EXACT INVERSION OF MIMO NONLINEAR POLYNOMIAL MIXTURES

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ABSTRACT

This paper deals with the inversion of MIMO mixing systems, which are instantaneous and nonlinear but polynomial. An exact inverse is searched in the class of polynomial systems. It is shown that Groebner bases techniques offer an attractive solution for testing the existence of such an inverse and computing it. If such an inverse does not exist we propose to test the existence of a polynomial relashionship between the sources and the observations and to compute simple polynomial functions, which each depend on one source only.

Relying on the fact that for finite alphabet sources, polynomials span the whole set of nonlinear mappings, we tackle the general nonlinear case. We generalize the first results to give a condition for the existence of an exact nonlinear inverse. The proposed method allows to compute this inverse in polynomial form.

Index Terms— nonlinear systems, polynomials, Groebner bases, source separation

1. INTRODUCTION

For the last decades, deconvolution and signal restoration issues have been active research fields. For instance in a multidimensional context, the problem of source separation — which consists in the restoration of several original signals from the observation of several mixtures of them— has received considerable attention, particularly in a blind context. Mixing models have often been restricted to linear ones, either instantaneous or convolutive. However, nonlinear distortions are likely to occur in many practical situations. Of course, nonlinear mixtures have already been considered for example in a blind context [1, 2], where specific non linear structures have been assumed.

In the linear context, perfect invertibility conditions of multi-input/multi-output (MIMO) linear time invariant systems are well-known and they reduce to left-invertibility of matrices: the elements of the mixing matrix are scalars in the instantaneous case or polynomials in the case of a finite impulse response (FIR) mixing filter. Polynomials may have several variables in the case where multi-dimensional (e.g. images,...) and multi-channel signals are considered [3, 4]. On the other hand, we are not aware of any such general result in the nonlinear case, although previous results include [5], which presents an invertibility criterion but no method for computing the inverse and [6] for a particular class of nonlinear systems.

The general class of nonlinear systems is often too wide to enable us to deal with the associated problem and we should preferably try to restrict to a smaller class of systems. Obtaining perfect invertibility conditions on a class of nonlinear mixture should help considering such models in a blind context. This motivates our interest for polynomial systems, which to a certain extent can be considered as one of the simplest form of nonlinearity. In this paper we show that methods exist which allow to compute a polynomial inverse (if it exists) of a polynomial MIMO mixing system. We illustrate their validity and effectiveness through examples.

Section 2 describes the issue which is addressed in the paper. Necessary definitions are introduced in Section 3. Section 4 explains how to compute a polynomial inverse. If it does not exist, an alternative solution method is shortly discussed. Section 5 is concerned with the case of finite alphabet sources.

2. PROBLEM STATEMENT

2.1. Source separation

We consider a set of Q sensors acquiring Q observation signals which compose the vector valued signal $(\mathbf{x}(n))_{n\in\mathbb{Z}} = ((x_1(n))_{n\in\mathbb{Z}}, \dots, (x_Q(n))_{n\in\mathbb{Z}})^{\mathrm{T}}$. In a source separation context, one assumes that these observations come from another set of signals, called the sources and denoted by the vector $(\mathbf{s}(n))_{n\in\mathbb{Z}} \triangleq ((s_1(n))_{n\in\mathbb{Z}}, \dots, (s_N(n))_{n\in\mathbb{Z}})^{\mathrm{T}}$. We assume a deterministic relation between the sources and the observations. More precisely, the paper focuses on instantaneous nonlinear transforms of the sources. Dropping the time index n, we thus write $\mathbf{x} = \mathbf{f}(\mathbf{s})$ where $\mathbf{f}$ is a nonlinear function $\mathbf{f} : \mathbb{C}^N \to \mathbb{C}^Q$. Componentwise, the corresponding mixing equations read:

$$\left\{ \begin{array}{lcl} x_1 & = & f_1(s_1, \ldots, s_N) \\ \vdots & & \vdots \\ x_Q & = & f_Q(s_1, \ldots, s_N) \end{array} \right. \tag{1}$$

where $f_1, \ldots, f_Q$ constitute the components of $\mathbf{f}$.

The source separation problem consists in recovering the sources $s_1, \ldots, s_N$ from the observations $x_1, \ldots, x_Q$. This is equivalent to finding the inverse MIMO system $\mathbf{g} : \mathbb{C}^Q \to \mathbb{C}^N$. In other words, we look for the components $g_i : \mathbb{C}^Q \to \mathbb{C}$ of $\mathbf{g} = \mathbf{f}^{-1}$ such that for all i:

$$s_i = g_i(x_1, \ldots, x_Q). \tag{2}$$

This contribution addresses the problem of computing an inverse for a known and given mixing system (1).

2.2. Non linear functions and polynomials

This paper focuses on the particular case where the functions $f_i, i \in \{1, \ldots, Q\}$ in (1) are polynomials, that is for all i, $f_i \in \mathbb{C}[\mathbf{s}]$, where $\mathbb{C}[\mathbf{s}]$ stands for the set of polynomials in variables $s_1, \ldots, s_N$ and with coefficients in $\mathbb{C}$. This restriction is partly justified by the difficulty to tackle the nonlinear case because of its generality. In addition, polynomials constitute an important class of nonlinear models which may represent acceptable approximations of certain nonlinearities. Finally, an important reason to deal with this model is the following one.

Consider the case where the multidimensional source vector belongs to a finite set: $\mathbf{s} \in \mathcal{A} = \{\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n_a)}\}$. Although seemingly restrictive, this situation is highly interesting since it occurs in digital communications, where the emitted source sequences belong to a finite alphabet depending on the modulation used.

An important observation is that *if* $\mathbf{s} \in \mathcal{A}$ *and* $\mathcal{A}$ *is finite, all instantaneous mixtures of the sources can be expressed as polynomial mixtures*. This follows immediately from the fact that any function on a finite set can be interpolated by a polynomial in a way similar to Lagrange polynomial interpolation [7]. It follows that polynomial mixtures constitute the general model of nonlinear mixtures in the case of sources belonging to a finite alphabet.

3. MATHEMATICAL PRELIMINARIES

The model (1) being polynomial, and in order be able to resort to algebraic techniques, we will restrict the separator to the class of polynomial functions in $x_1, \ldots, x_Q$, that is: $\forall i, g_i \in \mathbb{C}[\mathbf{x}]$. Algebra and Groebner basis techniques are powerful methods for the study of multivariate polynomials. They have been applied only recently in signal processing [8, 4, 9]. We recall some basic definitions required here for comprehension (see [7] for a detailed introduction to the subject). In the following, boldface letters denote N-tuples and for any $\boldsymbol{\alpha} \in \mathbb{N}^N$ we write: $\mathbf{s}^{\boldsymbol{\alpha}} \triangleq s_1^{\alpha_1} \ldots s_N^{\alpha_N}$.

Definition 1 *Let* $h_1, \ldots, h_p \in \mathbb{C}[\mathbf{s}]$ *be polynomials. The* ideal *generated by these polynomials is the subset of* $\mathbb{C}[\mathbf{s}]$ *which consists of all linear combinations* $a_1 h_1 + \ldots + a_p h_p$ *where* $a_1, \ldots, a_p$ *are polynomials. It is denoted by* $\langle h_1, \ldots, h_p \rangle$.

Definition 2 *A monomial order* $\prec$ *on* $\mathbb{C}[\mathbf{s}]$ *is a total ordering relation on the set of monomials such that:*

- *if* $\mathbf{s}^{\boldsymbol{\alpha}} \prec \mathbf{s}^{\boldsymbol{\beta}}$ *then* $\mathbf{s}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}} \prec \mathbf{s}^{\boldsymbol{\beta}+\boldsymbol{\gamma}}$
- $\prec$ *is a well-ordering, that is, every nonempty collection of monomials has a smallest element under* $\prec$.

A simple example of monomial order is the lexicographic order, where by definition $\mathbf{s}^{\boldsymbol{\alpha}} \prec \mathbf{s}^{\boldsymbol{\beta}}$ if and only if in the vector difference $\boldsymbol{\alpha} - \boldsymbol{\beta}$, the left-most nonzero entry is positive. Given a monomial order, we can define the leading term of a polynomial h which is the product $c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ where $\mathbf{x}^{\boldsymbol{\alpha}}$ is the largest monomial appearing in h in the order $\prec$. We can also define a division algorithm, which generalizes the division algorithm in the case of one variable:

Theorem 1 (division algorithm) *Let* $(h_1, \ldots, h_p)$ *be an ordered* p*-tuple of polynomials. Every polynomial* h *can be written as:*

$$h = a_1 h_1 + \ldots + a_p h_p + r \tag{3}$$

where a_i, r *are polynomials and* r *is a linear combination with coefficients in* $\mathbb{C}$ *of monomials, none of which is divisible by any leading term of* $h_1, \ldots, h_p$. *(possibly,* $r = 0$*).*

If we consider $\mathbf{I} = \langle h_1, \ldots, h_p \rangle$, the division algorithm provides a way to write any polynomial as the sum $h = h_{\mathbf{I}} + r$ where $h_{\mathbf{I}}$ lies in the ideal $\mathbf{I}$ and no term of r is divisible by any of the leading terms of $h_1, \ldots, h_p$. Unfortunately, the remainder r in this decomposition is not unique in general. A remarkable exception is when the set of generators satisfy the following definition.

Definition 3 *The set* $\{h_1, \ldots, h_p\}$ *is a* Groebner basis *of the ideal* $\mathbf{I} = \langle h_1, \ldots, h_p \rangle$ *if and only if the remainder* r *in* (3) *is uniquely determined.*

Importantly, there exist an algorithm, initially developped by Buchberger for converting a given generating set of an ideal to a Groebner basis.

4. INVERTIBILITY

4.1. Perfect invertibility

The problem of finding $g_i \in \mathbb{C}[\mathbf{x}]$ such that (2) is satisfied is a particular case of the subalgebra or subring membership problem [7]. Let $\mathbb{C}[\mathbf{f}]$ be the set of all polynomials in $\mathbb{C}[\mathbf{s}]$ which can be written as polynomial expressions in $f_1, \ldots, f_Q$ with coefficients in $\mathbb{C}$. We will make use of the following theorem [7]:

Theorem 2 *Fix a monomial order in $\mathbb{C}[\mathbf{s}, \mathbf{x}]$ where any monomial involving one of $s_1, \ldots, s_N$ is greater than all monomials in $\mathbb{C}[\mathbf{x}]$. Let G be a Groebner basis of the ideal $\langle f_1 - x_1, \ldots, f_Q - x_Q \rangle \subset \mathbb{C}[\mathbf{x}, \mathbf{s}]$. Given $h \in \mathbb{C}[\mathbf{s}]$, let g be the remainder of h on division by G. Then:*

(i) $h \in \mathbb{C}[\mathbf{f}]$ if and only if $g \in \mathbb{C}[\mathbf{x}]$.

(ii) if $h \in \mathbb{C}[\mathbf{f}]$, then $h = g(f_1, \ldots, f_Q)$ is an expression of h as a polynomial in $f_1, \ldots, f_Q$.

Monomial order satisfying the condition in the above theorem are called elimination order for $s_1, \ldots, s_N$. One should note that this condition is satisfied by the lexicographic order in $\mathbb{C}[\mathbf{s}, \mathbf{x}]$ but other monomial orders satisfy this condition [7]. The invertibility result will not depend on the chosen order, but different inverses may be found. The method for perfect inversion with a polynomial separator follows from the above proposition which can be applied successively to the polynomials $s_1, \ldots, s_N$ in $\mathbb{C}[\mathbf{s}]$. It reads:

1. Choose in $\mathbb{C}[\mathbf{s}, \mathbf{x}]$ an elimination order for $s_1, \ldots, s_N$ and define $\mathbf{I} = \langle f_1 - x_1, \ldots, f_Q - x_Q \rangle$.
2. Compute a Groebner basis G of $\mathbf{I}$.
3. For $i = 1 \ldots N$, compute the division of s_i by G. If the result g_i of the division is in $\mathbb{C}[\mathbf{x}]$, we have $s_i = g(f_1, \ldots, f_Q)$, otherwise, s_i cannot be recovered exactly by a polynomial in $f_1, \ldots, f_Q$.

4.1.1. Example

Throughout the paper, we will consider the example provided by the following equations:

$$\begin{cases} x_1 = f_1(s_1, s_2) = 3s_1^2 + 2s_1 s_2 + 4s_2^2 + 7s_1 + 4s_2 \\ x_2 = f_2(s_1, s_2) = -3s_1^2 + 5s_1 s_2 + 2s_1 + s_2 \\ x_3 = f_3(s_1, s_2) = -3s_1 + 6s_2 \\ x_4 = f_4(s_1, s_2) = 6s_1^2 - s_1 s_2 + 4s_2^2 + 3s_1 - 9s_2 \end{cases} \tag{4}$$

Groebner basis computation and polynomial division are implemented in many computer algebra systems. Using the lexicographic order in $\mathbb{C}[s_1, s_2, x_1, x_2, x_3, x_4]$, the following inverse of (4) has been computed:

$$\begin{cases} s_1 = \frac{17}{144}x_1 - \frac{1}{12}x_2 - \frac{1}{432}x_3^2 - \frac{91}{432}x_3 - \frac{7}{72}x_4 \\ s_2 = \frac{17}{288}x_1 - \frac{1}{24}x_2 - \frac{1}{864}x_3^2 + \frac{53}{864}x_3 - \frac{7}{144}x_4 \end{cases}$$

The method which has been described of course applies when the polynomials $f_1, \ldots, f_Q$ have total degree one. In this case, the mixture is actually a linear instantaneous one and consists in a simple matrix product. The above computation is then similar to a Gaussian elimination procedure.

4.2. Separability

Previous section gives a condition to be able to recover exactly one source with a polynomial expression of the observations. If not possible, it may sometimes be enough to recover a function (here, a polynomial function) of each source instead of recovering the source itself (e.g. in blind separation). A simple example of this particular case is given by the mixing system $x_1 = s_1^2 + s_2^2, x_2 = s_1^2 - s_2^2$ where one can easily recover s_1^2 and s_2^2 and may not be interested in s_1 and s_2. It would hence be interesting to be able to describe $\mathbb{C}[\mathbf{f}]$ which is the set of polynomials in $s_1, \ldots, s_N$ which can be obtained as polynomial expressions in the observations $x_1, \ldots, x_Q$. One would in this case be more particularly interested in knowing something about $\mathbb{C}[s_i] \cap \mathbb{C}[\mathbf{f}]$ which are the polynomials in s_i only which one can compute using only the observations. Unfortunately, $\mathbb{C}[\mathbf{f}]$ does not have the structure of an ideal and hence cannot be described by a set of generators. In the case where the system is not invertible by the previous method, we hence propose the following partial solution to this difficulty:

1. For $i \in \{1, \ldots, N\}$, test for algebraic dependence between $f_1, \ldots, f_Q$ and s_i, that is test whether there exist a polynomial δ such that $\delta(s_i, f_1, \ldots, f_Q) = 0$. This problem admits an algebraic solution [10].
2. For $i \in \{1, \ldots, N\}$, if $f_1, \ldots, f_Q$ and s_i are algebraically dependent, then try to determine whether simple polynomials in s_i only belong to $\mathbb{C}[s_i] \cap \mathbb{C}[\mathbf{f}]$.

4.2.1. Example

We consider the system in Equation (4) where only x_1, x_2 and x_3 are observed. (That is, the mixing system has 2 sources, 3 sensors and the last equation in (4) is ignored). In this case, one can check with the previous method that the system is no longer invertible. However, one can also check that there exist algebraic relations between the polynomials f_1, f_2, f_3 in Equation (4) and s_i for $i = 1, 2$. Going further, one can compute:

$$\begin{cases} s_1^2 + b_1 s_1 = (2b_1 - \frac{15}{7})s_2 + \frac{5}{28}x_1 - \frac{3}{14}x_2 - \frac{5}{252}x_3^2 \\ \qquad\qquad + (-\frac{b_1}{3} + \frac{23}{84})x_3 \\ s_2^2 + b_2 s_2 = (b_2 - \frac{7}{4})s_2 + \frac{1}{16}x_1 + \frac{1}{8}x_2 + \frac{1}{48}x_3^2 + \frac{11}{48}x_3 \end{cases}$$

Choosing $b_1 = 15/14$ (resp. $b_2 = 7/4$), one thus obtains the separation of the sources, that is a polynomial in s_1 (resp. s_2) only, which is expressed depending on x_1, x_2 and x_3 only.

5. FINITE ALPHABET SOURCES

Section 4.1 describes a method for computing a perfect inverse of a polynomial MIMO mixture. If the latter does not exist, Section 4.2 shows how, giving up the exact restitution of the sources, one can possibly separate them only. Using polynomials, even this is however not always possible. According to Section 2.2, the general case of nonlinear functions can be treated if the sources belong to a finite alphabet. We hence assume for all i, $s_i \in \mathcal{A}_i$, where $\mathcal{A}_i$ is a finite set. Equivalently, we write that for all i, the sources satisfy $q_i(s_i) = 0$ where $q_i(s_i) = \prod_{s \in \mathcal{A}_i}(s_i - s)$ is the reduced polynomial in the variable s_i, which roots are given by $\mathcal{A}_i$.

If we write $\mathcal{A} \triangleq \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$, it is sufficient that the polynomials $g_i, i \in \{1, \ldots, N\}$ of the inverse system satisfy:

$$\forall \mathbf{s} \in \mathcal{A} \qquad s_i = g_i(f_1(\mathbf{s}), \ldots, f_Q(\mathbf{s}))$$

and they need not verify the above equation on $\mathbb{C}^N$. In other terms, the inverse g_i should be such that $s_i - g_i(f_1, \ldots, f_Q)$ vanishes identically on $\mathcal{A}$. Since the ideal $\mathbf{I}_{\mathcal{A}} \triangleq \langle q_1, \ldots, q_N \rangle$ corresponds to the set of polynomials vanishing identically on $\mathcal{A}$ (this holds because the polynomials q_i are reduced), we are seeking for polynomials g_i such that

$$s_i - g_i(f_1, \ldots, f_Q) \in \mathbf{I}_{\mathcal{A}}.$$

We can prove a generalization of Theorem 2, which is conceptually equivalent to considering the quotient space $\mathbb{C}[\mathbf{s}]/\mathbf{I}_{\mathcal{A}}$:

Proposition 1 *Fix a monomial order in* $\mathbb{C}[\mathbf{s}, \mathbf{x}]$ *where any monomial involving one of* $s_1, \ldots, s_N$ *is greater than all monomials in* $\mathbb{C}[\mathbf{x}]$. *Let G be a Groebner basis of the ideal* $\langle f_1 - x_1, \ldots, f_Q - x_Q, q_1, \ldots, q_N \rangle \subset \mathbb{C}[\mathbf{x}, \mathbf{s}]$. *Given* $h \in \mathbb{C}[\mathbf{s}]$, *let g be the remainder of h on division by G. Then:*

(i) $g \in \mathbb{C}[\mathbf{x}]$ *if and only if there exists* $r \in \mathbf{I}_{\mathcal{A}}$ *such that* $h - r \in \mathbb{C}[\mathbf{f}]$.

(ii) if the above condition holds, then $g(f_1, \ldots, f_Q)$ *is an expression such that* $h - g(f_1, \ldots, f_Q) \in \mathbf{I}_{\mathcal{A}}$.

Similarly to Section 4.1, the method for computing an inverse follows from the above proposition. Let us stress that a nonlinear inverse exists if and only if a polynomial inverse exists.

5.0.2. Example

Assume that for all i the sources s_i belong to $\{\pm\frac{1}{2}; \pm\frac{3}{2}\}$. This is typically the case of PAM4 telecommunication sources. Defining x_1, x_2 and x_3 as in (4), the following equalities hold for all (s_1, s_2) in $\{\pm\frac{1}{2}; \pm\frac{3}{2}\}^2$:

$$\begin{cases}
s_1 = -\frac{32864}{52732215}x_2x_3^5 + \frac{17600}{10546443}x_2x_3^4 + \frac{153488}{3515481}x_2x_3^3 \\
\quad - \frac{132800}{1171827}x_2x_3^2 - \frac{900538}{1953045}x_2x_3 + \frac{44740}{43401}x_2 \\
\quad - \frac{417616}{807277479435}x_3^9 + \frac{16}{1055264679}x_3^8 + \frac{2223128}{29899165905}x_3^7 \\
\quad + \frac{127768}{1423769805}x_3^6 - \frac{121412}{31639329}x_3^5 - \frac{177568}{31639329}x_3^4 \\
\quad + \frac{10508731}{130279590}x_3^3 + \frac{474367}{27342630}x_3^2 - \frac{2547283}{5911920}x_3 + \frac{5715}{13616} \\
s_2 = -\frac{16432}{52732215}x_2x_3^5 + \frac{8800}{10546443}x_2x_3^4 + \frac{76744}{3515481}x_2x_3^3 \\
\quad - \frac{66400}{1171827}x_2x_3^2 - \frac{450269}{1953045}x_2x_3 + \frac{22370}{43401}x_2 \\
\quad - \frac{208808}{807277479435}x_3^9 + \frac{8}{1055264679}x_3^8 + \frac{1111564}{29899165905}x_3^7 \\
\quad + \frac{63884}{1423769805}x_3^6 - \frac{60706}{31639329}x_3^5 - \frac{88784}{31639329}x_3^4 \\
\quad + \frac{10508731}{260559180}x_3^3 + \frac{474367}{54685260}x_3^2 - \frac{576643}{11823840}x_3 + \frac{5715}{27232}
\end{cases}$$

This illustrates that with PAM4 sources, the mixture (4) can be exactly inverted using x_1, x_2 and x_3 only. Actually, using this method, it can be proved that for the considered discrete sources, no more than two of the observations in Equation (4) are required for exact inversion.

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