PEANO KERNELS OF FRACTIONAL DELAY SYSTEMS

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ABSTRACT

In this paper, design and analysis of fractional delay systems in time-domain is proposed. Under the condition of exactness, we prove that the difference between the desired output signal and the actual output of the system can be represented as the convolution of the derivative of the input signal and the Peano kernel. The Peano kernel are solved analytically in this article. We also solve the weighting coefficients in closed form when the degree of exactness is equal to the number of coefficients.

Index Terms— Fractional delay filters, Peano kernels, maximally flat filters.

1. INTRODUCTION

Fractional delay (FD) systems interpolate between samples. They are actually digital FD filters for which application and design methods has been widely studied and reported in literature [1]. Standard techniques for filter design in frequency domain such as windowing method, weighted-least-square, equiripple, or maximally flat (MF) approximation can be applied to designing FD systems [1, 2, 3]. Results on MF FD systems including finite impulse response (FIR) and infinite impulse response (IIR) filters are studied in [4].

In this paper, we propose a method for analyzing FD system. In stead of approximating the desired frequency response in frequency domain, we propose a time-domain analysis on the input and output signals. By this approach, we prove that for any FD system the difference between the ideal output and the actual output can be represented by convolution of the derivative of the input signal with the Peano kernel.

2. FORMULATION AND PRELIMINARIES

The ideal FD system in the continuous-time domain is specified by

$$y(t) = x(t-\tau) \tag{1}$$

where $x(t)$ is the input signal, $y(t)$ is the desired output signal that is the delayed version of $x(t)$, and τ is the desired time delay. Only the case of $\tau > 0$ is discussed in this paper. This system can be implemented by cascading a continuous-to-discrete-time converter, a linear time-invariant (LTI) discrete-time system, and a discrete-to-continuous-time converter [2]. The LTI system is represented by the difference equation

$$y_n = -\sum_{k=1}^{N} a_k y_{n-k} + \sum_{m=0}^{M} b_m x_{n-m}, \quad -\infty < n < \infty \tag{2}$$

where $x_n \equiv x(nT)$ and $y_n \equiv y(nT)$ are the sampled values of $x(t)$ and $y(t)$ at $t = nT$, respectively. T is the sampling period. a_k and b_m are the weighting coefficients. If the desired delay $\tau = m'T$ is an integer multiple of the sampling period T, we obtain an ideal delay system $y_n = x_{n-m'}$. However, this case rarely happens in a digital signal processing system since the sampling period T is usually determined by signal bandwidth. Eq. (2) explicitly express the output sample $y[n]$ as linear combination of input sample $x[n-m], 0 \le m \le M$ and the past computed outputs $y[n-k], 0 \le k \le N$. By assigning $a_0 = 1$, Eq. (2) can be also written into a compacter form

$$\sum_{k=0}^{N} a_k y_{n-k} = \sum_{m=0}^{M} b_m x_{n-m}. \tag{3}$$

For the system expressed by the above equation, the frequency response $H(e^{j\omega})$ are represented by

$$H(e^{j\omega}) = \frac{\displaystyle\sum_{m=0}^{M} b_m e^{-j\omega m}}{\displaystyle\sum_{k=0}^{N} a_k e^{-j\omega k}}.$$

In this paper, we will analyze the relationship among the input and the output samples in Eq. (3). It will be more intuitive for proceeding our discussion to express Eq. (3) as

$$\sum_{k=0}^{N} a_k x(nT - kT - \tau) = \sum_{m=0}^{M} b_m x(nT - mT) \tag{4}$$

in which the samples are written back as sampled continuous-time signals.

In the next section, we will expand $x(nT - kT - \tau)$ and $x(nT - mT)$ in Eq. (4) into their Taylor series, so we state the the Taylor's theorem here. Let $\mathcal{C}^n(-\infty,\infty)$ be the set of all functions f for which $f^{(n)}$ exists and is continuous everywhere. If $x(t) \in \mathcal{C}^{P+1}(-\infty,\infty)$, then

$$x(t+h) = \sum_{i=0}^{P} \frac{h^i}{i!}\, x^{(i)}(t) + R_P(t) \tag{5}$$

where

$$R_P(t) = \frac{1}{P!}\int_t^{t+h} (t+h-\theta)^P x^{(P+1)}(\theta)\, d\theta. \tag{6}$$

This is called Taylor's theorem with integral remainder [5]. The remainder $R_P(t)$ can be rewritten as the following form

$$R(t) = \frac{1}{P!}\int_0^{h} (h-\theta)^P x^{(P+1)}(\theta+t)\, d\theta. \tag{7}$$

which is more suitable for the derivation in the next section. We restrict the increment value $h \geq 0$ in the following derivation to keep the upper limit h is larger then the lower limit 0 in Eq. (7).

3. DEGREE OF EXACTNESS AND PEANO KERNEL

In this section, we will investigate Eq. (4) by Taylor series of $x(nT-kT-\tau)$ and $x(nT-mT)$. The expanding center is denoted by $t_0 = nT - KT$ where K is a positive integer. That is, $x(nT-kT-\tau)$ and $x(nT-mT)$ are expanded at past Kth samples for trying to keep the time increments $nT-kT-\tau-t_0$ and $nT-mT-t_0$ being positive. Let the increments be α_k and β_k, we need

$$\alpha_k = (K-k)T - \tau \geq 0, \quad 0 \leq k \leq N, \tag{8}$$

$$\beta_m = (K-m)T \geq 0, \quad 0 \leq m \leq M. \tag{9}$$

That is, we need $K \geq \max(N+d, M)$ where

$$d \equiv \frac{\tau}{T} \tag{10}$$

is the desired delay τ normalized by the sampling period T.

Now, $x(nT-kT-\tau)$ and $x(nT-mT)$ are ready to expand into the Taylor series and are expressed by

$$x(nT-KT+\alpha_k) = \sum_{i=0}^{P} \frac{\alpha_k^i}{i!}\, x^{(i)}(t_0) + r_k(nT) \tag{11}$$

$$x(nT-KT+\beta_m) = \sum_{i=0}^{P} \frac{\beta_m^i}{i!}\, x^{(i)}(t_0) + s_m(nT). \tag{12}$$

Using the formula of integral remainder given in Eq. (7), the remainders $r_k(nT)$ and $s_m(nT)$ can be explicitly expressed as

$$r_k(nT) = \frac{1}{P!}\int_0^{\alpha_k} (\alpha_k-\theta)^P x^{(P+1)}(\theta+t_0)\, d\theta \tag{13}$$

$$s_m(nT) = \frac{1}{P!}\int_0^{\beta_m} (\beta_m-\theta)^P x^{(P+1)}(\theta+t_0)\, d\theta. \tag{14}$$

By introducing the unit step function $u(\theta)$defined as $u(\theta) = 1, \theta \geq 0$ and $u(\theta) = 0, \theta < 0$, we can express Eqs. (13) and (14) by

$$r_k(nT) = \frac{1}{P!}\int_{-\infty}^{\infty} \tilde{r}_k(\theta) x^{(P+1)}(nT-\theta)\, d\theta \tag{15}$$

$$s_m(nT) = \frac{1}{P!}\int_{-\infty}^{\infty} \tilde{s}_m(\theta) x^{(P+1)}(nT-\theta)\, d\theta \tag{16}$$

where

$$\tilde{r}_k(\theta) = (\theta-kT-\tau)^P[u(\theta-kT-\tau) - u(\theta-KT)] \tag{17}$$

$$\tilde{s}_m(\theta) = (\theta-mT)^P[u(\theta-mT) - u(\theta-KT)] \tag{18}$$

The form of Eqs. (15) and (16) is more convenient than that of Eqs. (13) and (14) for our derivation because we can add these remainders together to represent the whole remainder of the system.

Since Eq. (4) is an approximation to the ideal delay system, the design error is represented by

$$e(nT) = \sum_{k=0}^{N} a_k x(nT-kT-\tau) - \sum_{m=0}^{M} b_m x(nT-mT). \tag{19}$$

Substituting Eqs. (11) and (12) for Eq. (19), we can write it as $e(nT) = a(nT) + r(nT)$ where

$$a(nT) = \sum_{i=0}^{P} \frac{T^i w_i}{i!}\, x^{(i)}(nT-KT), \tag{20}$$

$$r(nT) = \sum_{k=0}^{N} a_k r_k(nT) - \sum_{m=0}^{M} b_m s_m(nT). \tag{21}$$

with

$$w_i = \sum_{k=0}^{N} a_k (K-k-d)^i - \sum_{m=0}^{M} b_m (K-m)^i. \tag{22}$$

Substituting Eqs. (15) and (16) for Eq. (21), we obtain

$$r(nT) = \int_{-\infty}^{\infty} K(\theta) x^{(P+1)}(nT-\theta)\, d\theta \tag{23}$$

where

$$K(\theta) = \frac{1}{P!}\left[\sum_{k=0}^{N} a_k \tilde{r}_k(\theta) - \sum_{m=0}^{M} b_m \tilde{s}_m(\theta)\right]. \tag{24}$$

$a(nT)$ can be regarded as a primal term in the approximation error $e(nT)$ and $r(nT)$ is the remainder term. If $a(nT) = 0$, the derivatives $x^{(i)}(nT-KT), 0 \leq i \leq P$ vanish in $e(nT)$. This condition is called P-degree of exactness. It is obvious that P-degree of exactness holds if

$$w_i = 0, 0 \leq i \leq P. \tag{25}$$

The function $K(\theta)$ is called Peano kernel of the system [5]. It plays a key role in the systems with P-degree of exactness because the design error is explicitly represented by $K(\theta)$. We can estimate the error in terms of $K(\theta)$ and Pth derivative of $x(t)$ based on Eq. (23). It is interesting that this equation has a form of convolution. Specifically speaking, $e(nT)$ is sampled from $e(t) = K(t) * x^{(P+1)}(t)$. That is, $K(t)$ may be regarded as the impulse response of a continuous-time system that is able to generate the error signal when the input is $x^{(P+1)}(t)$. However, it seems that $K(t)$ depends on K that is an auxiliary parameter in our derivation. In the next subsection, we will prove that K will be canceled out in $K(t)$ for P-degree of exactness systems.

3.1. A Simplification of Peano Kernel

If the system is of P-degree of exactness, the Peano kernel in Eq. (24) can be simplified into a very compact form. Note that in Eq. (25) we have $M + N + 1$ unknows but only $P + 1$ equations. Thus, in general we can't solve a_k and b_m and use them to simplify $K(\theta)$. However, if the condition of P-degree of exactness holds, $K(\theta)$ can be simplified. Denote $(\theta - t)_+^p \equiv (\theta - t)^p u(\theta - t)$. We have the following property.

Property 1 *The Peano kernel of FD systems with P-degree of exactness is*

$$K(\theta) = \frac{1}{P!}\left[\sum_{k=0}^{N} a_k(\theta - kT - \tau)_+^P - \sum_{m=0}^{M} b_m(\theta - mT)_+^P\right]. \quad (26)$$

Proof: Substitute Eqs. (17) and (18) for Eq. (24), and expand $(\theta - kT - \tau)^P = [(\theta - KT) + (KT - kT - \tau)]^P$ and $(\theta - mT)^P = [(\theta - KT) + (KT - mT)]^P$ into the binomial series. By collecting terms according to the unit step functions, the coefficient of $u(\theta - KT)$ can be represented as

$$\sum_{i=0}^{P} \binom{P}{i} (\theta - KT)^{P-i}(-T)^i w_i.$$

Since $w_i = 0, 0 \le i \le P$ for P-degree of exactness, the desired result follows. □

It is possible to obtain the Peano kernel by applying the Peano kernel theorem [5]. However, in such way we need to express the difference equation Eq. (4) in continuous-time domain. Our approach in the section is more intuitive and straightforward. On another hand, the convolution form of the error function is a natural result in our derivation.

3.2. Maximal Degree of Exactness

In some case, we can solve a_k and b_m based on the exactness of the system. Specifically speaking, if $P = M + N$, we can solve a_k and b_m explicitly. This condition is called the maximal degree of exactness. In this section, we will solve the coefficients a_k and b_m in closed form based on the maximal degree of exactness. Substituting Eq. (22) for Eq. (25) and using the definition of $a_0 = 1$, the maximal exactness condition is expressed by a system of linear equations

$$-\sum_{k=1}^{N} a_k(K-k-d)^i + \sum_{m=0}^{M} b_m(K-m)^i = (K-d)^i. \quad (27)$$

The solution to the above equations are

$$a_k = (-1)^k \binom{N}{k} \frac{(d-M)_{M+1}}{(d-M+k)_{M+1}}, \quad 1 \le k \le N \quad (28)$$

$$b_m = \frac{(-1)^{M-m} N!(d-M)_{M+1}}{m!(M-m)!(d-m)_{N+1}}, \quad 0 \le m \le M. \quad (29)$$

The Pochhammer's symbol $(x)_n = x(x+1)\cdots(x+n-1)$ for $n > 0$ and $(x)_0 = 1$ [6]. In the next section, we will give some design examples based on the closed form solutions of a_k, b_m and $K(\theta)$.

4. EXAMPLES

In this section, three examples are represented. We will discuss the cases of $M = N$, $N = 0$ and $N = M + 1$.

Example 1. The first example is the FD system of $M = N$. In this case the coefficients in Eqs. (28) and (29) are

$$a_k = (-1)^k \binom{M}{k} \frac{(d-M)_{M+1}}{(d-M+k)_{M+1}} \quad (30)$$

$$b_m = (-1)^{M-m} \binom{M}{m} \frac{(d-M)_{M+1}}{(d-m)_{M+1}} \quad (31)$$

It is easy to show that $b_M = 1 = a_0$ and $b_{M-m} = a_m, 1 \le m \le M$. We conclude that the system is an allpass filter. Moreover, based on above closed-form expressions, the system of $M = N$ is actually Thiran's allpass filter with maximally flat group delay [1, 7].

Fig. 1 shows the Peano kernels $K(\theta)$ for $1 \le M = N \le 8$. The sampling period $T = 1$ and the desired delay $\tau = d = M + 0.25$. Because the amplitude of the kernel for increasing M drops very fast such that these 8 kernels is hard to show on the same figure, we show the scaled version of $K(\theta)$ and the corresponding scaling factors on two separate figures. Denote the scaled Peano kernel by $K(\theta)$ and the scaling factor by A. We have $\tilde{K}(\theta) = K(\theta)/A$ where $A = K[(N+d)/2]$ is the value of the Peano kernel at $\theta_0 = (N+d)/2$. We choose this value for scaling because Peano kernel is nearly maximal or minimal at $\theta_0 = (N+d)/2$ by inspection. Fig. 1(a) shows $\tilde{K}(\theta)$ and Fig. 1(b) shows $\log|A|$. In Fig. 1(a) we can find that the extremal values of $\tilde{K}(\theta)$ are very close to 1 or -1. That is to say, $A = K[(N+d)/2]$ is very close to the real extremal value of Peano kernel. Fig. 1(b) illustrates that A decays very fast as M increases. It means that the allpass MF group delay filter approximates the ideal FD system well as M is increasing.

Example 2. In the second example we consider the case of FIR system. That is, $N = 0$ and b_m is represented by

$$b_m = \prod_{\substack{i=0,\\ i\neq m}}^{M} \frac{-d+i}{-m+i}, \quad 0 \le m \le M. \tag{32}$$

This is actually the MF FD FIR filter [1, 8].

Figs. 2(a) and (b) show the scaled Peano kernels $\tilde{K}(\theta)$ and the scaling factor A for $1 \le M \le 8$. The sampling period $T = 1$ and the desired delay $\tau = d = M + 0.25$. The scaling scheme is the same as that used in Example 1. In Fig. 2(a) we find that $A = K[(N + d)/2]$ is close but not precise to the extremal value of Peano kernel. Fig. 2(b) indicates that the decay rate of A is less than that shown in Fig. 1(b). It is reasonable since the allpass system discussed in Example 1 has $2N$-degree of exactness but in this example the FIR system has only N-degree of exactness.

Example 3. In this last example we discuss the case of $N = M + 1$ for $1 \le M \le 8$. The sampling period and the desired delay are 1 and $\tau = d = M + 0.25$. Figs. 3(a) and (b) show the magnitude responses and the group delays, respectively. The magnitude responses drop from unity near $\omega = \pi$. It can be shown numerically that these filters are stable.

5. CONCLUSIONS

In this paper, we analytically solved the Peano kernel of the FD systems. The approximation error can be explicitly represented as the convolution of Peano kernel and derivative of the input signal. For the case of maximal exactness, we solved the weighting coefficients in closed form. The MF FIR FD filters and the allpass filters with MF group delay are proved to be special cases of the filter of maximal exactness. Stability should be investigated in the future.

6. REFERENCES

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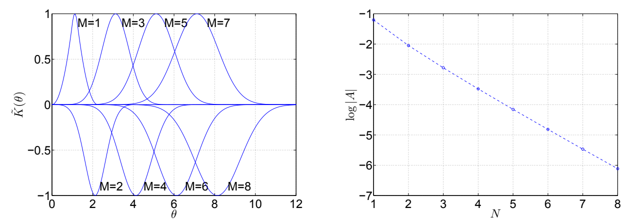
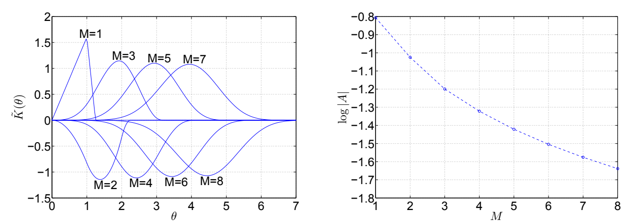


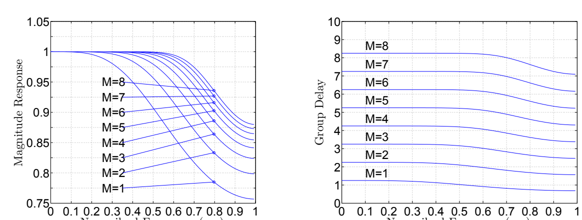
(a) (b)

Fig. 1. Peano kernels for $1 \le M = N \le 8$ in Example 1. (a) Scaled kernels. (b) Scaling factors.



(a) (b)

Fig. 2. Peano kernels for $N = 0, 1 \le M \le 8$ in Example 2. (a) Scaled kernels. (b) Scaling factors.



(a) (b)

Fig. 3. Frequency responses for $1 \le M \le 8, N = M + 1$ in Example 3. (a) Magnitude response. (b) Group delay.