SIGMA-DELTA QUANTIZATION OF GEOMETRICALLY UNIFORM FINITE FRAMES

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ABSTRACT

We consider sigma-delta quantization of Cyclic Geometrically Uniform (CGU) finites frames, family of frames containing finite harmonic frames (both in $\mathbb{C}^M$ and $\mathbb{R}^M$). For first- and second-order sigma-delta quantizers, we establish that the reconstruction minimum squares error (MSE) behaves as $\frac{1}{r^2}$ where r denotes the frame redundancy. This result is shown to be true both under the quantization model used in [1, 2] as well as under the widely used additive white quantization noise assumption. For the widely used L-th order noise shaping filter $G(z) = (1 - z^{-1})^L$, we show that the MSE behaves as $\frac{1}{r^2}$ irrespectively of the filter order L. More importantly, we prove also that in the case of tight and normalized CGU frame, when the frame length is too large compared to the filter order, the reconstruction MSE can decay as faster as $O(\frac{1}{r^{2L+1}})$.

***Index Terms*—** Sigma-Delta quantization, Overcomplete representations , CGU frames.

1. INTRODUCTION

Sigma-Delta (SD) quantization of overcomplete representations, first described in [3] in the context of oversampled filter banks (FBs), has recently attracted significant attention [1, 2]. Most notably, it has been demonstrated in [1, 2] that first- and second-order SD quantization of finite harmonic frames in $\mathbb{R}^M$ yields a reconstruction MSE behaving according to $\frac{1}{r^2}$, where r denotes the frame redundancy.

In the context of analog-digital (A/D) conversion a class of single-bit converters achieving exponential accuracy in the bit rate has recently been described in [4]. The frame expansions induced by A/D conversion have a very specific structure namely that of shift-invariant frames in $L^2(\mathbb{R})$ with the generator typically being a $\frac{\sin(t)}{t}$-function. The purpose of this paper is to address several problems in the context of finite frames exhibiting different structural properties, namely Cyclic Geometrically Uniform (CGU) frames [5]. Our specific contributions vis-a-vis previous work on quantization of finite frames reported in [1, 2] include:

- We consider a more general class of frames, namely CGU frames in $\mathbb{C}^M$, containing harmonic frames.
- Using techniques similar to those in [1, 2], we show that first- and second-order SD quantization of CGU frames achieves an MSE behaving as $\frac{1}{r^2}$.
- The results in [1, 2] use a deterministic model to describe the quantizer in the feedback loop. We demonstrate that the $\frac{1}{r^2}$-behavior of the reconstruction MSE continues to hold even if the impact of quantization is described by adding white Gaussian noise to the signal at the quantizer input, a widely used (but typically not accurate) model.
- For the widely used L-th order noise shaping filter $G(z) = (1 - z^{-1})^L$, we show that the MSE behavior depends on the nature of the considered frame. When the frame length N is too large compared to the filter order L ($N \gg L$), the MSE behaves as $\frac{1}{r^{2L+1}}$.

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2. NOTATION

The superscripts ^{T}, ^{H} and * denote the transpose, conjugate transpose and element wise conjugation, respectively. For a matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, we define $\|\mathbf{A}\| = \sqrt{\lambda_{\mathbf{max}}(\mathbf{A^H A})}$, where λ_{max} denotes the maximum eigenvalue. The expectation is denoted by $\mathcal{E}$.

3. CGU FRAMES, SYSTEM MODEL

In this section, we briefly describe CGU frames and we state the system model for first- and second-order SD quantizer used previously in [1, 2].

3.1. CGU Frames

We restrict our attention to frames in finite dimensional spaces. Let $\Omega = \{\phi_n\}_{n=1}^N$ denotes a set of N vectors in an M-dimensional Hilbert space $\mathcal{H}$, where $M \leq N$ and $\mathcal{H} = \mathbb{R}^M$

or $\mathbb{C}^M$. The vectors $\boldsymbol{\phi}_n$ constitute a frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that [6]

$$A\|\mathbf{x}\|^2 \leq \sum_{n=1}^{N} |\langle \mathbf{x}, \boldsymbol{\phi}_n\rangle|^2 \leq B\|\mathbf{x}\|^2, \ \forall \mathbf{x} \in \mathcal{H}. \tag{1}$$

When $A = B$ the frame is called tight [6]. If we normalize the $\boldsymbol{\phi}_n$ such that $\|\boldsymbol{\phi}_n\| = 1, \ \forall n$, we have $A \leq r \leq B$ [3] where $r = \frac{N}{M}$ denotes the redundancy. In particular, for a tight normalized frame, we obtain $A = B = r$. The frame operators corresponding to the frame vectors $\{\boldsymbol{\phi}_n\}_{n=1}^N$ is defined as $\mathbf{S} = \boldsymbol{\Psi}\boldsymbol{\Psi}^H$ where $\boldsymbol{\Psi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_N]$ and satisfies $A\mathbf{I} \leq \mathbf{S} \leq B\mathbf{I}$, where $\mathbf{I}$ is the identity operator on $\mathcal{H}$. The dual frame of Ω is defined as $\tilde{\Omega} = \{\tilde{\boldsymbol{\phi}}_n\}_{n=1}^N$ where $\tilde{\boldsymbol{\phi}}_n = \mathbf{S}^{-1}\boldsymbol{\phi}_n, \ n = 1, 2, \ldots, N$. Note that $\tilde{\Omega}$ is also a frame with frame bounds $\tilde{A} = \frac{1}{B}$ and $\tilde{B} = \frac{1}{A}$. Moreover, we have

$$\mathbf{x} = \sum_{n=1}^{N} \langle \mathbf{x}, \boldsymbol{\phi}_n\rangle \tilde{\boldsymbol{\phi}}_n = \sum_{n=1}^{N} \langle \mathbf{x}, \tilde{\boldsymbol{\phi}}_n\rangle \boldsymbol{\phi}_n, \ \forall \mathbf{x} \in \mathcal{H}.$$

In the sequel, we focus on a structured finite frame in $\mathbb{C}^M$, a sub-class of Geometrically Uniform (GU) frames, namely CGU frames. The definition of a general GU frame [5] is:

Definition 1 *Ω is a GU group, if $\boldsymbol{\phi}_n = \mathbf{V}_n\boldsymbol{\phi}, \ \forall n \in \{1, \ldots, N\}$, where $\boldsymbol{\phi} \in \mathbb{C}^M$ is an arbitrary generating vector and the matrices $\{\mathbf{V}_n\}_{n=1}^N$ are unitary and form an Abelian group. If the vectors $\{\boldsymbol{\phi}_n\}_{n=1}^N$ satisfy (1), then Ω is a GU frame.*

Definition 2 *A GU set Ω is CGU, if the unitary matrix $\mathbf{V}$ satisfies $\mathbf{V}_n = \mathbf{V}^n$ and $\mathbf{V}^N = \mathbf{I}$.*

In the following, we denote the eigendecomposition of $\mathbf{V}$ as $\mathbf{V} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^H$ where $\mathbf{R}\mathbf{R}^H = \mathbf{I}$ with $\mathbf{R} = [\mathbf{r}_1, \ldots, \mathbf{r}_M]$ and $\boldsymbol{\Lambda} = diag\{\lambda_n\}_{n=1}^M$, where $\lambda_k = e^{j\frac{2\pi}{N}k_n}$ with $k_n \in \{1, \ldots, N\}$. Assuming that the λ_k are distinct, it is shown in [7] that Ω is a tight CGU frame if and only if:

$$|\mathbf{r}_1^H\phi| = |\mathbf{r}_2^H\phi| = \ldots = |\mathbf{r}_M^H\phi| = \frac{1}{\sqrt{M}}. \tag{2}$$

A trivial solution of (2) is given by $\boldsymbol{\phi} = \frac{1}{\sqrt{M}}\mathbf{R}[(-1)^{l_1}, \ldots, (-1)^{l_M}]^T$. An example of a tight normalized CGU frame in $\mathbb{C}^M$ is a harmonic frame in $\mathbb{C}^M$, which is given by $\phi_{n,i} = \frac{1}{\sqrt{M}}W_N^{(n-1)(i-1)}$ $(i = 1, \ldots, M, \ n = 1, \ldots, N)$ or equivalently $\mathbf{V}= diag\{W_N^i\}_{i=0}^{M-1}$, $\mathbf{R} = \mathbf{I}$ and $\phi = \frac{1}{\sqrt{M}}[1, \ldots, 1]^T$ where $W_M = e^{j\frac{2\pi}{N}}$ and $\phi_{n,i}$ denotes the i^{th} element of $\boldsymbol{\phi}_n$. The harmonic frame in $\mathbb{R}^M$ employed in [1, 2], is shown to belong to the class of tight CGU frames [7]. In particular, we will see in the next section that the additional degrees of freedom obtained by considering the more general class of CGU frames can (provided the λ_k are chosen properly) result in a smaller reconstruction MSE when compared to harmonic frames in $\mathbb{R}^M$.

3.2. First and second order SD quantization

Following [1, 2] we define first- and second-order SD quantizers for finite frames as follows. Given a sequence $\{c_n\}_{n=1}^N \subset \mathbb{R}^M$ of frame coefficients, first-order SD quantizer produces the sequence of quantized frame coefficients $\{c_{q,n}\}_{n=1}^N$ via the following scheme:

$$u_n = u_{n-1} + c_n - c_{q,n}, \ c_{q,n} = Q(u_{n-1} + c_n) \tag{3}$$

where $\{u_n\}_{n=0}^N$ is a state sequence with $u_0 = 0$ and Q is an η-level mid-rise uniform quantizer with step size δ, where $\eta \in \mathbb{N}$ and $\delta > 0$. The corresponding mid-rise quantization alphabet is given by $\mathcal{A}_\eta^\delta = [(-\eta + \frac{1}{2})\delta, \ (-\eta + \frac{3}{2})\delta \ldots -\frac{1}{2}\delta, \ \frac{1}{2}\delta, \ \ldots(\eta - \frac{1}{2})\delta]$ consisting of 2η elements and $Q(u) = \arg \min_{q \in \mathcal{A}_\eta^\delta} |u - q|$.

The second-order SD quantizer, as introduced in [1, 2], is given by

$$u_n = u_{n-1} + c_n - c_{q,n}, \quad v_n = u_{n-1} + v_{n-1} + c_n - c_{q,n}$$
$$c_{q,n} = \frac{\delta}{2}\text{sign}(u_{n-1} + \gamma v_{n-1}), \quad n = 1, 2, \ldots, N \tag{4}$$

where $\gamma > 0$ is a fixed parameter and $\{u_n\}_{n=0}^N$ and $\{v_n\}_{n=0}^N$ are state sequences with $v_0 = u_0 = 0$. Like in [2] we assume that the SD quantizer is stable in the sense of [2] i.e., $(u_n, v_n) \in \frac{\delta}{2}([-2, 2] \times [-C, C]) \quad \forall n$ and C denotes a non-negative integer.

The equations (3) and (4) keep valid when the frame coefficients $\{c_n\}_{n=1}^N \subset \mathbb{C}^M$: we run the same quantizer on the real and imaginary parts of the signal to be quantized.

4. RECONSTRUCTION MSE FOR TIGHT CGU FRAMES

Throughout this section we assume that Ω is a tight CGU frame. Denoting the signal reconstructed from the quantized frame coefficients as $\tilde{\mathbf{x}} = \sum_{n=1}^{N} c_{q,n}\mathbf{S}^{-1}\boldsymbol{\phi}_n$, we shall study the behavior of the reconstruction MSE, $\|\mathbf{x} - \tilde{\mathbf{x}}\|^2$, for the first- and second-order SD quantizer as described in the previous section.

4.1. First-order SD quantizer

Consider the quantization scheme corresponding to (3), let $|u_0| \leq \delta/2$, and let $\mathbf{x} \in \mathcal{H}$ satisfy $\|\mathbf{x}\| \leq (\eta - 1/2)\delta$, it was shown in [1, 2] that:

$$\|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \|\mathbf{S}^{-1}\| \left(\sum_{n=1}^{N-1} |u_n|\|\boldsymbol{\phi}_n - \boldsymbol{\phi}_{n+1}\| + |u_0| + |u_N|\right). \tag{5}$$

As the frame is assumed to be tight and normalized we have $\|\mathbf{S}^{-1}\| = \frac{1}{r}$. To evaluate $\|\mathbf{x} - \tilde{\mathbf{x}}\|$, we need to compute the frame variation of Ω defined as follows [2]: $\sigma(\Omega) =$

$\sum_{n=1}^{N-1} \|\boldsymbol{\phi}_n - \boldsymbol{\phi}_{n+1}\|$. Since $\mathbf{V}^N = \mathbf{I}$, it follows that $\sum_{n=1}^{N} \boldsymbol{\phi}_n = 0$, which upon using Proposition III.2 in [1] and based on the structure of the frame Ω, it follows that

$$\| x - \tilde{x} \| \leq \begin{cases} \frac{1}{r}\frac{\delta}{2}\sigma(\Omega), & \text{for } N \text{ even} \\ \frac{1}{r}\frac{\delta}{2}(\sigma(\Omega)+1) & \text{for } N \text{ odd.} \end{cases} \quad (6)$$

In [7] it is shown that

$$\sigma(\Omega) \leq \frac{2\pi}{N}(N-1)k_{\text{min}}, \text{ where } k_{\text{min}} = \min(N - k_{max}, k_{\max}) \quad (7)$$

with $k_{\max} = \arg\max_{k_l} |\sin(\frac{\pi}{N}k_l)|$, which states that the frame variation is bounded. Using (5), we conclude that the reconstruction MSE can decay at least as faster as $\frac{1}{r^2}$. Note that results of the Corollary V.3 of [1] keep true in the case of a tight and normalized CGU frame.

The upper-bound obtained by inserting (7) into (5) is minimum w.r.t the choice of λ_k if

$$\begin{cases} k_l = l,\ k_{l+t} = N - k_{t-l+1}, & M = 2t,\ l = 1, \dots, t \text{ and } i = 1, \dots, t+1. \\ k_i = i,\ k_{t+l+1} = N - k_{t-l+1}, & M = 2t+1, \end{cases} \quad (8)$$

In Fig.1, we compare the upper-bound obtained by inserting (7) into (5) for the harmonic frame in $\mathbb{R}^M$ or $\mathbb{C}^M$ and a CGU frame in $\mathbb{C}^M$ with λ_{k_l} chosen according to (8). In both cases, we have $M = 4$. We can clearly see that the upper bound corresponding to the CGU is smaller than that for the harmonic frames. This is a direct consequence of the additional degrees of freedom available in a CGU frame.

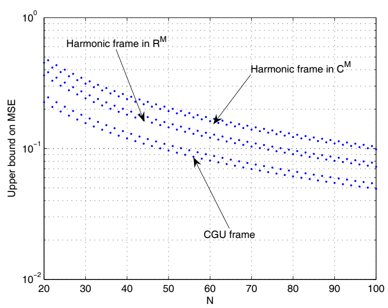

Fig. 1. The upper bound on the MSE given by the harmonic frame in $\mathbb{R}^M$ or $\mathbb{C}$ and the CGU frame.

4.2. Second order SD quantizer

Setting $a = 2C\pi\frac{k_{min}}{N}(\frac{2(N-2)}{N}\pi k_{min}+1)$ where k_{min} is defined in (7), it is shown in [7] that, in the case of CGU frames, for second-order SD quantizer given by (4), MSE satisfies:

$$\begin{cases} \|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \frac{1}{r}\frac{\delta}{2}a, & \text{for } N \text{ even,} \\ \frac{1}{r}\delta(1-a) \leq \|\mathbf{x} - \tilde{\mathbf{x}}\| \leq \frac{1}{r}\frac{\delta}{2}(1+a), & \text{for } N \text{ odd.} \end{cases} \quad (9)$$

Choosing k_l according to (8) again minimizes the MSE upper-bound resulting in:

$$a = \begin{cases} C\pi\left(\frac{N-2}{N}M\pi + 1\right)\frac{M}{N}, & \text{for } M \text{ even,} \\ C\pi\left(\frac{N-2}{N}(M+1)\pi + 1\right)\frac{M+1}{N} & \text{for } M \text{ odd.} \end{cases} \quad (10)$$

Like [1, 2], we have $O(\frac{1}{r^2}) \leq \|\mathbf{x} - \tilde{\mathbf{x}}\|^2 \leq O(\frac{1}{r^2})$. In the case of the harmonic frame in $\mathbb{R}^M$, it was shown in [2] that when M is even: $a = 2C\pi(\pi M + 1)\frac{1}{r}$ which compared with (10) shows that if k_l is chosen according to (8), the MSE upper bound for CGU frame in $\mathbb{C}^M$ is strictly smaller than for harmonic frame in $\mathbb{R}^M$, albeit in both cases we have a $O(\frac{1}{r^2})$ behavior. We conclude this section by noting that the techniques used in the derivation of the main results in this section follows closely the approach introduced previously in [1, 2]. Differences occur due to the more general structure of CGU frames which was shown to result in smaller reconstruction MSE compared to harmonic frame in $\mathbb{R}^M$.

5. RECONSTRUCTION MSE UNDER ADDITIVE WHITE QUANTIZATION NOISE

From the upper (and lower) bound on the reconstruction MSE derived in the previous section, it follows immediately that the $O(\frac{1}{r^2})$-behavior of the MSE depends crucially on the assumption on the quantizer. It is therefore interesting to ask whether the $O(\frac{1}{r^2})$-behavior for the reconstruction MSE can be obtained under the widely used additive white noise model for quantization. In this section, we answer this question in the affirmative.

In the following, we shall add one more level of generality by assuming an L-th order noise shaping filter. The overall system model can be summarized as (see Fig.2):

$$c_{q,n} = c_n + q_n - \hat{q}_n,\ \hat{q}_n = \sum_{k=1}^{L} g_k q_{n-k},\ n = 1, \dots, N \quad (11)$$

where $G(z) = 1 - \sum_{l=1}^{L} g_l z^{-l}$ with $g_l \in \mathbb{R}$ or $\mathbb{C}$ denotes the noise shaping filter cœfficients and q_i is the zero mean quantization error satisfying $\mathcal{E}\{q_n q_m^*\} = \sigma_q^2 \delta(n-m)$. Defining the $L+1 \times M$ matrix $[\mathcal{Q}]_{i,j} = q_{i-j}$, the reconstruction error is given by $\mathbf{w} = \mathbf{S}^{-1}\mathbf{\Psi}\mathcal{Q}\mathbf{g}$ where $\mathbf{g} = [-1, g_1, g_2, \dots, g_L]^T$. Consequently, we have $\sigma_{\mathbf{w}}^2 = \mathcal{E}\{\mathbf{g}^H \mathcal{Q}^H \mathbf{\Psi}^H (\mathbf{S}^{-1})^H \mathbf{S}^{-1}\mathbf{\Psi}\mathcal{Q}\mathbf{g}\}$. Minimizing $\sigma_{\mathbf{w}}^2$ as a function of the filter coefficients g_l for

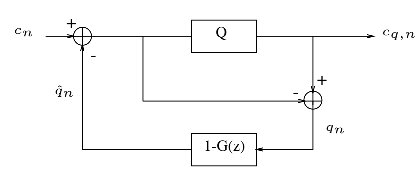

Fig. 2. General noise-shaping coder

arbitrary noise shaping filter order L is difficult. In the following, we shall therefore restrict our attention to first- and second-order noise shaping filters and then investigate the result for a specific L-th order filter which is widely used in SD quantizer based A/D conversion [8]. We again restrict our attention to CGU frames with parameters k_l chosen according to (8), when M is even (when M is odd we obtain the same results), and when the filter cœfficients $g_l \in \mathbb{R}$.

5.1. First-order SD-quantizer ($L = 1$)

For $L = 1$, it is shown in [7] that

$$\sigma_{\mathbf{w}}^2 = \frac{\sigma_q^2}{Mr^2}\left[(1+g_1^2)NM - 4g_1(N-1)\sum_{m=1}^{\frac{M}{2}}\cos(\frac{2\pi}{N}m)\right]. \quad (12)$$

For high redundancy r, i.e., N large, we have $\cos(\frac{2\pi}{N}m) \approx= 1 - \frac{1}{2}(\frac{2\pi}{N}m)^2$, which implies

$$\sigma_{\mathbf{w}}^2 \approx \frac{\sigma_q^2}{Mr^2}\left[(1-g_1)^2NM + 2g_1M + 2g_1(N-1)(\frac{2\pi}{N})^2\alpha\right]$$

where $\alpha = \sum_{m=1}^{\frac{M}{2}} m^2$. Setting $g_1 = 1$, we obtain

$$\begin{aligned} \sigma_{\mathbf{w}}^2 &\approx \frac{1}{r^2}\frac{\sigma_q^2}{M}\left[M + \left(\frac{8\pi^2}{M}\alpha\right)\frac{1}{r} + \left(-\frac{8\pi^2}{M^2}\alpha\right)\frac{1}{r^2}\right] \quad (13) \\ &\approx \frac{\sigma_q^2}{r^2} + O\left(\frac{1}{r^3}\right) \quad (14) \end{aligned}$$

which finally implies that $\sigma_{\mathbf{w}}^2$ decays according to $\frac{1}{r^2}$.

5.2. Second-order SD quantizer ($L = 2$)

For $L = 2$ and $g_1, g_2 \in \mathbb{R}$, it is shown in [7] that:

$$\begin{aligned} \sigma_{\mathbf{w}}^2 \approx \frac{\sigma_q^2}{Mr^2}&\left[(-1+g_1+g_2)^2 NM + 2(g_1 - g_1g_2 + 2g_2)M\right. \\ &\left. + 2((g_1 - g_1g_2 + 4g_2)N - a + g_1g_2 - 8g_2)(\frac{2\pi}{N})^2\alpha\right] \end{aligned}$$

where α is defined as previously. Setting $g_1 + g_2 = 1$, it follows that $\sigma_{\mathbf{w}}^2$ decays according to $\frac{1}{r^2}$.

5.3. The noise shaping filter $G(z) = (1 - z^{-1})^L$

As already mentioned above, it is difficult to obtain general results for high-order noise shaping filters. That's why we focus here on the widely used choice $G(z) = (1 - z^{-1})^L$ [8]. Starting with $g_l = (-1)^{l+1}C_L^l$ $(l = 0, 1, \ldots, L)$ where $C_L^l = \begin{pmatrix} L \\ l \end{pmatrix}$, for high redundancy r, we obtain [7]

$$\begin{aligned} \sigma_{\mathbf{w}}^2 \approx \frac{1}{r^2}\frac{\sigma_q^2}{M}&\left[\sum_{l=0}^{L}(C_L^l)^2NM\right. \\ &\left. + 2\sum_{l=1}^{L}(N-l)(-1)^l\sum_{t=0}^{L-l}C_L^tC_L^{t+l}\sum_{m=1}^{M}\Re(\lambda_m^l)\right] \quad (15) \end{aligned}$$

where $\lambda_m = e^{j\frac{2\pi}{N}k_m}$, and hence $\Re(\lambda_m^l) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k)!}(\frac{2\pi}{N})^{2k}(lm)^{2k}$.

It is shown in [7] that $\sum_{l=0}^{L}(C_L^l)^2 + 2\sum_{l=1}^{L}(-1)^l\sum_{t=0}^{L-l}C_L^tC_L^{t+l} = 0$. Consequently, we obtain

$$\sigma_{\mathbf{w}}^2 \approx \frac{2\sigma_q^2\sum_{l=1}^{L}(-1)^{l+1}l\sum_{t=0}^{L-l}C_L^tC_L^{t+l}}{r^2} + O\left(\frac{1}{r^3}\right). \quad (16)$$

The last expression would tend to deduce that the reconstruction MSE decays of $\frac{1}{r^2}$ independently of the noise shaping filter order L. This is not totally true and the frame length plays nonetheless a crucial role in the enhancement of the MSE decays. Indeed, when $N >> L$ and in the case of a causal shaping filter, the reconstruction MSE becomes

$$\sigma_w^2 \approx \left(\frac{2^{2L+1}\sigma_q^2}{M^{2L}}\pi^{2L}\sum_{m=1}^{\frac{M}{2}}m^{2L}\right)\frac{1}{r^{2L+1}}. \quad (17)$$

Hence, $MSE \approx O\left(\frac{1}{r^{2L+1}}\right)$. This result is consistent with the one obtained in the context of A/D conversion [8].

6. CONCLUSIONS

We established that CGU frames subject to first- and second-order SD quantization exhibit a reconstruction error behavior according to $\frac{1}{r^2}$. This result was shown to hold both under the quantizer model used in [1, 2] and under the widely used additive white quantization noise model. For general noise shaping filter coefficients, conditions on the cœfficients were provided in the case of first- and second-order SD quantizers guaranteeing a $\frac{1}{r^2}$ MSE behavior. Furthermore, it was demonstrate that the additional degrees of freedom by CGU frames, when compared to harmonic frame, result in redundancy r that remains unchanged. Finally, for the widely used noise shaping filter $G(z) = (1 - z^{-1})^L$ it was proved that, under additive white quantization noise model and when $N \gg L$, the reconstruction MSE is $O(\frac{1}{r^{2L+1}})$.

7. REFERENCES

[1] J. J. Benedetto, O. Yilmaz, and A. M. Powell, "Sigma delta quantization and finite frames," in *Proc. IEEE ICASSP-2004*, May 2004, pp. 17–21.

[2] J. J. Benedetto, M. Powell, and O. Yilmaz, "Second order sigma-delta quantization of finite frame expansion," *Elsevier Science*, 2004.

[3] H. Bölcskei, *Oversampled filter banks and predictive subband coders*, Ph.D. thesis, Vienna University of Technology, Nov. 1997.

[4] Z. Cvetcovic, I. Daubechies, and B. Logan, "Single-bit oversampled a/d conversion with exponential accuracy in the bitrate," *IEEE Trans. Inf. Theory*, submitted.

[5] Y. C. Eldar and H. Bölcskei, "Geometrically uniform frames," *IEEE Trans. Inf. Theory*, vol. 49, no. 4, Jan. 2003.

[6] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[7] F. Abdelkefi, "The first- and the second-order $\Sigma\Delta$ quantizer when using a finite GU frame," in preparation.

[8] R. DeVore and I. Daubechies, "Approximating a bandlimited function using very coarsely quantized data: A family of stable sigma-delta modulators of arbitrary order," *Annals of Mathematics*, vol. 17, no. 1, pp. 229–242, 2003.