A NEW VARIABLE STEP-SIZE LMS ALGORITHM WITH ROBUSTNESS TO NONSTATIONARY NOISE

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ABSTRACT

A new variable step-size least-mean-square (VSSLMS) algorithm is presented in this paper for applications in which the desired response contains nonstationary noise with high variance. The step size of the proposed VSSLMS algorithm is controlled by the normalized square Euclidean norm of the averaged gradient vector, and is henceforth referred to as the NSVSSLMS algorithm. As shown by the analysis and simulation results, the proposed algorithm has both fast convergence rate and robustness to high-variance noise signals, and performs better than Greenburg's sum method, which is a robust algorithm for applications with nonstationary noise.

***Index Terms*—** Adaptive filters, variable step size LMS algorithm

1. INTRODUCTION

The LMS algorithm has been extensively used in many applications as a consequence of its simplicity and robustness [1][2]. A key parameter in the design of LMS-based algorithms is the step size. It is well known that the VSSLMS algorithms can improve the performance of the LMS algorithm. In a summary of [2, p.255], several variable step-size algorithms designed to enhance the performance of the LMS algorithm have been given [3]-[6]. However, the algorithms in [3],[5] are very sensitive to interference noise, while the method in [4] needs the noise signal to be uncorrelated, and the method proposed in [6] is only suitable for stationary and low-level noise conditions; thus, they are limited in many applications. To the best of our knowledge, no variable step-size LMS algorithm has been proposed for a wide range of applications where the noise signal is correlated, potentially high variance, such as speech signals.

As shown in [7], some normalization terms can be utilized to modify the LMS algorithm, so as to overcome the interference of nonstationary noise. One such modified LMS algorithm, namely the sum method, is discussed by Greenburg in [7]. This algorithm can be deemed as a fixed step-size algorithm with normalized gradient vector which is designed to minimize the steady-state mean-square error (MSE). However, it is based on a constant convergence rate. Similar to the case of the LMS algorithm, a variable step-size algorithm is also necessary to obtain both fast convergence rate and small steady state MSE.

In this paper we propose a new variable step-size algorithm, a NSVSSLMS algorithm, which is robust to high-variance noise signals. In this algorithm, the step size is controlled by a normalized square Euclidean norm of the smoothed gradient vector. It will be shown that the proposed algorithm performs better than Greenberg's sum method for the scenario when the noise signal is nonstationary. It can be deemed as a variable step-size version of Greenberg's method.

The remainder of this paper is organized as follows: The proposed algorithm is described in Section 2. The analysis of the proposed algorithm in the context of stationary noise is introduced in Section 3. A simulation that confirms the analysis and the advantages of the proposed algorithm for nonstationary noise as compared with Greenburg's method is shown in Section 4. Section 5 provides conclusions.

2. ALGORITHM FORMULATION

2.1. Preliminary

In this section we will briefly review Greenburg's sum method [7], which is the foundation of the proposed algorithm. For the convenience of presentation, we formulate the LMS algorithm within the context of adaptive noise cancellation model, similar to the approach in [7]. In this case, the primary signal $d(n)$ can be formulated as follows:

$$d(n) = \mathbf{x}^{\mathrm{T}}(n)\mathbf{w}_{\mathrm{opt}} + t(n) \tag{1}$$

where $\mathbf{x}(n)$ is the input reference vector, $\mathbf{w}_{\mathrm{opt}}$ is the optimal filter vector, $t(n)$ is the target signal, while is also the noise signal for the LMS algorithm, n denotes the discrete-time index and $(\cdot)^{\mathrm{T}}$ denotes the vector transpose operator. The output error of the system $e(n)$ is the difference between the primary signal and the output of the adaptive filter:

$$e(n) = d(n) - \mathbf{x}^{\mathrm{T}}(n)\mathbf{w}(n) \tag{2}$$

where $\mathbf{w}(n)$ is the adaptive filter vector. The update equation of the LMS algorithm is then given by [1]:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu e(n)\mathbf{x}(n) \tag{3}$$

where μ is the step size.

In [7], a modified LMS algorithm named the sum method is shown to be suitable for nonstationary input and target signals. The update of the adaptive filter coefficients of this algorithm is as follows:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \frac{\mu_{\text{sum}} e(n)\mathbf{x}(n)}{\{L[\hat{\sigma}_e^2(n) + \hat{\sigma}_x^2(n)]\}} \tag{4}$$

where μ_{sum} is the step size for this sum method, $\hat{\sigma}_e^2(n)$ and $\hat{\sigma}_x^2(n)$ are time varying estimations of the output error signal variance and the input signal variance respectively, and L is the adaptive filter length [7]. As explained in [7], the step size in (4) is adjusted by the input and output error variance automatically, which reduces the influence brought by the fluctuation of the input and the target signals.

Next we will introduce the proposed new variable step size algorithm.

2.2. Proposed NSVSSLMS algorithm

The proposed NSVSSLMS algorithm can be formulated as follows:

$$\bar{\mathbf{g}}(n) = \beta\bar{\mathbf{g}}(n-1) + (1-\beta)\mathbf{x}(n)e(n) \tag{5}$$

$$\mu_{\text{NSVSS}}(n) = \frac{P\,\|\bar{\mathbf{g}}(n)\|_2^2}{\{L[\hat{\sigma}_e^2(n) + \hat{\sigma}_x^2(n)]\}^2} \tag{6}$$

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu_{\text{NSVSS}}(n)e(n)\mathbf{x}(n) \tag{7}$$

where $\bar{\mathbf{g}}(n)$ is the smoothed gradient vector, P is a positive constant, which can be easily chosen according to the analysis in the next section, $\mu_{\text{NSVSS}}(n)$ is the time-varying step size, and $\|\cdot\|_2^2$ denotes the squared Euclidean norm operator. The key step of this algorithm is (6), which is motivated as follows.

To develop a VSSLMS algorithm, the most important thing is to measure the proximity of the adaptive process to the desired solution. An ideal measure of the adaptive process is the mean square deviation (MSD), which is defined as $E\{\|\mathbf{w}_{\text{opt}} - \mathbf{w}(n)\|_2^2\}$, where $E\{\cdot\}$ represents statistical expectation. According to the formulation in [6], with a stationary input signal, the squared norm of the smoothed gradient vector which is formulated by (5) can track the variance of the MSD; thus, it is a good measure of the proximity of the adaptive process, and suitable to control the step size. The term $L[\hat{\sigma}_e^2(n) + \hat{\sigma}_x^2(n)]$ in (6) is motivated by [7]. The square of this term, as a novel normalization for the step size, is designed to make the steady-state excess mean-square error (EMSE) of the proposed algorithm robust to the target signal.

It will be shown in the following analysis that the proposed algorithm has robustness to the high variance of the target signal. Furthermore, the parameter P in the proposed algorithm can be easily determined according to the performance analysis in the next section.

3. APPROXIMATE PERFORMANCE ANALYSIS OF THE PROPOSED ALGORITHM

In this section we will give an approximate steady-state performance analysis of the proposed NSVSSLMS algorithm. For the convenience of analysis we make two assumptions:

A1. The input signal $x(n)$ is a zero-mean stationary white signal. The target signal $t(n)$ is also zero-mean stationary and independent of the input signal $x(n)$.

A2. At steady state the excess mean square error is much smaller than the target signal variance, and therefore the error signal $e(n)$ is approximately equal to the target signal $t(n)$.

Assumption *A1* is a general assumption for the analysis of the VSSLMS algorithm. We are justified in assuming the noise is stationary on the basis that signals such as speech can be assumed stationary over a certain interval. Assumption *A2* is true in the adaptive noise canceller, if the step size is not very large. Using these assumptions simplifies the analysis and gives insight into the performance of the algorithm.

Since the squared norm of the smoothed normalized gradient vector $\|\bar{\mathbf{g}}(n)\|_2^2$ is the key term for the proposed algorithm, we will give a steady-state performance analysis for this term first. From (5) we have

$$\bar{\mathbf{g}}(n) = (1-\beta)\sum_{i=1}^{n}\beta^{n-i}\mathbf{g}(i) \tag{8}$$

assuming $\bar{\mathbf{g}}(0) = \mathbf{0}$ and denoting $\mathbf{g}(i) = e(i)\mathbf{x}(i)$. The expected performance of the squared norm of the smoothed gradient vector can then be obtained

$$E\{\|\bar{\mathbf{g}}(n)\|_2^2\} = (1-\beta)^2\sum_{i=1}^{n}\sum_{j=1}^{n}C(i,j) \tag{9}$$

where $C(i,j)$ is defined as

$$C(i,j) = E\{\beta^{n-i}\mathbf{g}^{\mathrm{T}}(i)\beta^{n-j}\mathbf{g}(j)\}. \tag{10}$$

When n approaches infinity, the term β^{n-i} in (10) approaches to zero if i is finite. So when we calculate $E\{\|\bar{\mathbf{g}}(\infty)\|_2^2\}$, the term $C(i,j)$ can be ignored when i or j are not infinite. The following analysis will only consider this term at steady state, i.e., i and j are both steady-state time indexes.

At first we consider $C(i,j)$ when $i = j$. From assumption *A2* we have

$$e(i) \approx t(i). \tag{11}$$

With (11), the gradient vector $\mathbf{g}(i)$ can also be approximately written as

$$\mathbf{g}(i) \approx t(i)\mathbf{x}(i). \tag{12}$$

Substituting this into (10) we obtain

$$C(i,i) \approx E\{\beta^{2n-2i}\mathbf{x}^{\mathrm{T}}(i)\mathbf{x}(i)t^2(i)\}. \tag{13}$$

With assumption *A1*, (13) becomes

$$C(i,i) \approx \beta^{2n-2i}L\sigma_x^2\sigma_t^2 \tag{14}$$

where σ_t^2 and σ_x^2 are the variances of the target signal and input signal respectively. When $i \neq j$, similar derivation can be performed which yields

$$C(i,j) \approx 0 \text{ when } i \neq j \tag{15}$$

Substituting (14) and (15) into (9) we have

$$\lim_{n\to\infty}E\{\|\bar{\mathbf{g}}(n)\|_2^2\} \approx (1-\beta)^2\sum_{i=s}^{n}\beta^{2(n-i)}L\sigma_x^2\sigma_t^2 \tag{16}$$

where s is the time index beyond which the system is assumed at steady state. Equation (16) can be simplified as:

$$E\{\|\bar{\mathbf{g}}(\infty)\|_2^2\} \approx \frac{(1-\beta)}{(1+\beta)}L\sigma_x^2\sigma_t^2. \tag{17}$$

Now let's examine the steady-state performance of the proposed algorithm. Since the term $\{L[\hat{\sigma}_x^2(n)+\hat{\sigma}_t^2(n)]\}^2$ changes very slowly with stationary input and noise signals, we assume that it is a constant during the iteration. Taking the expectation on both sides of (6), we have

$$E\{\mu_{\text{NSVSS}}(n)\} = \frac{PE\{\|\bar{\mathbf{g}}(n)\|_2^2\}}{\{L[\hat{\sigma}_x^2(n)+\hat{\sigma}_t^2(n)]\}^2}. \tag{18}$$

Substituting (17) into (18) we have

$$E\{\mu_{\text{NSVSS}}(\infty)\} \approx \frac{P(1-\beta)L\sigma_x^2\sigma_t^2}{(1+\beta)\{L[\hat{\sigma}_x^2(n)+\hat{\sigma}_t^2(n)]\}^2}. \tag{19}$$

As described by equation (16) in [7], the steady-state EMSE of the LMS algorithm which is defined as $E\{[e(n)-t(n)]^2\}$ can be formulated as

$$J_{\text{ex,LMS}}(\infty) = \frac{\mu_{\text{LMS}}L\sigma_x^2\sigma_t^2}{2-\mu_{\text{LMS}}L\sigma_x^2}. \tag{20}$$

If we assume that μ_{LMS} is very small so that $\mu_{\text{LMS}}L\sigma_x^2 \ll 2$, we have

$$J_{\text{ex,LMS}}(\infty) \approx \frac{1}{2}\mu_{\text{LMS}}L\sigma_x^2\sigma_t^2. \tag{21}$$

Similarly, if we assume that at steady state the step size of the proposed algorithm is very small, and $\mu_{\text{NSVSS}}(\infty)L\sigma_x^2 \ll 2$, the EMSE of the proposed algorithm can then be formulated as

$$J_{\text{ex,NSVSS}}(\infty) \approx \frac{1}{2}E\{\mu_{\text{NSVSS}}(\infty)\}L\sigma_x^2\sigma_t^2. \tag{22}$$

Substituting (19) into (22) we obtain the steady-state EMSE for the proposed NSVSSLMS algorithm:

$$J_{\text{ex,NSVSS}}(\infty) \approx \frac{P(1-\beta)L^2\sigma_t^4\sigma_x^4}{2(1+\beta)\{L[\hat{\sigma}_x^2(n)+\hat{\sigma}_t^2(n)]\}^2}. \tag{23}$$

Since $\hat{\sigma}_t^2(n) \approx \sigma_t^2$, the following equation is obtained from (23)

$$\lim_{\sigma_t^2\to\infty} J_{\text{ex,NSVSS}}(\infty) \approx \frac{P(1-\beta)\sigma_x^4}{2(1+\beta)}. \tag{24}$$

It can be clearly seen from (24) that the EMSE obtained by the proposed algorithm will be independent of the target signal $t(n)$ when the variance of the target signal is very large. Although the analysis is based on the assumption that the target signal is stationary, an approximate indication of its general performance is also obtained. For some nonstationary target signals, such as speech, they can be deemed as stationary over some short interval. When the variance of some intervals of the target signal is much higher than the input signal at steady state, the EMSE will be independent of the variance of the target signal; thus, the proposed algorithm is robust for applications with nonstationary target signals.

Now let's consider the choice of the parameter P. Note that (24) also gives an upper bound of the steady state EMSE of the proposed algorithm with the variation of the target variance. To choose this parameter, we first need to determine an upper bound value of $J_{\text{ex,NSVSS}}$ according to the application. With this value and the variance of the input signal, P can be determined directly according to (24):

$$P = \frac{J_{\text{ex,NSVSS, max}}2(1+\beta)}{(1-\beta)\sigma_x^4} \tag{25}$$

where $J_{\text{ex,NSVSS, max}}$ is the upper bound value of the EMSE.

If the maximum short-interval variance of the nonstationary target signal can be obtained, a more accurate criterion for the choice of P similar to (25) can be obtained according to (23). In practice, since the target variance is not infinite, the parameter P can be a little larger than the value obtained from (25).

In the next section, all the above analysis and discussion will be supported by simulation in the context of a nonstationary target signal.

4. SIMULATION

In this simulation, we will compare the performance between Greenburg's method and the proposed algorithm within an adaptive noise canceller model. The input signal $x(n)$ is a pseudo-random, zero-mean unit-variance Gaussian signal with a length of 100,000 samples. The target signal $t(n)$ is the first $100,000$ samples of a speech signal which is available from
http://www.voiptroubleshooter.com/open_speech/american.html,
and the file name is "OSR_us_000_0016_8k.wav". This target signal is scaled to make the average SNR over the entire observation 0dB. The target signal and one representation of the input signal can be seen in Fig. 1.

The primary signal $d(n)$ is obtained as follows:

$$d(n) = x(n) * h(l) + t(n) \tag{26}$$

where $h(l)$ is the optimal filter obtained by

$$h(l) = e^{-0.05l}r(l), \;\; l = 1, ..., 100 \tag{27}$$

where $r(l)$ is drawn from a zero mean unit variance Gaussian sequence. One representation of $h(l)$ can be seen in Fig. 2(a).

In this simulation the proposed algorithm will be compared to Greenburg's method with different step sizes 0.1 and 0.02. The initial step sizes and adaptive filter vectors of the proposed algorithm are set to zero. The parameter β for the proposed algorithm is set to 0.999 to perform a sufficient smoothing operation. The parameter P in the proposed algorithm is set to 80. The parameter sets for the proposed algorithm are chosen to make its initial convergence rate approximately equal to that of Greenburg's method with a step size 0.1. The estimates $\hat{\sigma}_e^2(n)$ and $\hat{\sigma}_x^2(n)$ used in Greenburg's algorithm and the proposed algorithm are obtained by smoothing the input and error signals as

$$\hat{\sigma}_e^2(n) = 0.99\hat{\sigma}_e^2(n-1) + (1-0.99)e^2(n) \tag{28}$$

and

$$\hat{\sigma}_x^2(n) = 0.99\hat{\sigma}_x^2(n-1) + (1-0.99)x^2(n). \tag{29}$$

The initial values of $\hat{\sigma}_e^2(n)$ and $\hat{\sigma}_x^2(n)$ are set to zero and unit respectively. The evolutions of the EMSE curves for all the experiments are shown in Fig. 2(b). The results are obtained over 200 Monte Carlo trials of the same experiment.

It is clear seen in Fig. 2(b) that the proposed algorithm has a EMSE convergence rate similar to that of Greenburg's method with a parameter 0.1 at the early state of the process. The EMSE of both methods converges to -20dB at about 3,000 samples. However, the EMSE of Greenburg's method with parameter 0.1 fluctuates greatly with the variation of the target signal energy. The performance of Greenburg's method with parameter 0.02 has a small EMSE and slight fluctuation of the EMSE, but the convergence rate is very slow.

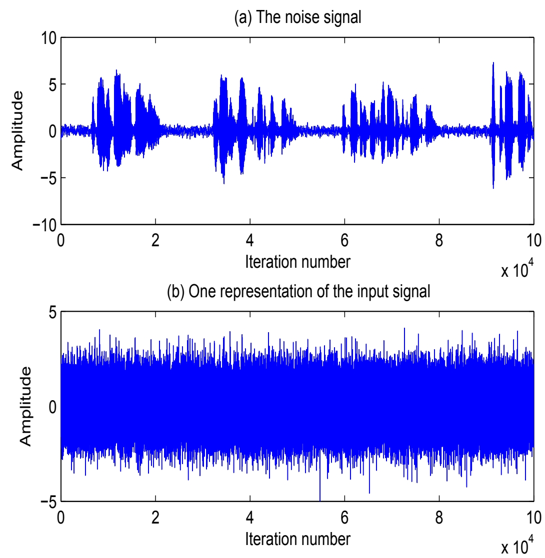

Fig. 1: The noise signal (a) and one representation of the input signal (b).

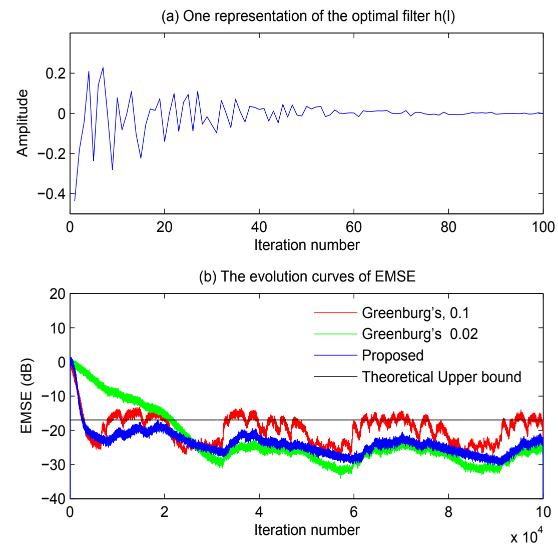

Fig. 2: One representation of the optimal filter (a) and the evolution curves of the EMSE for Greenburg's, Shin's, and the proposed NSVSSLMS algorithms (b).

The proposed algorithm has a fast convergence rate which is similar to Greenburg's method with parameter 0.1, and a small EMSE which is close to that of Greenburg's method with parameter 0.02. Therefore, the proposed NSVSSLMS algorithm performs better than Greenburg's method.

The theoretical upper bound of the EMSE of the proposed algorithm according to (24) is also shown in Fig. 2(b). It can be seen that over the interval $15,000$ to $20,000$, where the variance of the target signal is high, the EMSE of the simulation results is very close to this theoretical upper bound. Thus (24) can give a good upper bound of the steady-state EMSE for the proposed algorithm, and we conclude that with a given upper bound of the steady-state EMSE, the parameter P can be properly chosen according to (25).

Note that all the analysis and simulation are based on a white input signal. When the input signal is correlated, the analysis results obtained from (19) and (23) are both incorrect, and smaller than the pratical results. In this case, the parameter P should be chosen smaller than the value obtained from (24). Finally, if both input and noise signals are nonstationary signals, the smoothed gradient vector can not measure the proximity of the adaptive process, and the proposed algorithm has no advantage as compared with Greenburg's method. A new variable step-size approach is needed in such cases.

5. CONCLUSIONS

A new VSSLMS algorithm, namely the NSVSSLMS algorithm, has been presented in this paper. According to our analysis and simulation results, the proposed algorithm performs better than Greenburg's sum method with stationary input and nonstationary noise signals. Simulations show that this algorithm can obtain both fast convergence and small EMSE with robustness to nonstationary noise signals. Future work will focus on the issues when both the input signal and the noise signal are nonstationary.

6. REFERENCES

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