CONVERGENCE OF ADAPTIVE ESTIMATORS OF TIME-VARYING LINEAR SYSTEMS USING BASIS FUNCTIONS: CONTINUOUS TIME RESULTS

Mohammad Abu-Naser and Geoffrey A. Williamson

Department of Electrical and Computer Engineering
Illinois Institute of Technology
Chicago, IL U.S.A.
E-mail: {abunmoh}{williamson}@iit.edu

ABSTRACT

The convergence properties of adaptive filtering algorithms are investigated in situations where the optimal filter is modeled as a time-varying linear system whose parameters are expanded over basis functions. This type of model is one approach when parameters cannot be considered as slowly varying, and is appropriate for modeling certain mobile radio channels and in the identification of the dynamics of vascular autoregulation in kidneys. Appropriate adaptive algorithms are developed in a continuous-time setting, and the local convergence of these algorithms is studied. Conditions for convergence are shown to include an excitation condition on the algorithm regressor and a passivity condition on an algorithm operator. The excitation conditions are interpreted in terms of system signals and the parameter basis functions using previously established results in the discrete-time case. A test for the passivity condition is developed whose application is presented via an illustrative example.

Index Terms— Identification, parameter estimation, adaptive estimation, time-varying systems, time-varying filters

1. INTRODUCTION

In certain applications of adaptive system estimation, time variations in the system being identified are too rapid for an assumption of slowly varying parameters to be valid, as is usually required for appropriate behavior of adaptive filters [1]. These situations require a more explicit representation of the time-varying system model. The idea of a basis function expansion can be utilized to adapt to systems with fast time variations as discussed in [2]. With this approach, the coefficients of moving average (MA) or auto-regressive moving average (ARMA) models are expressed as a linear combination of basis functions. Doing so converts the time-varying parameter estimation to a time-invariant parameter estimation, which enables us to import all the theory of time-invariant parameter estimation. This approach has been used to model rapidly fading mobile radio channels [3] as well as the autoregulation dynamics in the microvasculature of kidneys [4].

Good results in the application of these adaptive systems requires an understanding of the conditions under which the algorithms converge. In [5], we studied convergence properties of adaptive algorithms applied to time-varying parameter estimation using basis functions in the discrete-time setting. We showed that local convergence of the algorithms required a persistent excitation (PE) condition on a system regressor and a passivity condition on a time-varying operator that arose in the algorithm. Work in [5] decomposed the PE condition into two parts, one depending on system signals and the other on the parameter basis functions, and illustrated their interactions. Here we will extend the PE results to the case when the model and the adaptive algorithms are in continuous time, and we also develop results for the satisfaction of the operator condition.

The analysis proceeds by first exposing the relationship that the prediction error has with the estimator regressor and parameter error. This prediction error motivates a class of adaptive algorithms described in [6]. As in the discrete-time case, sufficient conditions for the local exponential convergence of these algorithms include a PE condition on the regressor and a strict passivity condition on a particular operator. We extend the PE conditions that were developed in [5] for the continuous-time adaptive algorithms. Also, sufficient conditions for the strict passivity of the operator will be established. These conditions depend on having the ‘frozen” operators at each point in time be strictly passive of a given degree and having the time rate of change of the operator satisfy a bound determined by that degree. We present an example that illustrates the application of this result.

2. PREDICTION ERROR STRUCTURE

Consider the continuous time-varying ARMA model described by

$$y^{(n)}(t) = \sum_{i=1}^{n} a_i(t) y^{(n-i)}(t) + \sum_{j=0}^{m} b_j(t) u^{(m-j)}(t) \qquad (1)$$

where $n > m$ and $^{(i)}$ denotes the ith derivative. This model is the continuous-time version of the discrete-time time-varying model described in [5]. Let $\{f_\ell(t), \ell = 1, \dots, L\}$ be a set of linearly independent functions, called the basis functions. Assume that each time-varying parameter can be expressed as a linear combination of these basis functions, so that

$$a_i(t) = \sum_{\ell=1}^{L} \alpha_{i\ell} f_\ell(t), \quad b_j(t) = \sum_{\ell=1}^{L} \beta_{j\ell} f_\ell(t). \qquad (2)$$

Substituting (2) into (1), we obtain

$$y^{(n)}(t) = \sum_{i=1}^{n}\sum_{\ell=1}^{L} \alpha_{i\ell} f_\ell(t) y^{(n-i)}(t) + \sum_{j=0}^{m}\sum_{\ell=1}^{L} \beta_{j\ell} f_\ell(t) u^{(m-j)}(t). \qquad (3)$$

Notice that this converts the system model from one with time-varying parameters to one whose parameters are time-invariant, albeit the number of parameters has increased by a factor of L.

Using estimates $\hat{\alpha}_{i\ell}(t)$ and $\hat{\beta}_{j\ell}(t)$ of the basis function expansion coefficients, we obtain an estimated output $\hat{y}(t)$

$$\hat{y}^{(n)}(t) = \sum_{i=1}^{n}\sum_{\ell=1}^{L}\hat{\alpha}_{i\ell}(t)f_\ell(t)\overline{y}^{(n-i)}(t) + \sum_{j=0}^{m}\sum_{\ell=1}^{L}\hat{\beta}_{j\ell}(t)f_\ell(t)u^{(m-j)}(t). \tag{4}$$

When $\overline{y}(k-i) = y(k-i)$ in (4) we have an equation error estimator, and when $\overline{y}(k-i) = \hat{y}(k-i)$ we have an output error estimator.

Parallel to the development in [6] of a regression-based expression for time-invariant system dynamics, we may also express (3) and (4) as linear regressions

$$y(t) = \boldsymbol{\psi}^T(t)\boldsymbol{\theta},$$
$$\hat{y}(t) = \overline{\boldsymbol{\psi}}^T(t)\hat{\boldsymbol{\theta}}(t)$$

where

$$\begin{aligned}
\boldsymbol{\psi}(t) =\ & [f_1(t)P_{n-m}u(t) \quad \cdots \quad f_L(t)P_nu(t) \\
& f_1(t)P_1y(t) \quad \cdots \quad f_L(t)P_ny(t)]^T \\
\boldsymbol{\theta} =\ & [\beta_{01} \quad \cdots \quad \beta_{mL} \quad \alpha_{11}+\gamma_{11} \quad \cdots \quad \alpha_{nL}+\gamma_{nL}]^T \\
\overline{\boldsymbol{\psi}}(t) =\ & [f_1(t)P_{n-m}u(t) \quad \cdots \quad f_L(t)P_nu(t) \\
& f_1(t)P_1\overline{y}(t) \quad \cdots \quad f_L(t)P_n\overline{y}(t)]^T \\
\hat{\boldsymbol{\theta}}(t) =\ & [\hat{\beta}_{01} \quad \cdots \quad \hat{\beta}_{mL} \quad \hat{\alpha}_{11}+\gamma_{11} \quad \cdots \quad \hat{\alpha}_{nL}+\gamma_{nL}]^T
\end{aligned}$$

and $P_i(D,t), i = 1, \cdots, n$ is a family of stable operators of the form

$$P_i(D,t) = \frac{D^{n-i}}{C(D,t)} = \frac{D^{n-i}}{D^n + c_1(t)D^{n-1} + \cdots + c_n(t)}. \tag{5}$$

In (5), D is the differential operator and

$$c_i(t) = \sum_{\ell=1}^{L}\gamma_{i\ell}f_\ell(t) \tag{6}$$

are user-chosen variables such that (5) is stable. As will be seen, the choice of $C(D,t)$ enters into the convergence requirements.

We also let $\tilde{\boldsymbol{\theta}}(t) = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(t)$. Then in the same fashion as in [5], one can develop an expression for the prediction error as a filtered version of the inner product between the regressor vector and the parameter error vector.

Proposition 1. *The prediction error $e(t) = y(t) - \hat{y}(t)$ is given by*

$$e(t) = \boldsymbol{\psi}^T(t)\tilde{\boldsymbol{\theta}}(t) \tag{7}$$

in the equation error case, and it is given by

$$e(t) = H(D,t)\left[\hat{\boldsymbol{\psi}}^T(t)\tilde{\boldsymbol{\theta}}(t)\right] \tag{8}$$

in the output error case. In (8)*, the operator $H(D,t)$ is*

$$H(D,t) = \frac{C(D,t)}{D^n - a_1(t)D^{n-1} - \cdots - a_n(t)}. \tag{9}$$

Proof. In the equation error case, (7) follows from $\overline{\boldsymbol{\psi}}(t) = \boldsymbol{\psi}(t)$. In the output error case, we can show that

$$\begin{aligned}
e(t) &= y(t) - \hat{y}(t) \\
&= \boldsymbol{\psi}^T(t)\boldsymbol{\theta} - \hat{\boldsymbol{\psi}}^T(t)\hat{\boldsymbol{\theta}}(t) \\
&= \boldsymbol{\psi}^T(t)\boldsymbol{\theta} - \hat{\boldsymbol{\psi}}^T(t)[\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}(t)] \\
&= [\boldsymbol{\psi}^T(t) - \hat{\boldsymbol{\psi}}^T(t)]\boldsymbol{\theta} + \hat{\boldsymbol{\psi}}^T(t)\tilde{\boldsymbol{\theta}}(t).
\end{aligned} \tag{10}$$

Noting that

$$\boldsymbol{\psi}(t) - \hat{\boldsymbol{\psi}}(t) = [\mathbf{0} \quad \cdots \quad \mathbf{0} \quad \boldsymbol{v}_1^T(t) \quad \cdots \quad \boldsymbol{v}_n^T(t)]^T \tag{11}$$

where

$$\boldsymbol{v}_i(t) = [f_1(t)P_i(D,t)e(t) \quad \cdots \quad f_L(t)P_i(D,t)e(t)]^T, \tag{12}$$

we can write the first term on the right hand side of (10) as

$$\begin{aligned}
[\boldsymbol{\psi}^T(t) - \hat{\boldsymbol{\psi}}^T(t)]\boldsymbol{\theta} &= \sum_{i=1}^{n}\sum_{\ell=1}^{L}[\alpha_{i\ell}(t) + \gamma_{i\ell}(t)]f_\ell(t)P_i(D,t)e(t) \\
&= \sum_{i=1}^{n}[a_i(t) + c_i(t)]P_i(D,t)e(t).
\end{aligned} \tag{13}$$

Substituting (13) into (10) and rearranging yields

$$e(t) = \frac{\sum_{i=1}^{n}[a_i(t) + c_i(t)]D^{n-i}}{C(D,t)}e(t) + \hat{\boldsymbol{\psi}}^T(t)\tilde{\boldsymbol{\theta}}(t). \tag{14}$$

So

$$e(t) = \frac{C(D,t)}{D^n - a_1(t)D^{n-1} - \cdots - a_n(t)}\left[\hat{\boldsymbol{\psi}}^T(t)\tilde{\boldsymbol{\theta}}(t)\right]. \tag{15}$$

□

Notice that in both (7) and (8), $e(t)$ is a filtered version of an inner product of a regressor and parameter error vector.

3. ALGORITHMS AND CONVERGENCE

Algorithms appropriate for $e(t)$ in (15) have the form

$$\dot{\hat{\theta}}(t) = \mu F(D,t)[\overline{\boldsymbol{\psi}}(t)]G(D,t)[e(t)] \tag{16}$$

$$e(t) = H(D,t)[\overline{\boldsymbol{\psi}}^T(t)\tilde{\boldsymbol{\theta}}(t)] \tag{17}$$

in continuous-time analogy with the algorithms of [7]. As a trivial example, the basic equation error algorithm is obtained by letting $G(D,t) = 1$ and $F(D,t) = 1$.

Sufficient conditions for the (local) exponential convergence of (16) includes a persistence-of-excitation condition on the regressor $\overline{\boldsymbol{\psi}}(t)$ and strict passivity of the operator $G(D,t)H(D,t)F^{-1}(D,t)$ [6]. Clearly, the passivity requirement is guaranteed in the equation error case.

4. CONDITIONS FOR PE REGRESSORS

Definition 1. *A function $\boldsymbol{\psi}(t)$ is said to be* persistently exciting *(PE) if there exist T and $\eta_2 \geq \eta_1 > 0$ such that*

$$\eta_1 I \leq \int_{t=t_0}^{t_0+T} \boldsymbol{\psi}(t)\boldsymbol{\psi}^T(t)dt \leq \eta_2 I \qquad \forall t_0 \in \Re^+ \tag{18}$$

where I is the identity matrix.

Note that for time-varying ARMA systems, at $\tilde{\boldsymbol{\theta}}(t) = 0$ the regressor $\overline{\boldsymbol{\psi}}(t)$ becomes equal to $\boldsymbol{\psi}(t)$, so that the PE condition applies to that vector sequence.

For identifying time-varying ARMA systems using a basis function approach, we must interpret the PE condition in that setting. First, define the ARMA regressor $\boldsymbol{\phi}(t)$ to be the vector whose first $M+1$ elements are filtered versions of $\{u(t)\}$ and whose remaining N elements are filtered versions of $\{y(t)\}$ (this is the standard regressor in the time-invariant ARMA setting) and define the basis function regressor as

$$\boldsymbol{f}(t) = [\ f_1(t) \quad \cdots \quad f_L(t)\ ]^T. \tag{19}$$

The three regressors $\boldsymbol{f}(t)$, $\boldsymbol{\phi}(t)$, and $\boldsymbol{\psi}(t)$ are related by the following equation [2]

$$\boldsymbol{\psi}(t) = \boldsymbol{\phi}(t) \otimes \boldsymbol{f}(t) \tag{20}$$

where $\otimes$ denotes the Kronecker product operation.

In the same fashion as in [5], one can establish PE relationships between the three regressors. In this paper, we will develop the results in the case of bounded regressors. In the case of unbounded and random regressors, the results will be similar to their discrete-time counterparts developed in [5]. Let $\boldsymbol{\phi}(t)$ and $\boldsymbol{f}(t)$ be bounded. To establish a PE relationship between the three regressors, four different cases that represent all the possible combinations of $\boldsymbol{f}(t)$ and $\boldsymbol{\phi}(t)$ will be studied. Those cases are listed in Table 1.

Table 1. PE relationships for bounded $\boldsymbol{f}(t)$ and $\boldsymbol{\phi}(t)$

Case No.	$\boldsymbol{f}(t)$	$\boldsymbol{\phi}(t)$	$\boldsymbol{\psi}(t)$
1)	PE	PE	PE/not PE
2)	PE	not PE	Not PE
3)	not PE	PE	Not PE
4)	not PE	not PE	Not PE

The characterization of $\boldsymbol{\psi}(t)$ in Table 1 as either PE or not PE is accomplished in Theorem 1 below. The proof of Theorem 1 proceeds in a fashion analogous to the proof of the similar result in [5].

Theorem 1. *If the basis function regressor $\boldsymbol{f}(t)$ and the ARMA regressor $\boldsymbol{\phi}(t)$ are bounded, a necessary condition for the regressor $\boldsymbol{\psi}(t)$ to be PE is that both of $\boldsymbol{f}(t)$ and $\boldsymbol{\phi}(t)$ be PE.*

A result that would naturally be desirable is that $\boldsymbol{\psi}(t)$ is PE if and only if both $\boldsymbol{f}(t)$ and $\boldsymbol{\phi}(t)$ are PE. However, it is shown via a counterexample in [5] that this is not the case. For a discussion of the PE relationships of unbounded and random regressors see [5]

5. OPERATOR CONDITIONS

The other requirement for local stability of the adaptive algorithm (16) involves the operator conditions, and central to these is the strict passivity requirement. These conditions parallel the similar strict positive reality (SPR) condition applied in output error IIR filtering [8], together with the attendant algorithm operator selection. Many of the same types of choices are available for the time-varying ARMA identification, but a further understanding of when the passivity condition is satisfied for general time-varying operators is needed.

Consider the following state variable realization of the operator $G(D,t)H(D,t)F^{-1}(D,t)$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)w(t) \tag{21}$$

$$v(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)w(t) \tag{22}$$

where $\boldsymbol{x}$ is an l-dimensional vector.

Definition 2 ([9],[10]). *The linear time-varying system with state variable realization $\{\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)\}$ is strictly passive if and only if $\exists \delta > 0$ such that*

$$\int_{t_0}^{t_1} v(t)u(t)dt > \delta \int_{t_0}^{t_1} u^2(t)dt, \qquad \forall t_1 \geq t_0. \tag{23}$$

Further, the system of (21) and (22) is passive with degree of passivity $\sigma \geq 0$, or σ-passive if the system with state variable realization $\{\sigma I + \boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)\}$ is strictly passive.

Our analysis will be based on the Kalman-Yakubovich-Popov (KYP) lemma [10]-[13].

Theorem 2 ([12]). *Consider the linear time invariant system with state variable realization $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ having an exponentially stable zero input response. The system is σ-SPR if and only if there exists at least one positive definite symmetric matrix $\boldsymbol{P}$ such that*

$$\begin{bmatrix} -\boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A} - 2\sigma\boldsymbol{P} & \boldsymbol{P}\boldsymbol{B} - \boldsymbol{C}^T \\ (\boldsymbol{P}\boldsymbol{B} - \boldsymbol{C}^T)^T & \boldsymbol{D} + \boldsymbol{D}^T \end{bmatrix} > 0 \tag{24}$$

Based on results in [11] and [13] we have established the following theorem. The proof is omitted because of space limitation.

Theorem 3. *Consider the time-varying system with state variable realization $\{\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)\}$ having an exponentially stable zero input response. Assume that $\boldsymbol{D}(t) + \boldsymbol{D}^T(t)$ is nonsingular for all t. Then the system is strictly passive if there exists at least one positive definite symmetric matrix $\boldsymbol{P}(t)$ such that for all $t \geq 0$, the matrix*

$$\begin{bmatrix} -\boldsymbol{A}^T(t)\boldsymbol{P}(t) - \boldsymbol{P}(t)\boldsymbol{A}(t) - \dot{\boldsymbol{P}}(t) & \boldsymbol{P}(t)\boldsymbol{B}(t) - \boldsymbol{C}^T(t) \\ [\boldsymbol{P}(t)\boldsymbol{B}(t) - \boldsymbol{C}(t)^T]^T & \boldsymbol{D}(t) + \boldsymbol{D}^T(t) \end{bmatrix} \tag{25}$$

is positive definite.

Our main result in the paper is expressed as the following.

Theorem 4. *The time varying system $\{\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)\}$ will be passive if the following conditions are satisfied:*

- *The frozen systems are σ-SPR;*
- *$\boldsymbol{P}(t)$ is differentiable; with $\dot{\boldsymbol{P}}(t) < 2\sigma\boldsymbol{P}(t)$.*

Proof. The Theorem follows from direct application of Theorems 2 and 3. □

Example 1. *Assume the linear time-varying system of (1) is first order with*

$$\dot{y}(t) + a(t)y(t) = \dot{u}(t) + b(t)u(t), \tag{26}$$

with $a(t) = 2 + \sin(\omega t)$. (We do not define in this example the basis functions $\boldsymbol{f}(t)$ since we illustrate only the satisfaction of the operator condition.) Let $C(D,t) = D + c_1$, with c_1 a constant. In (16) we let $F(D,t) = G(D,t) = 1$; in (17) we have

$$H(D,t) = \frac{D + c_1}{D + a_1(t)}. \tag{27}$$

The convergence of the algorithm is then guaranteed by $H(D,t)$ being strictly passive and with satisfaction of the appropriate PE condition. To examine passivity of $H(D,t)$, let (21) and (22) be the realization with

$$\boldsymbol{A}(t) = -a(t) \tag{28}$$
$$\boldsymbol{B}(t) = 1 \tag{29}$$
$$\boldsymbol{C}(t) = c_1 - a(t) \tag{30}$$
$$\boldsymbol{D}(t) = 1. \tag{31}$$

The frozen systems are σ-passive if there is $\boldsymbol{P}(t) > 0$ such that

$$\begin{bmatrix} 2[a(t)-\sigma]\boldsymbol{P}(t) & \boldsymbol{P}(t) + a(t) - c_1 \\ \boldsymbol{P}(t) + a(t) - c_1 & 2 \end{bmatrix} > 0. \tag{32}$$

This is satisfied when

$$4[a(t) - \sigma]\boldsymbol{P}(t) - [\boldsymbol{P}(t) + a(t) - c_1]^2 > 0. \tag{33}$$

Choosing

$$\boldsymbol{P}(t) = a(t) + c_1 - 2\sigma \tag{34}$$

maximizes this determinant, yielding

$$4[a(t)-\sigma][a(t) + c_1 - 2\sigma] - [2a(t) - 2\sigma]^2 \tag{35}$$
$$= 4(c_1 - \sigma)[a(t) - \sigma]. \tag{36}$$

Notice that the minimum value $a(t)$ can take is 1. Hence we need $\sigma < 1$ and $\sigma < c_1$ for the determinant to be strictly positive. Hence $H(D,t)$ is σ-passive $\forall t$ for any $\sigma \in [0, \min(1, c_1))$.

Let $c_1 > 1$. We then have

$$\sigma = 1 - \epsilon \tag{37}$$
$$\boldsymbol{P}(t) = a(t) + c_1 - 2\sigma \tag{38}$$
$$\dot{\boldsymbol{P}}(t) = \omega \cos(\omega t). \tag{39}$$

For the passivity of the time varying operator $H(D,t)$, we need $\dot{\boldsymbol{P}}(t) < 2\sigma \boldsymbol{P}(t)$, i.e.,

$$\omega \cos(\omega t) < 2(1-\epsilon)[a(t) + c_1 - 2(1-\epsilon)] \tag{40}$$
$$\omega \cos(\omega t) < 2(1-\epsilon)[c_1 + 2\epsilon + \sin(\omega t)]. \tag{41}$$

This is always satisfied if

$$\omega < 2(1-\epsilon)(c_1 - 1 + 2\epsilon). \tag{42}$$

Taking ϵ arbitrarily small, we obtain the following sufficient condition for the strict passivity of $H(D,t)$:

$$\omega < 2(c_1 - 1). \tag{43}$$

Example 1 illustrates Theorem 4. Notice that there is an upper bound on the fastest parameter variations in order to guarantee passivity using Theorem 4. Notice also that by pushing the zero of this operator far in the left-half-plane, the operator can tolerate more rapid variations and maintain its passivity.

6. CONCLUSIONS

In this paper we have shown that the persistence-of-excitation (PE) conditions that were developed in the discrete-time setting can be extended to the continuous-time case. The first main result in this paper, stated in Theorem 1, shows that PE of the basis functions and the ARMA regressors is required for the estimator regressor to be PE. The second main result, stated in Theorem 4, is a sufficient condition that guarantees the passivity of the estimator operator. Both PE and passivity are necessary conditions for the convergence of the algorithm. Future work will develop similar passivity results for the discrete-time case.

7. REFERENCES

[1] A. H. Sayed, *Fundamentals of Adaptive Filtering*. Hoboken, NJ: Wiley, 2003.

[2] M. Niedzwiecki, *Identification of Time-varying Processes*. New York: Wiley, 2000.

[3] M. K. Tsatsanis and G. B. Giannakis, "Modeling and equalization of rapidly fading channels," *Int. J. Adaptive Contr. Signal Process.*, vol. 10, pp. 159–176, 1996.

[4] K. H. Chon, H. Zhao, R. Zou, and K. Ju, "Multiple time-varying dynamic analysis using multiple sets of basis functions," *IEEE Trans. Biomed. Eng.*, vol. 52, no. 5, pp. 956–960, May 2005.

[5] M. Abu-Naser and G. A. Williamson, "Convergence properties of adaptive estimators of time-varying linear systems using basis functions," in *Proc. 12th IEEE Digital Signal Processing Workshop*, 2006, pp. 336–341.

[6] B. D. O. Anderson, R. R. Bitmead, C. R. Johnson, Jr., P. V. Kokotovic, R. L. Kosut, M. Y. Mareels, L. Praly, and B. D. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*. Cambridge, MA: MIT Press, 1986.

[7] W. A. Sethares, B. D. O. Anderson, and C. R. Johnson, Jr., "Adaptive algorithms with filtered regressor and filtered error," *Math. Contr., Signals, Syst.*, vol. 2, pp. 381–403, Jul. 1989.

[8] C. R. Johnson, Jr., M. Larimore, J. Treichler, and B. D. O. Anderson, "Sharf convergence properties," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. 29, no. 3, pp. 659–670, Jun. 1981.

[9] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic Press, 1975.

[10] S. Dasgupta, G. Chockalingam, and M. Fu, "Stability and passivity analysis of systems with time-varying parameters," in *Proc. 35th IEEE Decision and Control Conf.*, 1996, pp. 1857–1862.

[11] Y. D. Landau, *Adaptive Control: The Model Reference Approach*. New York: Marcel Dekker, Inc., 1979.

[12] S. Dasgupta, G. Chockalingam, B. D. O. Anderson, and M. Fu, "Lyapunov functions for uncertain systems with applications to the stability of time varying systems," *IEEE Trans. Circuits Syst. I*, vol. 41, no. 2, pp. 93–106, Feb. 1994.

[13] B. D. Anderson and P. J. Moylan, "Synthesis of linear time-varying passive networks," *IEEE Trans. Circuits Syst.*, vol. 21, no. 4, pp. 678–687, Sep. 1974.