.2685	2.3996	-0.0926	11.4145	19.5583	-351.5546
0	0.0304	-0.1233	-1.4363	-3.4693	1.6909	-22.7874	-37.9130	701.6147
1	0.0169	0.3663	1.2500	2.4258	-0.5807	12.5960	19.3919	-349.1916

5. APPLICATIONS

As we have proposed the OEMD and BOEMD algorithms, we can use them for various applications. One application considered here is signal denoising. Suppose we are given a noisy observation. The goal is to remove the noise components in the signal so that the denoised signal $\hat{x}(n)$ is as close to the original noise-free signal $x_o(n)$ as possible. The following example shows the denoising using the OEMD and BOEMD algorithms and compares them with the partial reconstruction EMD (PAR-EMD) in [5]. The denoising method by PAR-EMD is based on the IMF signal energy and the reconstructed signal is given by the partial summation of those IMFs whose energy exceeds the threshold.

The original signal is a moving average of order 10 (MA(10)) process. Additive Gaussian noise with variance 0.0066 is added to the signal so that the SNR=10 dB, where SNR is defined as the ratio of signal power and noise variance. The total signal length is 1200 and the first 1000 samples are used as the desired signal $d(n)$ to estimate the OEMD and BOEMD coefficients a_i's and b_{ij}'s in (6) and (16). Once these coefficients are determined by the algorithms, the remaining non-training samples are tested for denosing performance. The denoised signal is obtained by substituting the optimal coefficients into the reconstruction formulae (6) and (16). Since the OEMD and BOEMD are supervised algorithms, we need a desired signal to train the algorithms. However, as long as the chosen desired signal has the same statistical characteristics as the underlying signal, we can always achieve optimal denoising performance. In the following, the denoising performance is evaluated by the mean square error given by

$$\text{MSE} = \frac{1}{L_2 - L_1 + 1}\sum_{n=L_1}^{L_2}\left[x_o(n) - \hat{x}(n)\right]^2, \tag{25}$$

where L_1 and L_2 are starting and ending indices of non-training samples, and $x_o(n)$ and $\hat{x}(n)$ are original noise-free and denoised signals, respectively.

In the following, the signal memory M in the BOEMD is chosen to be 1. Eight IMFs are obtained after the EMD decomposition. Hence, the total number of coefficients a_i is 8 and the total number of coefficients b_{ij} is 24. The optimal coefficients a_i^* and b_{ij}^* obtained by the OEMD and BOEMD are listed in Table 1 and 2, respectively. It can be observed that the first several weighting coefficients for the OEMD are less than 1 but as the IMF order increases, the coefficients a_i's also increase to some values close to one. The result is also in agreement with that of the PAR-EMD that the lower-order IMFs contain more noise components than the higher-order IMFs. Consequently, lower-order IMFs should be assigned small weights in denoising. The BOEMD coefficients contain both positive and negative numbers, which is a result of subband filtering.

The denoising results are shown in Fig. 1 where we also show the result of the PAR-EMD algorithm. The noisy signal is shown in Fig. 1(a) in which non-training samples from 1000-1200 are shown. Figs. 1(b), 1(c), and 1(d) show the denoised signals reconstructed by the PAR-EMD, OEMD and BOEMD, respectively and compare the resulting signals with the original signal. It can be seen that the

$$\begin{bmatrix} R_2(1,1;-M,-M) & R_2(1,1;-M+1,-M) & \cdots & R_2(N,1;M,-M) \\ R_2(1,1;-M,-M+1) & R_2(1,1;-M+1,-M+1) & \cdots & R_2(N,1;M,-M+1) \\ \vdots & \vdots & \ddots & \vdots \\ R_2(1,N;-M,M) & R_2(1,N;-M+1,M) & \cdots & R_2(N,N;M,M) \end{bmatrix} \begin{bmatrix} b_{1,-M} \\ b_{1,-M+1} \\ \vdots \\ b_{N,M} \end{bmatrix} = \begin{bmatrix} p_2(1,-M) \\ p_2(1,-M+1) \\ \vdots \\ p_2(N,M) \end{bmatrix}. \quad (22)$$

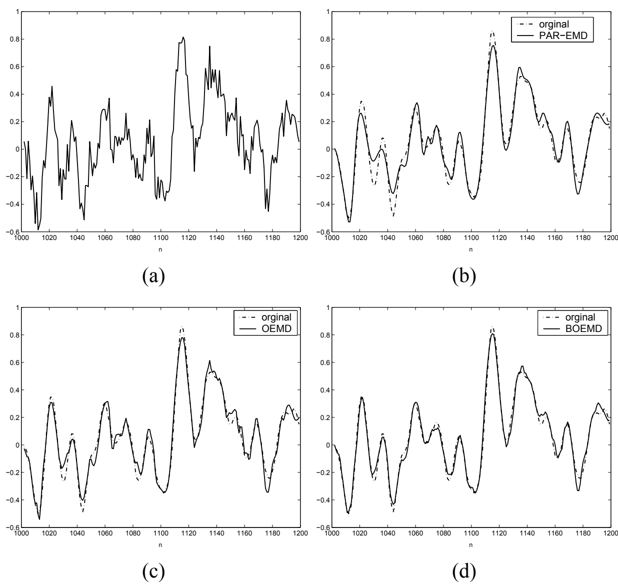

Fig. 1. Denoising using different kind of EMD. Shown in dash-dotted lines are the original signal and the solid lines are denoised signals. (a) Noisy signal, (b) denoising by the PAR-EMD, (c) denoising by the OEMD, (d) denoising by the BOEMD.

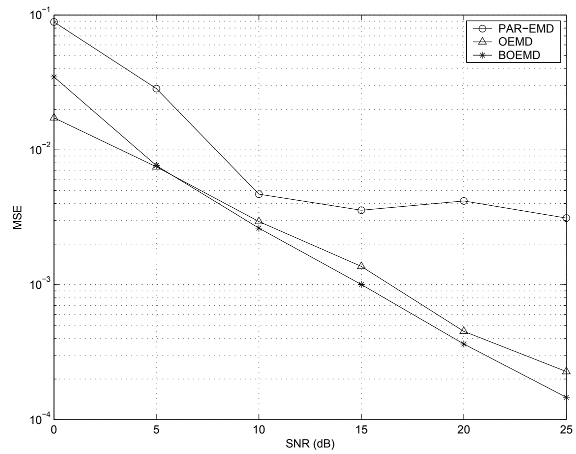

Fig. 2. MSE vs. SNR for three different denoising algorithms

OEMD and BOEMD produce signals closer to the original signal than that of the PAR-EMD. However, the BOEMD performs slightly better than the OEMD since the residual error is smaller. The reason is that the BOEMD takes the signal correlation into account. Calculating the MSE by (25), we find the MSE for these algorithms are 0.0053 for the PAR-EMD, 0.0024 for the OEMD, and 0.0015 for the BOEMD, which again shows the relative goodness of these different EMD algorithms.

A more thorough study using a wide range of different realizations of stochastic signals is carried out by Monte Carlo simulation. Fig. 2 shows the MSE versus SNR for the three EMD algorithms: PAR-EMD, OEMD, and BOEMD. At each SNR, 500 runs are performed to obtain an averaged MSE as shown in the figure. We see that the OEMD and BOEMD algorithms outperforms the PAR-EMD in the entire SNR range. Except for low SNR, the performance of the BOEMD is better than that of the OEMD as expected.

6. CONCLUSION

The empirical mode decomposition is a tool for analyzing nonlinear and nonstationary signals. Conventional EMD, however, does not warrant any optimality conditions. In this paper, several novel EMD algorithms that are optimal in the minimum mean square error sense are proposed. The first algorithm, optimal EMD, estimates a given signal by linear weighting of the IMFs. The coefficients are determined by solving a set of linear equations. To consider the temporal structure of a signal, a bidirectional optimal EMD is then proposed. The weighting of the BOEMD is carried out not only in the IMF scale direction, but also in the temporal direction. It is able to compensate for the time correlation between adjacent samples. An application of the proposed algorithms to signal denoising demonstrates that both the OEMD and BOEMD have better performance than the traditional partial reconstruction EMD. In addition, the BOEMD improves the performance of the OEMD further.

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