PARAMETER ESTIMATION OF POSITIVE ALPHA-STABLE DISTRIBUTION BASED ON NEGATIVE-ORDER MOMENTS

Zengguo Sun, Chongzhao Han

School of Electronics and Information Engineering
Xi'an Jiaotong University
Xi'an, Shaanxi, P. R. China

ABSTRACT

Positive alpha-stable distribution is used to model the non-negative quantity with impulsive property. Based on negative-order moments, three methods are used to estimate the parameters of positive alpha-stable distribution in this paper. First, a ratio estimator based on the ratio of negative-order moments is presented whose performance is significantly determined by the choice of order. Second, a new estimator with explicit closed form is presented and it is shown to be robust compared to the ratio estimator. Last, an iterative estimator is proposed and it achieves better performance only using fewer samples in each step computation. Monte Carlo simulation results demonstrate that the proposed iterative estimator is high efficient for the positive alpha-stable distribution.

Index Terms— Positive alpha-stable distribution, negative-order moments, symmetric alpha-stable distribution, iterative parameter estimator, Monte Carlo simulation

1. INTRODUCTION

Alpha-stable distribution is widely used for the modeling of impulsive signals and phenomena such as underwater acoustic signals, low-frequency atmospheric noise and many types of man-made noise [1]. The shape of stable distribution is determined by characteristic parameter α and symmetry parameter β. When α is restricted to the range $(0,1)$ and β is fixed at 1, the alpha-stable distribution reduces to the positive alpha-stable ($P\alpha S$) distribution. It is worth pointing out that $P\alpha S$ distribution is totally skewed with all of the probability concentrated on $(0,\infty)$, so it is reasonable to model the non-negative quantity such as energy and power with $P\alpha S$ distribution [2]. It is well known that any symmetric alpha-stable ($S\alpha S$) distribution, which is bell-shaped and symmetric about the origin, can be represented as the product of a zero mean Gaussian distribution and the square root of a $P\alpha S$ distribution. This property is usually used for the generation of $S\alpha S$ distributed random samples [3].

Based on negative-order moments introduced in [1,2], we present three methods to estimate the parameters of $P\alpha S$ distribution in this paper. First, we present a ratio estimator based on the ratio of negative-order moments. This estimator is affected by the choice of p. Once p is chosen inappropriately, performance of the estimator is degraded seriously. Second, we present an alternative $\log P\alpha S$ estimator whose idea is similar to the $\log|S\alpha S|$ estimator used for the parameter estimation of $S\alpha S$ distribution [1]. This estimator is completely determined by samples, and it is easy to compute because of the explicit closed form. Last, we propose an iterative $\log P\alpha S$ estimator that achieves better performance by fewer samples according to the iterative computation of sample block. Monte Carlo simulation demonstrates the high efficiency of the iterative $\log P\alpha S$ estimator.

2. RATIO ESTIMATOR FOR $P\alpha S$ DISTRIBUTION

The $P\alpha S$ distribution can be described by its characteristic function

$$\varphi(t) = \exp\left(-\gamma|t|^{\alpha}\left[1+i\,\mathrm{sgn}(t)\tan(\alpha\pi/2)\right]\right), \qquad (1)$$

where $0<\alpha<1$ is the characteristic parameter and $\gamma>0$ is the scale parameter. Here, $\mathrm{sgn}(t)$ denotes the sign function. If X is a $P\alpha S$ random variable, the negative-order moment of X can be written as

$$E\left(X^{p}\right) = \frac{\gamma^{p/\alpha}\Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\Gamma(-p)}\left[1+\tan^{2}\left(\frac{\pi\alpha}{2}\right)\right]^{\frac{p}{2\alpha}}, \quad p<0. \qquad (2)$$

Here, $\Gamma(x)$ denotes the Gamma function [2]. Then, we can take the ratio

$$\frac{E\left(X^{2p}\right)}{E^{2}\left(X^{p}\right)} = \frac{\Gamma(-p)}{\Gamma\left(\frac{1}{2}-p\right)}\frac{\alpha 2^{2p-\frac{2p}{\alpha}}\Gamma\left(\frac{1}{2}-\frac{p}{\alpha}\right)}{\Gamma\left(-\frac{p}{\alpha}\right)}. \qquad (3)$$

It is seen that α can be numerically evaluated from (3). Once α is obtained, γ can be estimated from (2) as follows

$$\gamma=\left\{\frac{E\left(X^{p}\right)\Gamma(-p)\alpha}{\Gamma(-p/\alpha)\left[1+\tan^{2}(\pi\alpha/2)\right]^{\frac{p}{2\alpha}}}\right\}^{\frac{\alpha}{p}}. \tag{4}$$

In the ratio estimator (3) and (4), $E\left(X^{p}\right)$ can be estimated by the empirical moments calculated from the samples. It can be shown that the estimator proposed by Pierce is just a special case of (3) when $p=-0.5$. Table 1 illustrates the average and standard deviation values (in parentheses) of Monte Carlo simulation results based on the ratio estimator. Various numbers of samples from a standard (unit scale parameter) $P\alpha S$ distribution were generated using the method introduced in [3] and the experiment was repeated 100 times independently. Obviously, performance of this estimator greatly relies on the choice of p. The standard deviation of estimators decreases as p increases for a fixed sample size. When p is fixed, generally, sufficient samples lead to good performance.

Table 1 Performance of the ratio estimator (True $\alpha=0.5$, $\gamma=1$)

Number of Samples	$p=-1.5$		$p=-1$		$p=-0.5$	
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$
1000	0.5019 (0.0168)	0.9957 (0.0602)	0.5004 (0.0126)	1.0025 (0.0422)	0.4998 (0.0074)	1.0029 (0.0308)
2000	0.5012 (0.0126)	0.9980 (0.0406)	0.4998 (0.0075)	0.9994 (0.0281)	0.4998 (0.0062)	0.9993 (0.0252)
5000	0.4993 (0.0079)	1.0021 (0.0270)	0.5009 (0.0053)	0.9972 (0.0188)	0.4999 (0.0040)	1.0002 (0.0148)

3. $\log P\alpha S$ ESTIMATOR FOR $P\alpha S$ DISTRIBUTION

Let X be a random variable with $P\alpha S$ distribution, then its negative-order moment satisfies (2). We can rewrite $E\left(X^{p}\right)$ as $E\left(\mathrm{e}^{p\log X}\right)$ and define a new random variable $Y=\log X$. Therefore,

$$E\left(X^{p}\right)=E\left(\mathrm{e}^{p\log X}\right)=E\left(\mathrm{e}^{pY}\right),\quad p<0 \tag{5}$$

where $E\left(\mathrm{e}^{pY}\right)$ can be regarded as the moment-generating function of Y.Expanding it into the Taylor series, we can write $E\left(\mathrm{e}^{pY}\right)$ as

$$E\left(\mathrm{e}^{pY}\right)=\sum_{k=0}^{\infty}E\left(Y^{k}\right)\frac{p^{k}}{k!}=D(p,\alpha)\gamma^{\frac{p}{\alpha}}. \tag{6}$$

Here, $D(p,\alpha)=\dfrac{\Gamma(-p/\alpha)}{\alpha\Gamma(-p)}\left[1+\tan^{2}\left(\dfrac{\pi\alpha}{2}\right)\right]^{\frac{p}{2\alpha}}$. Then, moments of Y of any order can be obtained by

$$E\left(Y^{k}\right)=\left.\frac{\mathrm{d}^{k}\left(D(p,\alpha)\gamma^{p/\alpha}\right)}{\mathrm{d}\,p^{k}}\right|_{p=0}. \tag{7}$$

Considering the first-order and second-order moments of Y, after some manipulation, we can obtain

$$E(Y)=C_{\mathrm{e}}\left(\frac{1}{\alpha}-1\right)+\frac{\log\gamma}{\alpha}+\frac{\log\left(\sec^{2}(\pi\alpha/2)\right)}{2\alpha} \tag{8}$$

and

$$Var(Y)=\frac{\pi^{2}\left(1-\alpha^{2}\right)}{6\alpha^{2}}. \tag{9}$$

Here, C_{e} denotes the Euler's constant. Since α is isolated in (9), it can be easily obtained by $Var(Y)$, and γ can be estimated from (8) immediately.

In (8) and (9), mean and variance of Y can be estimated from the sample mean and the sample variance, respectively. Compared to the ratio estimator (3) and (4), the $\log P\alpha S$ estimator are only determined by data sample and it is computationally efficient owing to the explicit closed form. Table 2 shows the Monte Carlo simulation results based on the $\log P\alpha S$ estimator and the experimental conditions are the same as the Table 1. It must be stressed that performance of $\log P\alpha S$ estimator may be inferior to the ratio estimator with appropriate choice of p (e.g., 1000 samples and $p=-0.5$ for the ratio estimator), but performance of the ratio estimator is seriously degraded when p is chosen improperly (e.g., 1000 samples and $p=-1.5$ for the ratio estimator). So we believe that the $\log P\alpha S$ estimator is robust compared to the ratio estimator.

Table 2 Performance of $\log P\alpha S$ estimator (True $\alpha=0.5$, $\gamma=1$)

Number of Samples	$\hat{\alpha}$	$\hat{\gamma}$
1000	0.5002 (0.0157)	0.9969 (0.0322)
2000	0.5009 (0.0109)	0.9975 (0.0231)
5000	0.5001 (0.0073)	1.0022 (0.0171)

4. ITERATIVE $\log P\alpha S$ ESTIMATOR FOR $P\alpha S$ DISTRIBUTION

It is seen that $\log P\alpha S$ estimator processes all samples at one time. However, a reasonable choice is to update the estimated parameter values iteratively in order to achieve memory efficiency. Let total samples be divided into B

non-overlapping blocks and each block contains M samples. Denoting $\hat{\alpha}(k)$ and $\hat{\gamma}(k)$ as the estimated parameter values derived from the first k ($1 \le k \le B$) blocks, we hope that $\hat{\alpha}(k)$ and $\hat{\gamma}(k)$ can be obtained from the previous estimated values $\hat{\alpha}(k-1)$ and $\hat{\gamma}(k-1)$. Donating $E_{1,k}$ as the sample mean calculated from the first k blocks, then

$$\begin{aligned} E_{1,k} &= \frac{1}{kM}\sum_{i=1}^{kM} Y_i = \frac{1}{kM}\left[\sum_{i=1}^{(k-1)M} Y_i + \sum_{i=(k-1)M+1}^{kM} Y_i\right] \\ &= \frac{(k-1)M}{kM}\frac{1}{(k-1)M}\sum_{i=1}^{(k-1)M} Y_i + \frac{M}{kM}\frac{1}{M}\sum_{i=(k-1)M+1}^{kM} Y_i \\ &= \frac{k-1}{k}E_{1,k-1} + \frac{1}{k}E_k \end{aligned} \quad (10)$$

Here, E_k denotes the sample mean calculated from the k th block. Similarly, denoting $V_{1,k}$ as the sample variance calculated from the first k blocks, then

$$\begin{aligned} V_{1,k} &= \frac{1}{kM-1}\sum_{i=1}^{kM}\left(Y_i - E_{1,k}\right)^2 \\ &= \frac{1}{kM-1}\sum_{i=1}^{(k-1)M}\left(Y_i - E_{1,k}\right)^2 + \frac{1}{kM-1}\sum_{i=(k-1)M+1}^{kM}\left(Y_i - E_{1,k}\right)^2 \\ &= \frac{1}{kM-1}\sum_{i=1}^{(k-1)M}\left[\left(Y_i - E_{1,k-1}\right) + \left(\frac{1}{k}E_{1,k-1} - \frac{1}{k}E_k\right)\right]^2 \\ &\quad + \frac{1}{kM-1}\sum_{i=(k-1)M+1}^{kM}\left[\left(Y_i - E_k\right) + \left(\frac{k-1}{k}E_k - \frac{k-1}{k}E_{1,k-1}\right)\right]^2 \end{aligned} \quad (11)$$

Expanding the square-items in above series, after some manipulation, we can obtain

$$V_{1,k} = \frac{(k-1)M-1}{kM-1}V_{1,k-1} + \frac{M-1}{kM-1}V_k + \frac{(k-1)M}{k(kM-1)}\left(E_{1,k-1} - E_k\right)^2, \quad (12)$$

where V_k denotes the sample variance calculated from the k th block. Generally, $kM \gg 1$ and $(k-1)M \gg 1$, so (12) can reduces to

$$V_{1,k} = \frac{k-1}{k}V_{1,k-1} + \frac{M-1}{kM}V_k + \frac{k-1}{k^2}\left(E_{1,k-1} - E_k\right)^2. \quad (13)$$

Substituting (8) and (9) into (10) and (13), respectively, we can obtain the iterative $\log P\alpha S$ estimator by

$$\begin{aligned} \frac{\log\hat{\gamma}(k)}{\hat{\alpha}(k)} &= \frac{k-1}{k}\left\{C_e\left[\frac{1}{\hat{\alpha}(k-1)} - 1\right] + \frac{\log\hat{\gamma}(k-1)}{\hat{\alpha}(k-1)} + \frac{\log\left(\sec^2\left(\pi\hat{\alpha}(k-1)/2\right)\right)}{2\hat{\alpha}(k-1)}\right\} \\ &\quad + \frac{E_k}{k} - C_e\left[\frac{1}{\hat{\alpha}(k)} - 1\right] - \frac{\log\left(\sec^2\left(\pi\hat{\alpha}(k)/2\right)\right)}{2\hat{\alpha}(k)} \end{aligned} \quad (14)$$

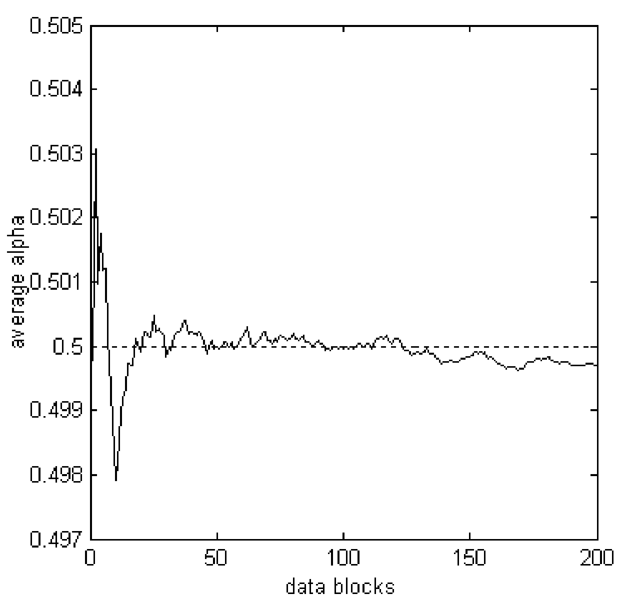
and

$$\begin{aligned} \frac{\pi^2\left[1-\hat{\alpha}^2(k)\right]}{6\hat{\alpha}^2(k)} &= \frac{k-1}{k}\frac{\pi^2\left[1-\hat{\alpha}^2(k-1)\right]}{6\hat{\alpha}^2(k-1)} + \frac{M-1}{kM}V_k \\ &\quad + \frac{k-1}{k^2}\left\{C_e\left[\frac{1}{\hat{\alpha}(k-1)} - 1\right] + \frac{\log\hat{\gamma}(k-1)}{\hat{\alpha}(k-1)} + \frac{\log\left(\sec^2\left(\pi\hat{\alpha}(k-1)/2\right)\right)}{2\hat{\alpha}(k-1)} - E_k\right\}^2 \end{aligned}. \quad (15)$$

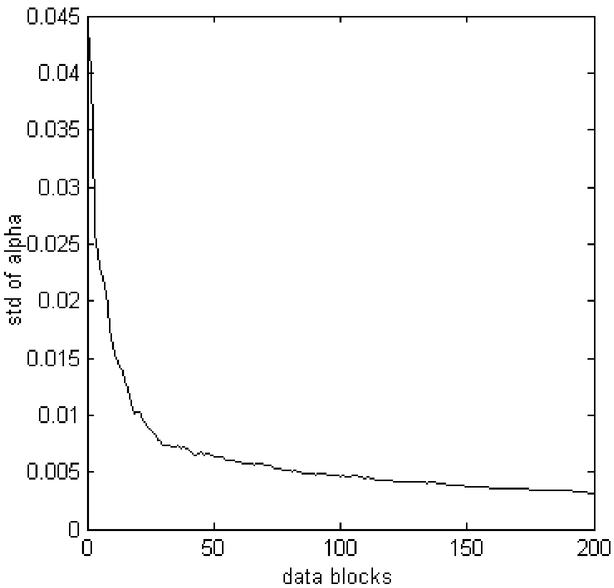
Firstly, $\hat{\alpha}(k)$ can be obtained from (15). Then, $\hat{\gamma}(k)$ can be obtained from (14). Table 3 shows the Monte Carlo simulation results based on the iterative $\log P\alpha S$ estimator with 100 independent realizations. We can see that the number of data block is a key factor determining the performance of the iterative estimator, which is illustrated in Fig. 1. Obviously, better performance is achieved with bigger block size. Compared to the $\log P\alpha S$ estimator presented above, the proposed iterative one gets similar estimated results in the same condition of all samples (e.g., 5000 samples), but much fewer samples are required in each step computation. This demonstrates the high efficiency of the iterative $\log P\alpha S$ estimator.

Table 3 Performance of the iterative $\log P\alpha S$ estimator (true $\alpha = 0.5$, $\gamma = 1$)

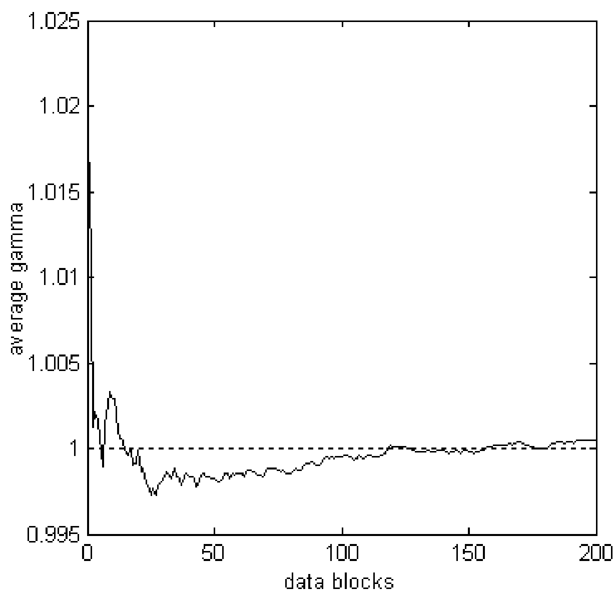
Samples of each Block	$B = 50$		$B = 100$		$B = 200$	
	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\gamma}$
50	0.4997 (0.0086)	1.0014 (0.0177)	0.5007 (0.0071)	0.9970 (0.0156)	0.5007 (0.0052)	0.9989 (0.0107)
100	0.5006 (0.0063)	0.9995 (0.0150)	0.5003 (0.0042)	1.0000 (0.0085)	0.5000 (0.0034)	1.0000 (0.0070)
200	0.5005 (0.0049)	0.9990 (0.0108)	0.5000 (0.0035)	0.9993 (0.0077)	0.5001 (0.0026)	1.0006 (0.0055)



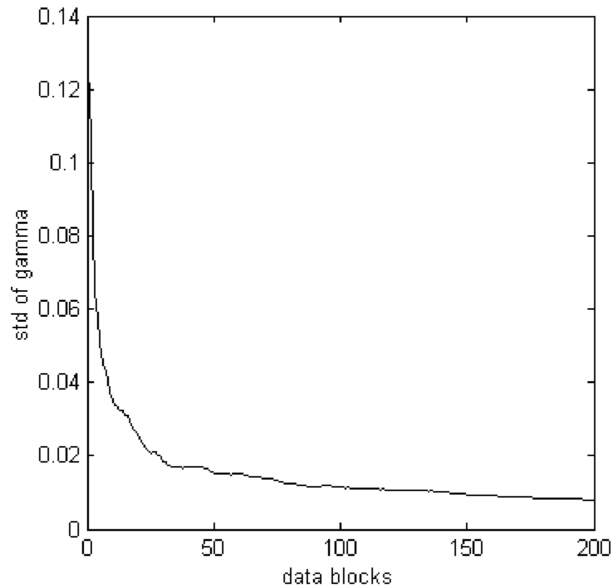
(a) Average of $\hat{\alpha}$ compared with the true value $\alpha = 0.5$



(b) Standard deviation of $\hat{\alpha}$



(c) Average of $\hat{\gamma}$ compared with the true value $\gamma = 1$



(d) Standard deviation of $\hat{\gamma}$

Fig. 1. Performance of the iterative estimator with respect to the number of data blocks

5. CONCLUSIONS

In this paper, we use the negative-order moments to estimate parameters of $P\alpha S$ distribution and present three methods. First, we present ratio estimator based on the ratio of negative-order moments and we can see that the estimator proposed by Pierce is just a special case of (3). It can be shown that the ratio estimator results in bad performance when the order p is chosen inappropriately. Second, we present the $\log P\alpha S$ estimator with explicit closed form. This estimator is completely determined by samples and it is robust compared to the ratio estimator. Last, we propose the iterative $\log P\alpha S$ estimator that achieves much better performance only by fewer samples in each step computation. Monte Carlo simulation results demonstrate the high efficiency of the proposed iterative estimator.

6. REFERENCES

[1] C. L. Nikias and M. Shao, *Signal Processing with Alpha-Stable Distribution and Application*, Wiley, New York, 1995.

[2] R. D. Pierce, "Application of the positive alpha-stable distribution," in *Proceedings of the IEEE Signal Processing Workshop on Higher-Order Statistics*, 1997, pp. 420-424.

[3] W. J. Szajnowski and J. B. Wynne, "Simulation of dependent samples of symmetric alpha-stable clutter," *IEEE Trans. Signal Processing Letters*, vol. 8, no. 5, pp. 151-152, May 2001.