A SLICE BASED 3-D SCHUR COHN STABILITY CRITERION

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ABSTRACT

In this paper a new necessary and sufficient criterion of BIBO stability of 3-D filters is given, based on a recent n-D extension of the Schur coefficients associated to a polynomial. The criterion relies upon the slice functions mechanism, and is given as an extended 3-D Jury Table. Several examples are provided to show how the new procedure reduces the complexity of stability tests that are using the Anderson Jury criterion of stability.

Index Terms— Multidimensional systems, Stability criteria

1. INTRODUCTION

A 3-D rational digital filter defined by its $z-$transform:

$$T(z_1, z_2, z_3) = \frac{A(z_1, z_2, z_3)}{P(z_1, z_2, z_3)}$$

is called BIBO stable if the output of a bounded input is bounded. Assume that T does not have any non essential singularities of the second kind. It is known that in this case the BIBO stability is assured if all the zeros of the denominator P are outside the closed unit polydisk $\overline{\mathbb{D}}^3$.

Testing this condition on the zeroes of a polynomial P in more than one variable is a difficult problem, as multivariable polynomials may have no root factorization. The two dimensional case was considered in [3], [5], [8], [9]. Also a sufficient but not necessary condition of stability was obtained in [1] by the means of the 2-D reflection coefficients. Several multidimensional stability tests were developed in [2], [4], [6], [7], [13].

Recently a new multidimensional stability criterion has been established by the authors in [12], troughout an n-D extension of the 1-D Schur coefficients, called "functional Schur coefficients". This extension is based on the properties of the so called "slice functions" [11], a powerful tool in the multidimensional theory of analytic functions.

The purpose of this paper is to present an algorithm for checking the stability condition of a 3-D filter, based on the slice functions mechanism and the functional Schur coefficients associated to a three variable polynomial $P(z_1, z_2, z_3)$. The paper is organized as follows: in Section 2 the functional Schur coefficients are defined and the 3-D stability criterion is given, under the form of an extended 3-D Jury Table. The end of the section 2 contains a reduction of conditions in the proposed criterion to some simpler conditions involving positivity of real polynomials. Section 3 contains a comparative analysis of the proposed algorithm with stability tests that use the Anderson Jury criterion of stability. A couple of examples are given to illustrate how the proposed algorithm reduces the computational complexity.

2. 3-D SLICED-BASED SCHUR-COHN CRITERION

Let $P(z_1, z_2, z_3)$ be a polynomial in three variables of degree n ($n \leq n_1 + n_2 + n_3$):

$$P(z_1, z_2, z_3) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} p_{ijk} z_1^i z_2^j z_3^k \qquad (1)$$

Denote by $\mathbb{D}$ the set $\{z \in \mathbb{C} : |z| < 1\}$, and by $\mathbb{T}$ the set $\{z \in \mathbb{C} : |z| = 1\}$.

For each point $v = (v_1, v_2)$ on the polytorus $\mathbb{T}^2$ let D_v be the one-dimensional disk that "slices" $\mathbb{D}^3$ through the origin and through $(1, v_1, v_2)$:

$$D_v = \{\lambda(1, v_1, v_2) : \lambda \in \mathbb{D}\} \qquad (v = (v_1, v_2) \in \mathbb{T}^2) \quad (2)$$

Consider the restriction of the three variables polynomial P to the one-dimensional disk D_v, which can be regarded as a one variable polynomial:

$$P_v(\lambda) = P(\lambda, \lambda v_1, \lambda v_2) \qquad (\lambda \in \mathbb{D}). \qquad (3)$$

P_v is called *the slice of P through* $(1, v_1, v_2)$ [11].
We can now give the 3-D criterion of stability.

3-D Schur-Cohn type criterion of stability
The following statements are equivalent:
A) P has no zeros in $\overline{\mathbb{D}}^3$:

$$P(z_1, z_2, z_3) \neq 0 \quad \cap_{i=1}^3 |z_i| \leq 1 \qquad (4)$$

B) P_v has no zeros in $\overline{\mathbb{D}}$, for all $v = (v_1, v_2) \in \mathbb{T}^2$:

$$P_v(\lambda) = P(\lambda, \lambda v_1, \lambda v_2) \neq 0, \quad |\lambda| \leq 1, \cap_{i=1}^2 |v_i| = 1 \quad (5)$$

The equivalence between A) and B) was obtained in [12] using the *functional Schur coefficients*. The functional Schur coefficients are Schur coefficients associated to the one variable polynomial $P_v(\lambda)$, parametered by $v \in \mathbb{T}^2$. In order to construct a Schur-type sequence associated to $P(z_1, z_2, z_3)$, consider P_v the slice of P and the polynomial transpose of $P_v^T(\lambda) = \lambda^n \overline{P}(1/\overline{\lambda})$. Define (P_v^k, Q_v^k) by putting $P_v^0 = P_v^T$, $Q_v^0 = P_v$ and let

$$P_v^k(\lambda) = \frac{1}{\lambda}(P_v^{k-1}(\lambda) - \gamma_{k-1} Q_v^{k-1}(\lambda)) \qquad (k \geq 1)$$

$$Q_v^k(\lambda) = Q_v^{k-1}(\lambda) - \overline{\gamma}_{k-1} P_v^{k-1}(\lambda)) \qquad (k \geq 1)$$

Then the functional Schur coefficients of $P(z_1, z_2, z_3)$ are:

$$\gamma_k(v_1, v_2) = P_v^k(0)/Q_v^k(0) \qquad (k \geq 0) \tag{6}$$

The condition B) in (5) holds if and only if all functional Schur coefficients are no greater than 1 in absolute value, for all $v \in \mathbb{T}^2$ (see [12]):

$$C)\ |\gamma_k(v_1, v_2)| < 1 \quad k = 0, ..., n-1, \quad (v_1, v_2) \in \mathbb{T}^2 \tag{7}$$

Moreover, the last condition is equivalent with the nonnegativity of the Schur-Cohn matrix associated to $P_v(\lambda)$. This is a consequence of the connection between the Schur parameters and principal leading minors of the Schur Cohn matrix. In the one dimensional case one of the methods used to compute the principal leading minors of the Schur Cohn matrix is the Jury Table [9]. We give in the following a 3-D extension of the Jury table.

First, it is easy to see from (1) that the slice of P through v is:

$$P_v(\lambda) = \sum_{l=0}^{n} c_l(v)\lambda^l \qquad (\lambda \in \mathbb{D}), \tag{8}$$

where the coefficients c_l are two variable polynomials given by:

$$c_l(v_1, v_2) = \sum_{i+j+k=l} p_{ijk} v_1^j v_2^k \qquad (0 \leq l \leq n). \tag{9}$$

Extended 3-D Jury Table

1. For $i = 0, \ldots, n$ let $b_i^0(v_1, v_2) = c_i(v_1, v_2)$.
2. For $k = 1, \ldots n$ let m be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the k^{th} row of the table with the entries b_i^k for $i = 0, \ldots, n-k-1$ defined by:

$$b_i^k(v) = \left(\frac{1}{b_0^{k-1}(v)}\right)^m \begin{vmatrix} b_0^{k-1}(v) & b_{n-k+1-i}^{k-1}(v) \\ \overline{b_{n-k+1}^{k-1}(v)} & \overline{b_i^{k-1}(v)} \end{vmatrix} \tag{10}$$

3. $P(z) \neq 0$ for all $|z| \leq 1$ if and only if $b_0^k(v_1, v_2) > 0$ for all $k = 1, \ldots, n$ and $(v_1, v_2) \in \mathbb{T}^2$.

Finally, we obtain the following Schur-Cohn type criterion of stability:

3-D Schur-Cohn criterion of stability

Let $P(z_1, z_2, z_3)$ be a three variable polynomial, consider $\gamma_k(v_1, v_2)$ for $k = 1, \ldots, n$ the functional Schur coefficients associated to P, and $b_0^k(v_1, v_2)$ the first entries of each k-th row of the 3-D Jury Table.

The following assertions are equivalent:

$$A)\ P(z_1, z_2, z_3) \neq 0 \quad \cap_{i=1}^3 |z_i| \leq 1; \tag{11}$$

$$B)\ P(\lambda, \lambda v_1, \lambda v_2) \neq 0,\ |\lambda| \leq 1, |v_1| = |v_2| = 1; \tag{12}$$

$$C)\ |\gamma_k(v_1, v_2)| < 1, \quad k = 1, \ldots, n,\ |v_1| = |v_2| = 1; \tag{13}$$

$$D)\ b_0^k(v_1, v_2) > 0, \quad k = 1, \ldots, n,\ |v_1| = |v_2| = 1. \tag{14}$$

In conclusion, checking a polynomial stability comes down to checking a set of positivity conditions of type:

$$\Delta(v_1, v_2) = \sum_{i=-p}^{p} \sum_{j=-q}^{q} \delta_{ik} v_1^i v_2^j > 0 \tag{15}$$

when $|v_1| = |v_2| = 1$. $\Delta(v_1, v_2)$ is a trigonometric polynomial, real valued on the bitorus, with $\delta_{-i,-k} = \overline{\delta_{ik}}$. In [4] and [7] procedures for testing this type of condition are given.

In the last years significant advances were made in the domain of checking positivity of real polynomials, and some very useful tools for optimization as SeDuMi [14], SOSToolbox [10] are available. In order to use them when checking conditions of type (15), we propose to use the following change of variable:

$$v_1 = \frac{t_1 - i}{t_1 + i} \quad \text{and} \quad v_2 = \frac{t_2 - i}{t_2 + i} \tag{16}$$

Then with a simple calculation one can see that

$$\Delta(v_1, v_2) = \sum_{i=-p}^{p} \sum_{j=-q}^{q} \delta_{ik} \left(\frac{t_1 - i}{t_1 + i}\right)^i \left(\frac{t_2 - i}{t_2 + i}\right)^j$$
$$= \frac{1}{(t_1^2 + 1)^p (t_2^2 + 1)^q} Q(t_1, t_2). \tag{17}$$

The sign of $\Delta(v_1, v_2)$, when $|v_1| = |v_2| = 1$ is the same as the sign of $Q(t_1, t_2)$ when $t_1, t_2 \in \mathbb{R}$. Moreover, $Q(t_1, t_2)$ is a real polynomial, and testing the positivity of Q can be done for instance with SOSToolbox, as shown in the next example:

Example: Let $\Delta(v_1, v_2) = 16 - |1 + v_1^2 + v_2^2|^2$. Using (17), we have:

$Q(t_1, t_2) = 7t_1^4 t_2^4 + 46t_1^4 t_2^2 + 7t_1^4 - 32t_1^3 t_2^3 + 32t_1^3 t_2 + 46t_1^2 t_2^4 + 28t_1^2 t_2^2 + 46t_1^2 + 32t_1 t_2^3 - 32t_1 t_2 + 7t_2^4 + 46t_2^2 + 7$.

Using SOSToolbox we get that Q is a sum of squares, so it is positive.

3. EXAMPLES

It is natural to compare the new Schur-Cohn criterion (5) with the Anderson Jury criterion [2], which we formulate in the 3-D case in the following.

Anderson Jury criterion

The condition (4) holds if and only if all the following 3 conditions are satisfied:

$$P(z_1,0,0) \neq 0 \quad |z_1| \leq 1 \tag{18}$$
$$P(z_1,z_2,0) \neq 0 \quad |z_1| = 1, |z_2| \leq 1 \tag{19}$$
$$P(z_1,z_2,z_3) \neq 0 \quad |z_1| = |z_2| = 1, |z_3| \leq 1 \tag{20}$$

Note that when using the Jury Anderson criterion verifying condition (4) implies checking the three conditions (18),(19) and (20). The first condition is a classical one dimensional stability problem. There are several algorithms developed for testing the second condition (see [8], [9] [3],[5]).

The condition (20) is similar with the condition (5). Mainly, verifying this condition involves testing the positivity of the n_3 leading principal minors of the Schur-Cohn matrix associated to $P_{z_1,z_2}(z_3) = \sum_{k=0}^{n_1} a_k(z_1,z_2) z_3^k$, where

$$a_k(z_1,z_2) = \sum_{i=0}^{n_2} \sum_{j=0}^{n_3} p_{ijk} z_1^i z_2^j$$

As the choice of z_3 is arbitrary, it is convenient to choose the variable z_i for which $n_i = \min\{n_1,n_2,n_3\}$. Therefore, in the following we will assume that $n_3 = \min\{n_1,n_2,n_3\}$.

When using the proposed 3-D Schur-Cohn criterion only (5) needs to be checked. The first example provided bellow illustrate how using (5) significantly reduces the computational complexity of the stability chekcking of P. We shall inspect the stability of P first using the Anderson-Jury criterion and then the proposed 3-D Schur-Cohn criterion.

Example 1.

Let $P(z_1,z_2,z_3) = 4 + z_1 + z_2 + z_3 + z_1^2 + z_2^2 + z_3^2$.

I. Using the Jury-Anderson criterion the polynomial P is devoid of zero inside the closed unit polydisk iff (18), (19) and (20) are satisfied:

$$P(z_1,0,0) = 4 + z_1 + z_1^2 \neq 0 \quad |z_1| \leq 1 \tag{21}$$
$$P(z_1,z_2,0) = 4 + z_1 + z_2 + z_1^2 + z_2^2 \neq 0 \; |z_1| = 1, |z_2| \leq 1 \tag{22}$$
$$P(z_1,z_2,z_3) = 4 + z_1 + z_2 + z_3 + z_1^2 + z_2^2 + z_3^2 \neq 0 \quad |z_1| = |z_2| = 1, |z_3| \leq 1 \tag{23}$$

In order to verify conditions (21), (22) and (23) the following steps are necessary:

Step 1. Checking (21) can be done for instance using the one-dimensional Schur-Cohn algorithm.

Step 2. The condition (22) is a two-dimensional problem of stability. Using for example the 2-D Jury Table one needs to check if for $|z_1| = 1$ the following are satisfied:

$$17 + 5(z_1 + \overline{z}_1) + 4(z_1^2 + \overline{z}_1^2) > 0 \tag{24}$$
$$180 + 103(z_1 + \overline{z}_1) + 79(z_1^2 + \overline{z}_1^2) + 20(z_1^3 + \overline{z}_1^3) > 0 \tag{25}$$

Step 3. Condition (23) holds iff for all $|z_1| = |z_2| = 1$ we have $d_0^1(z_1,z_2) > 0$ and $d_0^2(z_1,z_2) > 0$, where d_0^1 and d_0^2 are the entries on the first column from the 2-D Jury-Table:

$$d_0^1(z_1,z_2) = |4 + z_1 + z_1^2 + z_2^2 + z_2|^2 - 1 > 0 \tag{26}$$
$$d_1^1(z_1,z_2) = 3 + z_1 + z_2 + z_1^2 + z_2^2$$
$$d_0^2(z_1,z_2) = |d_1^1(z_1,z_2)|^2 - |d_0^1(z_1,z_2)|^2 > 0 \tag{27}$$

II. Using the proposed 3-D Schur-Cohn criterion (5) the polynomial P is devoid of zero inside the closed unit polydisk iff for all $|v_1| = |v_2| = 1$ and $|\lambda| \leq 1$ the following holds:

$$P_v(\lambda) = 4 + \lambda(1 + v_1 + v_2) + \lambda^2(1 + v_1^2 + v_2^2) \neq 0 \tag{28}$$

This condition is similar with condition (23). It is equivalent with the following two conditions of positivity:

$$b_0^1(v_1,v_2) = 16 - |1 + v_1^2 + v_2^2|^2 > 0 \tag{29}$$
$$b_1^1(v_1,v_2) = 4 + 4\overline{v}_2 + 4\overline{v}_1 - (1 + v_1 + v_2)(1 + v_1^2 + v_2^2)$$
$$b_0^2(z_1,z_2) = |b_1^1(z_1,z_2)|^2 - |b_0^1(z_1,z_2)|^2 > 0 \tag{30}$$
$$\text{for all} \quad |v_1| = |v_2| = 1$$

where b_0^1 and b_0^2 are the entries of the first column of the 3-D Jury Table, computed with (10).

Note that the conditions (26) and (27) are similar with condition (29) and (30). When using the Anderson-Jury criterion, the stability of P is equivalent with three conditions that need to be tested. In our example, the set of three conditions is equivalent with a set of five conditions of positivity: step 1 and conditions (24) to (27). Using the proposed 3-D Schur-Cohn criterion, only one condition is sufficient and necessary for the stability of P, which is equivalent in our example with a set of two conditions (29) and (30). We conclude that the overall computational cost is significantly reduced, as two conditions in stead of five are necessary.

Let us consider another example that further illustrate the differences of implementation between the Anderson-Jury criterion and the proposed 3-D Schur-Cohn criterion.

Example 2

Let $P(z_1,z_2,z_3) = z_1^2 z_2 + z_1 - z_2 + z_3 + 4$.

I. Using the Jury-Anderson criterion the polynomial P is devoid of zero inside the closed unit polydisk iff (18), (19) and (20) are satisfied:

$$P(z_1,0,0) = z_1 + 4 \neq 0 \quad |z_1| \leq 1 \tag{31}$$
$$P(z_1,z_2,0) = z_1^2 z_2 + z_1 - z_2 + 4 \neq 0 \quad |z_1| = 1, |z_2| \leq 1 \tag{32}$$
$$P(z_1,z_2,z_3) = z_1^2 z_2 + z_1 - z_2 + z_3 + 4 \neq 0 \quad |z_1| = |z_2| = 1, |z_3| \leq 1 \tag{33}$$

Step 1. Checking (31) is direct.

Step 2. (32) is equivalent with:

$$|z_1 + 4|^2 - |z_1^2 - 1|^2 > 0, \quad \text{for} \quad |z_1| = 1 \qquad (34)$$

Step 3. Finally, condition (33) is equivalent with

$$|z_1^2 z_2 + z_1 - z_2 + 4|^2 - 1 > 0 \qquad (35)$$

II. Using the proposed 3-D Schur-Cohn criterion (5), the polynomial P is devoid of zeros in the closed unit polydisk iff:

$$P_v(\lambda) = \lambda^3 v_1 + \lambda(1 - v_1 + v_2) + 4 \neq 0 \qquad (36)$$

when $|v_1| = |v_2| = 1$ and $|\lambda| \leq 1$.

Using the Jury Table as described in (10) for $P_v(\lambda)$ we obtain that the condition (36) is equivalent with:

$$b_0^1(v_1, v_2) = 16 - |v_1|^2 > 0 \qquad (37)$$

$$b_0^2(v_1, v_2) = 225 - |1 - \bar{v}_1 - v_2\bar{v}_1|^2 > 0 \qquad (38)$$

$$b_0^3(v_1, v_2) = |222 + \bar{v}_1 v_2 + v_1\bar{v}_2 - v_2 - \bar{v}_2 + v_1 + \bar{v}_1|^2 - $$
$$- |52 - 8v_2 - 56\bar{v}_1 + 60\bar{v}_2 + 8\bar{v}_1 v_2 + 4v_1 + 4\bar{v}_1 v_2^2|^2 > 0 \qquad (39)$$

Remark that the degree n of polynomial P is equal 3. Therefore when using the 3-D Schur-Cohn criterion three tests of positivity are needed: (37), (38) and (39). The Anderson Jury criterion also involves three conditions of positivity: step 1, (34) and (35). Even if the condition of stability is simpler in our criterion than the one in Anderson-Jury criterion, the computational complexity for this example is bigger when using the 3-D Schur-Cohn criterion. This is due to the significant difference between the global polynomial degree $n = 3$ and the degree $n_3 = 1$ of P in z_3.

In conclusion, checking the stability of a polynomial P of degree n with the proposed 3-D Schur-Cohn criterion involves the test of n positivity conditions of two-variables polynomials. The use of the Anderson Jury criterion leads to the test of $n_1 + n_2 + n_3$ conditions: n_1 scalars, n_2 positivity conditions of one-variable polynomials and n_3 positivity conditions of two-variables polynomials. Whenever the computational cost of testing the n conditions is lower than the cost of testing the $n_1 + n_2 + n_3$ conditions in the Anderson Jury criterion the proposed method should be used. This is always the case for instance if the polynomial degree n is equal to the minimum of degrees of P on each variable.

4. CONCLUSION

In the paper a new necessary and sufficient condition of BIBO stability for 3-D systems is proposed. The criterion is based on the slice functions mechanism and a recent n-D extension of the 1-D Schur coefficients associated to a polynomial. The criterion is stated in the form of an extended 3-D Jury Table, but can also be used in conjunction with other known methods, as for instance the polynomial array techniques proposed in [7]. In this sense, a reduction to a real polynomial positivity condition is also presented.

As shown in the last section, the proposed criterion implies in general a lower computational cost than Anderson-Jury type criteria.

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