FREQUENCY SELECTIVE KYP LEMMA AND ITS APPLICATIONS TO IIR FILTER BANK DESIGN

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ABSTRACT

For a transfer function/filter $F(e^{\jmath\omega})$ of order n, Kalman-Yakubovich-Popov (KYP) lemma characterizes the intractable semi-infinite programming (SIP) condition $F(e^{-\jmath\omega})\ 1\ \Theta\,[F(e^{\jmath\omega})\ 1]^T \geq 0\ \forall\,\omega$ in frequency domain by a tractable semi-definite programming (SDP) in state-space domain. Some recent results generalize this lemma to SDP for SIP of frequency selectivity (FS-SIP). All these SDP characterizations are given at the expense of the introduced Lyapunov matrix variable of dimension $n \times n$, making them impractical for high order problem. Moreover, the existing SDP characterizations for FS-SIP do not allow to formulate synthesis/design problems as SDPs. In this paper, we propose a completely new SDP characterization of general FS-SIP, which is of moderate size and is free from Lyapunov variables. Extensive examples are provided to validate the effectiveness of our result.

Index Terms— KYP lemma, semi-definite programming (SDP), IIR filter banks

1. INTRODUCTION

The celebrated Kalman-Yakubovich-Popov (KYP) lemma with its variations such as positive real lemma and bounded real lemma (see e.g. [1–5]) are certainly among the most fascinating results in modern control and signal processing. They allow to express a computationally intractable semi-infinite programming (SIP) constraint of a transfer function/filter $F(e^{\jmath\omega})$ in frequency domain by a computationally tractable semi-definite programming (SDP) in state-space domain.

The most general KYP lemma (see e.g. [3]) states that given a Hermitian indefinite matrix $\boldsymbol{\Theta} \in C^{2\times 2}$, the SIP condition

$$\begin{pmatrix} F(e^{\jmath\omega}) \\ 1 \end{pmatrix}^H \boldsymbol{\Theta} \begin{pmatrix} F(e^{\jmath\omega}) \\ 1 \end{pmatrix} \geq 0 \quad \forall\,\omega \in [0, 2\pi] \tag{1}$$

for an $n-$order transfer function/filter $F(e^{\jmath\omega})$ is characterized by a SDP involving its state-space realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ and a Lyapunov matrix function variable of dimension $n \times n$. As a matter of fact, the variable dimension $n \times n$ of Lyapunov variable, which is equivalent to $n(n+1)/2$ scalar variables, increases dramatically as order n increases moderately. Consequently, the resultant SDPs are of large dimension and difficult to solve using existing SDP solvers such as [6]. For instance, a 100-order transfer function/filter requires a Lyapunov variable of dimension 100×100, which is equivalent to 5000 scalar variables. This "curse of dimensionality" may not be so visible in control as many physical plants are fixed low orders, so SDP in tandem with KYP and Lyapunov machineries are the most dominant approach for control problems. However, many signal processing applications require filters of very high order so such kind of SDP is not applicable. Hence, traditional methods such as interpolation (frequency sampling) and window techniques are still popular though there is not much freedom in design specifications [5].

Furthermore, many problems in signal processing involve SIPs of a certain frequency selectivity rather the entire frequency range. For example, one of the filter design specification is the peak error between the filter response and the ideal response in a passband $[0, \omega_p]$ and stopband $[\omega_s, \pi]$, i.e. SIPs for $\omega \in [0, \omega_p]$ or $\omega \in [\omega_s, \pi]$ only, thus an extension for the mentioned KYP lemma, commonly referred to as frequency selective KYP (FS-KYP) lemma, is needed.

A number of FS-KYP lemma [7, 8] experience the similar drawback of the original KYP lemma: even more matrix variables of dimension $n \times n$ are involved in the SDP formulation. The formulation of [8] does not allow to formulate a design problem as a SDP. More precisely, it leads to a bilinear matrix inequality (BMI) formulation for the problem. In [9], we obtain a new SDP characterization of the FS inequality $|F(e^{\jmath\omega})| \leq \gamma\ \forall\omega \in [a, b]$ for finite impulse response (FIR) $F(e^{\jmath\omega})$. This SDP formulation is of substantially reduced order and its dual formulation does not involve any additional variables, hence opens a new way for effective solution of large dimensional digital systems. Our new SDP-based method can not only compete well with the traditional methods but it offers much more flexibility.

The main objective of this paper is to derive a FS-KYP lemma for infinite impulse response (IIR) systems. In contrast to other results, our resultant SDP are of much smaller dimension, enabling very high order IIR filters to be solved effectively on a standard personal computer. Needless to say, IIR filter and filter bank design is a fundamental problem in signal processing [5]. For a given specification, an IIR filter requires much lower order as compared to a FIR filter. However, IIR filter and filter bank design is a challenging task. Most digital IIR filters are either Chebyshev, Butterworth, or Elliptic that are derived from their analog counterparts via bilinear transformation or the impulse invariant method [10]. The peak errors as well as cut-off frequencies are not easily controlled. Thus the purpose of the paper is two-fold

- To develop an effective FS-KYP lemma which requires a SDP of moderate size for its solutions;
- To apply the obtained FS-KYP lemma to the problem of designing IIR filter bank and QMF bank. As it is known [11], using the traditional KYP lemma these problems are formulated as BMIs, which are highly non-convex from the optimization view point. Nevertheless, they are formulated as SDPs in the context of our newly developed FS-KYP lemma and thus are computationally tractable.

The structure of the paper is as follows. The reduced order formulation is discussed in Section II. Based on this result, some IIR filter bank design problems are developed in Section III and Section

IV. Finally concluding remarks are presented in Section V.

The following notation is used in the paper. Vectors and matrices will be represented by italicized bold lower case and upper-case letters, respectively. The superscript "T" denotes the transpose (without conjugation) whereas the superscript "H" denotes Hermitian transpose. The conventional symbols $\mathbb{R}^N$ and $\mathbb{C}^N$ are used to denote real and complex spaces. The standard notation $\boldsymbol{X} \geq 0$ represents a positive semidefinite Hermitian matrix and $\langle \boldsymbol{X}, \boldsymbol{Y} \rangle$ denotes the inner product of the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$. For a set C, conv$C$ (cone(C), resp.) is its convex hull (conic hull, resp.) i.e. it is the smallest convex set (smallest cone) containing C (conv(C), resp.)

2. REDUCED ORDER SDP FORMULATION FOR FREQUENCY SELECTIVE KYP LEMMA

2.1. Mathematical background

In this paper, we adopt the concept of the trigonometric curve and its convex hull introduced in [9].

Let $\boldsymbol{\varphi}_n(\omega) = (1, \cos\omega, \cos 2\omega, ..., \cos n\omega)^T$, then a trigonometric curve $C_{a,b} \in \mathbb{R}^{n+1}$ is defined as $C_{a,b} := \{\boldsymbol{\varphi}_n(\omega) : \cos\omega \in [\cos a, \cos b]\} \subset \mathbb{R}^{n+1}$ and its polar $C^*_{a,b}$ is given by $C^*_{a,b} = \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \langle \boldsymbol{u}, \boldsymbol{v} \rangle \geq 0 \ \forall \ v \in C_{a,b}\}$. The k-th moment trigonometric matrix $\boldsymbol{T}_k$ of size $(k+1) \times (k+1)$ is defined as the positive semi-definite matrix $\boldsymbol{T}_k(\omega) = \boldsymbol{\varphi}_k(\omega)\boldsymbol{\varphi}_k^T(\omega)$ and accordingly, the matrix $\boldsymbol{T}_k(\boldsymbol{y})$ is created from $\boldsymbol{T}_k(\omega)$ by the variable change

$$\cos h\omega \leftarrow y_h, \ h = 0, 1, 2, ..., \tag{2}$$

i.e. $\boldsymbol{T}_k(\omega) = \boldsymbol{T}_k(\boldsymbol{\varphi}_k(\omega))$. It is straightforward to show that

$$\boldsymbol{T}_k(\boldsymbol{y}) = \begin{bmatrix} y_0 & y_1 & \dots & y_k \\ y_1 & \frac{y_2+y_0}{2} & \dots & \frac{y_{k+1}+y_{k-1}}{2} \\ \dots & \dots & \dots & \dots \\ y_k & \frac{y_{k+1}+y_{k-1}}{2} & \dots & \frac{y_{2k}+y_0}{2} \end{bmatrix}.$$

Define also $\boldsymbol{T}_{1k}(\omega) = \cos(\omega)\boldsymbol{T}_k(\omega)$ and accordingly $\boldsymbol{T}_{1k}(\boldsymbol{y}) = \boldsymbol{T}_{1k}(y_1, y_2, ..., y_{2k+1})$ is created from $\boldsymbol{T}_{1k}(\omega)$ by the variable change (2). Our main results in this paper are based on the following LMI characterization for the convex hull of the trigonometric curve $C_{a,b}$:

Theorem 1 *([12]) The conic hull* $\mathrm{cone(C_{a,b})}$ *of the trigonometric curve $C_{a,b}$ is fully characterized by LMIs: $\boldsymbol{y} \in \mathrm{cone}(C_{a,b})$ if and only if it satisfies the LMIs*

$$\cos b\boldsymbol{T}_{[n/2]}(\boldsymbol{y}) \geq \boldsymbol{T}_{1[n/2]}(\boldsymbol{y}) \geq \cos a\boldsymbol{T}_{[n/2]}(\boldsymbol{y}) \tag{3}$$

The convex hull $\mathrm{conv}(C_{a,b})$ of $C_{a,b}$ is also fully characterized by LMIs: $y \in conv(C_{a,b})$ if and only if it satisfies the LMIs (3) with $y_0 = 1$.

Note that for n even, by the definition, $\boldsymbol{T}_{1[n/2]}(\boldsymbol{y})$ is a matrix function of $(y_0, y_1, ..., y_{n+1})$ and accordingly LMIs (3) are understood for some $\boldsymbol{y} \in \mathbb{R}^{n+2}$.

From the above result, it is easy and effective to transform SIP optimizations to SDPs. For instance, consider the following general optimization

$$\min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} \text{ s.t. } \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{d}_i \in C^*_i, \ i = 1, 2, ..., m, \tag{4}$$

where $\boldsymbol{Q} > 0$ is given and $C_i = \mathrm{cone}(C_{a_i,b_i})$). Then, by Theorem 1, its dual is

$$\max_{\boldsymbol{y}_i \in C_i} \min_{\boldsymbol{x}} [\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x} - \sum_{i=1}^{m}(\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{d}_i)^T\boldsymbol{y}_i] =$$
$$\begin{cases} \max_{\boldsymbol{y}_i} -\sum_{i=1}^{m}\boldsymbol{y}_i^T\boldsymbol{d}_i - \frac{1}{4}(\boldsymbol{c} - \sum_{i=1}^{m}\boldsymbol{A}_i^T\boldsymbol{y}_i)^T\boldsymbol{Q}^{-1}(\boldsymbol{c} - \sum_{i=1}^{m}\boldsymbol{A}_i^T\boldsymbol{y}_i) : \\ \text{(3) for } a_i \leftarrow a, \ b_i \leftarrow b, \ i = 1, 2, ..., m, \end{cases} \tag{5}$$

which can also be rewritten as a linear conic optimization [6] for efficient computation. The optimal solution $\boldsymbol{x}^*$ of (4) is directly retrieved from the optimal solution $\boldsymbol{y}^*_i$ of (5) by the solution of the following linear equation system: $\boldsymbol{Q}\boldsymbol{x}^* = -\frac{1}{2}(\boldsymbol{c} - \sum_{i=1}^{m}\boldsymbol{A}_i^T\boldsymbol{y}^*_i)$. Thus the optimal solution of the semi-infinite program (4) can be easily found from the program (5) involving just $(n+1)m$ scalar variables.

2.2. FS-KYP lemma

Now, any n-order IIR filter is represented as

$$F(e^{j\omega}) = \frac{N(e^{j\omega})}{D(e^{j\omega})} = \frac{\sum_{k=0}^{n} n_k e^{-jk\omega}}{\sum_{k=0}^{n} d_k e^{-jk\omega}} = \frac{\boldsymbol{\psi}_n^T(\omega)\boldsymbol{n}}{\boldsymbol{\psi}_n^T(\omega)\boldsymbol{d}} \tag{6}$$

where $\boldsymbol{\psi}_n(\omega) = [1, e^{-\jmath\omega}, ..., e^{-\jmath n\omega}]^T$, $\boldsymbol{n} = [n_0, n_1, \dots, n_n]^T$, $\boldsymbol{d} = [d_0, d_1, \dots, d_n]^T$. With the same Hermitian indefinite matrix $\boldsymbol{\Theta} \in C^{2\times 2}$ of (1), a FS-SIP is stated as

$$\begin{pmatrix} F(e^{j\omega}) \\ 1 \end{pmatrix}^H \boldsymbol{\Theta} \begin{pmatrix} F(e^{j\omega}) \\ 1 \end{pmatrix} \geq 0 \ \forall \ \cos\omega \in [\cos a, \cos b]. \tag{7}$$

The most popular particular cases of (7) are

• Frequency selective bounded realness (FS-BR) with

$$\boldsymbol{\Theta} = \begin{pmatrix} -1 & 0 \\ 0 & \gamma^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma & \gamma \end{pmatrix}^T - \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^T \tag{8}$$
$$\text{i.e.} \quad |F(e^{j\omega})| \leq \gamma \ \forall \cos\omega \in [\cos a, \cos b]. \tag{9}$$

Without FS restrictions, it merely says that the H_∞-norm of F is less than or equal γ.

• Frequency selective positive realness (FS-PR) with

$$\boldsymbol{\Theta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^T - \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}^T \tag{10}$$
$$\text{i.e.} \quad F(e^{-\jmath\omega}) + F(e^{\jmath\omega}) \geq 0 \ \forall \cos\omega \in [\cos a, \cos b]. \tag{11}$$

Without FS restriction, it merely says that the function F is positive real.

A linear algebra based efficient method has been proposed in [13] for computing H_∞-norm of F. This method, however, cannot be extended for verifying (9). Also, another heuristic method to solve LMIs in the original KYP lemma has been proposed in [14]. However, both methods do not work for the synthesis problem, i.e. one has to design F satisfying (9). Now, based on the result of Theorem 1, we provide a new look at FS-SIP (7). Clearly, (7) is equivalent to:

$$\begin{pmatrix} N(e^{j\omega}) \\ D(e^{j\omega}) \end{pmatrix}^H \boldsymbol{\Theta} \begin{pmatrix} N(e^{j\omega}) \\ D(e^{j\omega}) \end{pmatrix} \geq 0 \ \forall \cos\omega \in [\cos a, \cos b]. \tag{12}$$

Defining $\boldsymbol{A} = \begin{matrix} \boldsymbol{n}^T \\ \boldsymbol{d}^T \end{matrix}$, $\boldsymbol{\mathcal{M}} = \boldsymbol{A}^T\boldsymbol{\Theta}\boldsymbol{A}$, $\boldsymbol{\mathcal{T}}_n(e^{j\omega}) = \boldsymbol{\psi}_n(\omega)\boldsymbol{\psi}_n^H(\omega)$, $\boldsymbol{R}_n(\omega) = Re(\boldsymbol{\mathcal{T}}_n(e^{j\omega}))$, (12) is rewritten by

$$
\begin{aligned}
& \boldsymbol{\psi}_n^H(\omega)\boldsymbol{\mathcal{M}}\boldsymbol{\psi}_n(\omega) \geq 0 \quad \forall \cos\omega \in [\cos a, \cos b] \\
\Leftrightarrow \quad & \langle \boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{T}}_n(e^{j\omega}) \rangle \geq 0 \quad \forall \cos\omega \in [\cos a, \cos b] \\
\Leftrightarrow \quad & \langle \boldsymbol{\mathcal{M}}, \boldsymbol{R}_n(\omega) \rangle \geq 0 \quad \forall \cos\omega \in [\cos a, \cos b] \qquad (13) \\
\Leftrightarrow \quad & \langle \boldsymbol{\mathcal{M}}, \boldsymbol{R}_n(\boldsymbol{y}) \rangle \geq 0 \quad \forall y \in \text{conv}(C_{a,b}) \qquad (14)
\end{aligned}
$$

Now, based on LMI characterization for conv$(C_{a,b})$ in Theorem 1 we can state the following result

Proposition 1 *The FS-SIP (7) can be verified by the following SDP*

$$\min_{\boldsymbol{y}} \langle \boldsymbol{\mathcal{M}}, \boldsymbol{R}_n(\boldsymbol{y}) \rangle : \ (3), \ y_0 = 1. \qquad (15)$$

Namely, FS-SIP holds true if and only if the optimal value of SDP (15) is nonnegative.

In contrast with all previous results on LMI characterization for SIP (1) or (7) requiring a matrix variable $\boldsymbol{X}$ of dimension $n \times n$, the above SDP involves merely $n+1$ scalar variables and thus is applicable to F of large order n for practical interest.

One can see, SDP formulation is suitable only for analysis problem, i.e. to verify FS-SIP (7) for a given F. For a synthesis/design problem, one has to design F to satisfy FS-SIP (7), so $\boldsymbol{n}$ and $\boldsymbol{d}$ in (6) are design variables and then (15) is a BMI in $(\boldsymbol{n}, \boldsymbol{d}, \boldsymbol{y})$. We now reformulate (15) in a form that is more convenient for design.

Since $\boldsymbol{\Theta}$ is indefinite, by eigenvalue decomposition, there are $\boldsymbol{p} = (\alpha, \beta)^T, \boldsymbol{q} = (\lambda, \mu)^T$ and $\langle \boldsymbol{p}, \boldsymbol{q} \rangle = 0$ such that $\boldsymbol{\Theta} = \boldsymbol{p}\boldsymbol{p}^T - \boldsymbol{q}\boldsymbol{q}^T$ (see (8) and (10)). Then (13) is expressed as

$$
\begin{aligned}
& \sum_{i=0}^{n} (x_i - t_i) \cos i\omega \geq 0 \, \forall \cos\omega \in [\cos a, \cos b] \\
\Leftrightarrow & \boldsymbol{x} - \boldsymbol{t} \in C_{a,b}^*, \qquad (16)
\end{aligned}
$$

where $\boldsymbol{x} = (x_0, .., x_n)^T$ and $\boldsymbol{t} = (t_0, ..., t_n)^T$ are the autocorrelation sequences of $\boldsymbol{u} = \alpha\boldsymbol{d} + \beta\boldsymbol{n}$ and $\boldsymbol{v} = \lambda\boldsymbol{d} + \mu\boldsymbol{n}$. For instances, according to (8)

$$(9) \quad \Leftrightarrow \quad \gamma\bar{\boldsymbol{d}} - \bar{\boldsymbol{n}} \in C_{a,b}^*, \qquad (17)$$

which will be used frequently from now on. Here $\bar{\boldsymbol{n}}$ and $\bar{\boldsymbol{d}}$ are the autocorrelation sequences of $\boldsymbol{n}$ and $\boldsymbol{d}$.

3. IIR FILTER BANK DESIGN

Consider the Quadrature Mirror Filter (QMF) bank shown in the Fig. 1.

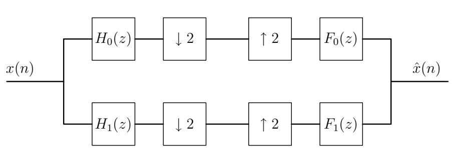

Fig. 1. The QMF bank structure

Suppose that $H_i(z)$ and $F_i(z)$ are all IIR filters, then alias-free condition requires:

$$[F_0(z), F_1(z)] = [H_0(z^{-1}), H_1(z^{-1})], \ H_1(z) = z^{2k-1}H_0(-z^{-1}) \qquad (18)$$

Let $P(z) = H_0(z)H_0(z^{-1})$, the design of QMF bank reduces to designing $P(z)$ to be a halfband low-pass filter. Once $P(z)$ is derived, $H_0(z)$ is obtained from $P(z)$ by factorization, and the other three filters are derived from $H_0(z)$ according to (18). To satisfy halfband condition, it is shown that $P(z)$ must admit the following representation [15]:

$$P(\omega) := P(e^{j\omega}) = \frac{X(\omega)}{X(\omega) + X(\omega - \pi)},$$

for a positive polynomial $X(\omega) = \boldsymbol{\varphi}_n^T(\omega)\boldsymbol{x}$ (i.e. $\mathbf{x} \in \mathbf{C}_{\pi,\mathbf{0}}^*$). Thus under the peak-error δ and ϵ constraints for the frequency responses of $P(z)$ in a passband $[0, \omega_p]$ and stopband $[\omega)s, \pi]$, $P(\omega)$ the design problem can be formulated as

$$
\begin{aligned}
\min_{\boldsymbol{x} \in C_{\pi,0}^*} \quad & w_1 \int_0^{\omega_p} (P(\omega) - 1)^2 d\omega + w_2 \int_{\omega_s}^{\pi} P(\omega)^2 d\omega \\
\text{s.t.} \quad & 1 - \delta \leq P(\omega) \forall \, \omega \in [0, \omega_p] \Leftrightarrow \mathbf{x} - (1-\delta)e_1 \in C_{\omega_p,0}^*, \\
& P(\omega) \leq \epsilon \, \forall \, \omega \in [\omega_s, \pi] \Leftrightarrow -\mathbf{x} + \epsilon e_1 \in C_{\pi,\omega_s}^*,
\end{aligned}
$$

where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^{n+1}$. As there is no analytical formula for the objective function, we uses the following quadratic function that also an indicator of deviation from the desired response

$$
\begin{aligned}
& \int_0^{\omega_p} [X(\omega - \pi)]^2 \, d\omega + \gamma_1 \int_0^{\omega_p} [X(\omega) + X(\omega - \pi) - 1]^2 \, d\omega \\
& + \int_{\omega_s}^{\pi} [X(\omega)]^2 \, d\omega + \gamma_2 \int_{\omega_s}^{\pi} [X(\omega) + X(\omega - \pi) - 1]^2 \, d\omega \qquad (19)
\end{aligned}
$$

Clearly, with this modified objective, by some simple manipulations, the IIR filter bank design is rewritten compactly as (4) and is thus solved through the dual SDP (5).

Example: To illustrate the effectiveness of the proposed method, we simulate a filter bank which comprises the order-4 IIR filters in comparison a with Butterworth filter bank of the same length [15]. In contrast to Butterworth method, our approach allows direct control of transition band width as well as passband ripple and stopband attenuation. Fig. 2 shows that our designed filters not only possess better transition band but also are capable of retaining both the flat passband and good stop band attenuation.

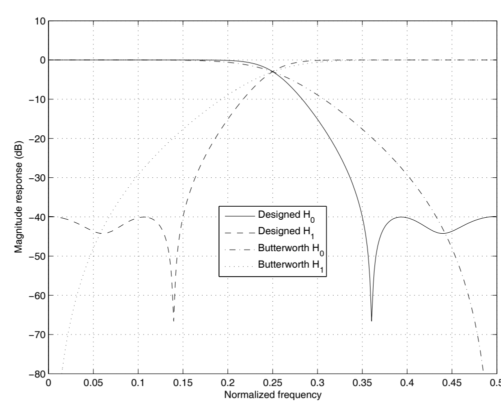

Fig. 2. Frequency response of the analysis filters

4. QMF BANK SYNTHESIS

In this section, we consider the design of IIR synthesis filters provided that analysis filters are given. It is shown in [16] that the synthesis filters $G_0(z), G_1(z)$ must be chosen as

$$G_0(z) \quad G_1(z) \;= G(z) \;\; H_1(-z) \quad -H_0(-z) \tag{20}$$

with some $G(z)$ such that the H_∞-norm of the distortion is minimized:

$$\min_{G(z)} \|z^{-n_0} - H(z)G(z)\|_\infty \tag{21}$$

where $H(z) = H_0(z)H_1(-z) - H_1(z)H_0(-z)$. Assuming $H(z) = N(z)/D(z)$ and $G(z) = P(z)/Q(z)$, (21) is equivalent to the following SIP:

$$\min_{P(z),Q(z),\epsilon} \quad \epsilon \tag{22}$$

$$\text{s.t.} \quad 1-\epsilon \leq \frac{|N(z)|^2|P(z)|^2}{|D(z)|^2|Q(z)|^2} \leq 1+\epsilon, \ \forall z = e^{j\omega}$$

Denoting autocorrelation sequences of the coefficients of $|N(z)P(z)|^2$ and $|D(z)Q(z)|^2$ by $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ respectively. Since $N(z)$ and $P(z)$ are known, it can be shown that $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ are linear function of $\boldsymbol{p}$ and $\boldsymbol{q}$, viz. $\bar{\boldsymbol{p}} = \boldsymbol{C}\boldsymbol{p} - \boldsymbol{e}$ and $\bar{\boldsymbol{q}} = \boldsymbol{D}\boldsymbol{q} - \boldsymbol{e}$.

Besides, it should be noted that minimizing the following quadratic function:

$$\min_{\bar{\boldsymbol{p}},\bar{\boldsymbol{q}}} \; \alpha_1 \int_0^\pi \; \boldsymbol{\varphi}_n^T(\omega)\bar{\boldsymbol{p}} - \boldsymbol{\varphi}_n^T(\omega)\bar{\boldsymbol{q}} \;\; ^2 d\omega + \alpha_2 \int_0^\pi \; \boldsymbol{\varphi}_n^T(\omega)\bar{\boldsymbol{q}} \;\; ^2 d\omega \tag{23}$$

makes the rational function $(\boldsymbol{\varphi}_n^T(\omega)\bar{\boldsymbol{p}} + 1)/(\boldsymbol{\varphi}_n^T(\omega)\bar{\boldsymbol{q}} + 1)$ smooth and close to 1, thus also minimizing ϵ. Now, with the augmented variable $\boldsymbol{x} = [\boldsymbol{p}, \boldsymbol{q}]^T$, we can rewrite (22) as $\min_{\boldsymbol{x} \in \mathbb{R}^{2(n+1)}} \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^T\boldsymbol{x}$ subject to $\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{d}_i \in C^*_{0,\pi}, \quad i = 1, \ldots, 4$ which again belongs to the optimization class (4) and then is solved through the SDP (5).

Example: In this example, we design the synthesis bank provided that the analysis bank consists of two Chebyshev filters:

$$H_0(z) = \frac{0.1412 + 0.3805z^{-1} + 0.3805z^{-2} + 0.1412z^{-3}}{1 - 0.3011z^{-1} + 0.3694z^{-2} - 0.0250z^{-3}}$$

and $H_1(z) = H_0(-z)$. The magnitude response of the synthesis bank of order 7 is presented in Fig. 3. Note that the magnitude responses of the designed filters of order 7 are already better than those of order 14 and 15 in [11] and [17]. It should be not surprised because our solution is globally optimal while the solutions given by [11] and [17] are locally optimal only in their class.

5. CONCLUSION

In this paper, a new form of FS-KYP lemma has been derived for both analysis and synthesis/dsign purposes. The attractive feature of the proposed FS-KYP lemma is that the resultant SDP is of minimal order, thus enabling high dimensional problems to be solved efficiently using general purpose solvers on a standard PC. Numerous IIR filter bank design problems have been formulated as reduced-order SDP and successfully demonstrated by a number of numerical examples.

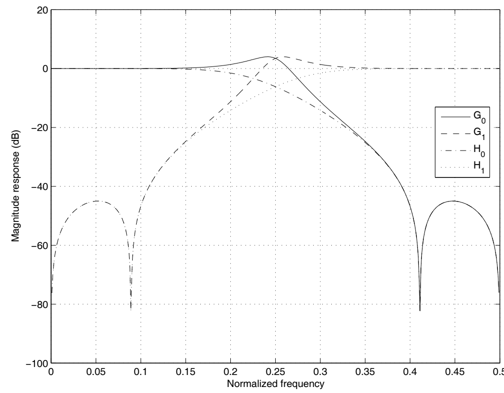

Fig. 3. Frequency response of a length-8 synthesis filters

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