MEAN-SQUARE ERROR ANALYSIS OF A NON-UNIFORM SUBBAND ADAPTIVE FILTERING STRUCTURE WITH CRITICAL SAMPLING

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ABSTRACT

Adaptive subband structures have been proposed with the objective of increasing the convergence speed and/or reducing the computational complexity of conventional adaptive algorithms, mainly for applications which require a large number of adaptive coefficients. In this paper, we present a non-uniform subband structure with critical sampling, which is able of modeling an arbitrary FIR system with reduced aliasing. An LMS-type adaptation algorithm with normalized step-sizes, which works at the lowest downsampling rate and minimizes the average of the subband squared-errors, is derived for the proposed structure. A convergence analysis of the adaptation algorithm is presented, from which the steady-state mean-square error can be estimated.

Index Terms— Adaptive filters, Filter banks, Convergence analysis, Digital signal processing, Multirate processing

1. INTRODUCTION

Subband structures are attractive in applications that require high-order adaptive filters, such as acoustic echo cancellation and channel equalization, because of their properties of acceleration of the adaptation convergence and reduction of the computational complexity. Several adaptive subband algorithms have been proposed recently, using oversampled or critically sampled decomposed signals [1], [2]. In the critically sampled structures, there are necessarily extra filters between the subbands to eliminate the aliasing effects [1]. In the structure proposed in [2], the extra filters are not updated independently, but are directly derived from the main subfilters.

Most of the proposed subband adaptive structures employ uniform filter banks. A recent work [3], however, has shown that non-uniform subband adaptive structures may be able to outperform the uniform ones. The non-uniform subband structure presented in [3] employs oversampled subband signals. A non-uniform critically-sampled structure was derived in [4]. In this paper, we present a convergence analysis of the mean-square error obtained with the structure of [4]. Because of the non-uniform frequency decomposition of the input signal, the distinct adaptive subfilters work at different rates, which leads to some particularities in the adaptation algorithm.

In Section 2 the non-uniform subband structure with critical sampling is described and an LMS-type adaptive algorithm for the update of the coefficients of the subfilters is presented. An expression for the optimal coefficients of the proposed structure in terms of the unknown system impulse response is derived in Section 3. Section 4 contains the mean-square error analysis. In Section 5 computer simulations are presented in order to illustrate the convergence behavior and verify the theoretical analysis results. In Section 6 concluding remarks are made.

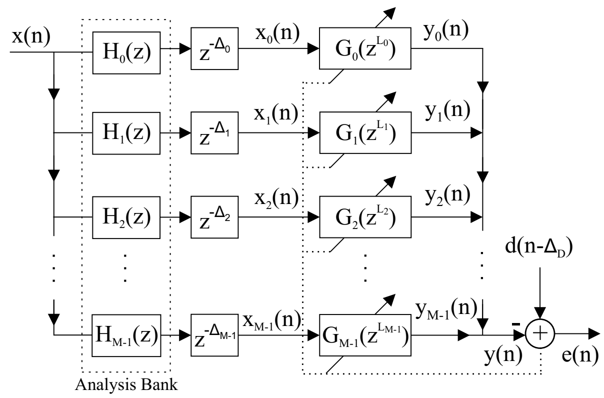

Fig. 1. Adaptive subband structure composed by a non-uniform analysis filter bank and sparse subfilters.

2. ADAPTIVE NON-UNIFORM SUBBAND STRUCTURE WITH CRITICAL SAMPLING

The non-uniform subband structure is derived from an adaptive filter that employs an analysis filter bank to decompose the input signal and sparse adaptive filters in the subbands [5]. Such structure is illustrated in Fig. 1, where $x(n)$ is the input signal, $H_k(z)$ are the subfilters of an M-channel non-uniform analysis bank, $G_k(z^{L_k})$ are the sparse adaptive subfilters, $d(n)$ is the desired signal, and $e(n)$ is the error signal used on the adaptation algorithm.

For an M-channel octave-band filter bank [6], the sparsity factors are $L_0 = 2^{M-1}$ and $L_k = 2^{M-k}$ for $1 \leq k < M-1$. The delays Δ_k in Fig. 1 were introduced with the purpose of matching the delays of the different length analysis filters.

The non-uniform critically sampled structure is derived from the sparse subband structure of Fig. 1 by including a non-uniform perfect-reconstruction multirate system following each adaptive subfilter. Considering that the analysis filters are sufficiently selective to assume that their frequency responses only overlap with those of the adjacent subbands and moving the sparse subfilters $G_k(z^{L_k})$ to the right of decimators [6], we obtain the non-uniform critically-sampled structure with its k-th band illustrated in Fig. 2, where $H_{i,j}(z) = H_i(z)H_j(z)$. Observe that the adaptive filters now work at a rate which is $(1/L_k)$-th or $(1/L_{k+1})$-th of the input rate.

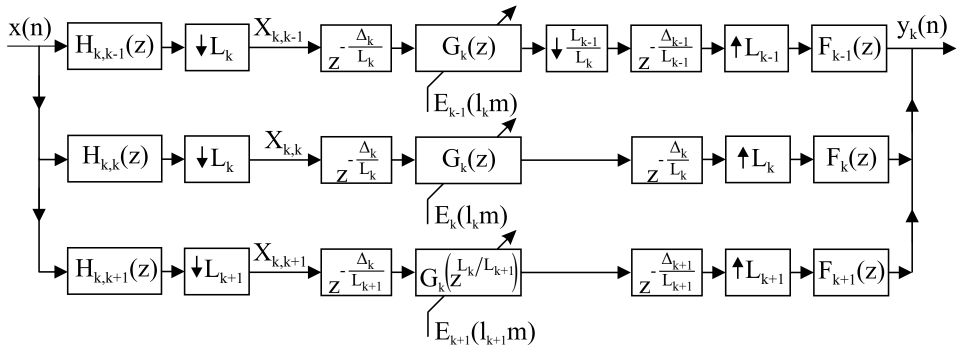

Fig. 2. The k-th subband with adaptive filters at lower rates.

Denoting as $\mathbf{G}_i(m)$ the vector containing the N_{G_i} coefficients of the subfilter $G_i(z)$ at iteration m, as $X_{i,j}(m)$ the downsampled signal at the output of $H_{i,j}$, and as $\mathbf{X}_{i,j}(m)$ the vector containing the most recent N_{G_i} samples of $X_{i,j}(m)$ (downsampled by a factor $L_{i,j} = \min(L_i, L_j)$ with respect to the input signal $x(n)$), and considering as cost function the sum of the subband average squared-errors, that is,

$$J(m) = \sum_{k=0}^{M-1} \frac{1}{l_k} \sum_{q=0}^{l_k-1} E_k^2(m_k^{'}), \tag{1}$$

with $l_k = L_0/L_k$,$m_k^{'} = l_k m + q$ and $q = 0, ..., l_k - 1$, the updating equation for the coefficients of the k-th subfilter is given by

$$\mathbf{G}_k(m+1) = \mathbf{G}_k(m) + \frac{\mu_{k,k}}{l_k} \sum_{q=0}^{l_k-1} \mathbf{X}_{k,k}(m_k^{'} - \frac{\Delta_k}{L_k}) E_k(m_k^{'})$$
$$+ \frac{\mu_{k-1,k}}{l_{k-1}} \sum_{q=0}^{l_{k-1}-1} \mathbf{X}_{k,k-1}(\frac{L_{k-1}}{L_k} m_{k-1}^{'} - \frac{\Delta_k}{L_{k-1}}) E_{k-1}(m_{k-1}^{'})$$
$$+ \frac{\mu_{k,k+1}}{l_{k+1}} \sum_{q=0}^{l_{k+1}-1} \mathbf{X}_{k,k+1}^{\downarrow}(m_{k+1}^{'} - \frac{\Delta_k}{L_{k+1}}) E_{k+1}(m_{k+1}^{'}). \tag{2}$$

The vector $\mathbf{X}_{k,k+1}^{\downarrow}$ contains N_{G_k} samples of the signal $X_{k,k+1}$ taking every (L_k/L_{k+1})-th sample, because of the sparsity factor L_k/L_{k+1} of the adaptive subfilter $G_k\left(z^{\frac{L_k}{L_{k+1}}}\right)$ in Fig. 2. The subband errors are given by

$$\begin{aligned} E_k(m_k^{'}) &= D_k(m_k^{'} - \Delta_{D_k}) - \mathbf{X}_{k,k}(m_k^{'} - \frac{\Delta_k}{L_k})^T \mathbf{G}_k(m) \\ &- \mathbf{X}^{\downarrow}{}_{k-1,k}(m_k^{'} - \frac{\Delta_{k-1}}{L_k})^T \mathbf{G}_{k-1}(m) \\ &- \mathbf{X}_{k+1,k}(2m_k^{'} - \frac{\Delta_{k+1}}{L_k})^T \mathbf{G}_{k+1}(m), \end{aligned} \tag{3}$$

and the delays Δ_{D_k} are required in order to match the delays introduced by the different length analysis filters of desired signal.

Observe that the adaptation algorithm operates at the lowest subband rate, which results in savings in the computational complexity when compared to the fullband algorithm. To improve the convergence rate of the adaptation algorithm for colored noise input signal, each step-size of Eq. (2) is made inversely proportional to the sum of the powers of the input signals of the filters responsible for the corresponding error.

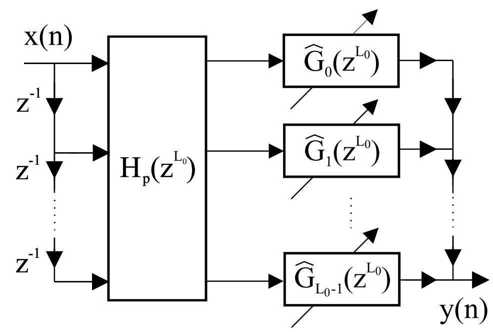

Fig. 3. Adaptive structure of Fig. 1 with polyphase representation of the non-uniform filter bank.

3. OPTIMAL COEFFICIENTS

Assuming that selective analysis filters are used (such that their frequency responses only overlap with those of the adjacent subbands), the coefficients of the critically-sampled structure of Fig. 2 are identical to those of the sparse structure of Fig. 1.

The sparse subfilter $G_k(z^{L_k})$ of Fig. 1 can be represented in terms of delayed versions of its l_k polyphase components [6], resulting in l_k subfilters of sparsity L_0. Denoting as $\mathbf{G}_k(z)$ the vector that contains these l_k polyphase components, that is:

$$\mathbf{G}_k(z) = \left[\begin{array}{cccc} G_{k,0}(z) & G_{k,1}(z) & \cdots & G_{k,l_k-1}(z) \end{array}\right]^T \tag{4}$$

and $\widehat{\mathbf{G}}(z)$ the vector formed by the polyphase vectors of the M subfilters $G_k(z)$:

$$\begin{aligned} \widehat{\mathbf{G}}(z) &= \left[\begin{array}{cccc} \widehat{G}_0(z) & \widehat{G}_1(z) & \cdots & \widehat{G}_{L_0-1}(z) \end{array}\right]^T \\ &= \left[\begin{array}{ccc} \mathbf{G}_0(z)^T & \mathbf{G}_1(z)^T & \cdots \mathbf{G}_{M-1}(z)^T \end{array}\right]^T, \end{aligned} \tag{5}$$

the subband structure of Fig. 1 can be redrawn in terms of the L_0 components of $\widehat{\mathbf{G}}(z)$, as shown in Fig. 3. In this figure, $\mathbf{H}_p(z)$ is the $L_0 \times L_0$ matrix that contains the polyphase components of the analysis filters, given by

$$\mathbf{H}_p(z) = \left[\begin{array}{cccc} \mathbf{H}_0(z)^T & \mathbf{H}_1(z)^T & \cdots & \mathbf{H}_{M-1}(z)^T \end{array}\right]^T, \tag{6}$$

where $\mathbf{H}_k(z)$ is the $l_k \times L_0$ matrix with the i-th row formed by the L_0 polyphase components of $z^{-(\Delta_k + iL_k)} H_k(z)$.

From Fig. 3, the transfer function implemented by the non-uniform subband adaptive structure of Fig. 1 is given by

$$T(z) = \widehat{\mathbf{G}}(z^{L_0})^T \mathbf{H}_p(z^{L_0}) \left[\begin{array}{cccc} 1 & z^{-1} & \cdots & z^{-(L_0-1)} \end{array} \right]^T. \quad (7)$$

In a system identification application, the coefficients of the subfilters $G_k(z^{L_k})$ are adapted such as to model an unknown FIR system. The unknown system transfer function, denoted here by $S(z)$, can be expressed in terms of its L_0 polyphase components as

$$S(z) = \left[\begin{array}{cccc} S_0(z^{L_0}) & S_1(z^{L_0}) & \cdots & S_{L_0-1}(z^{L_0}) \end{array} \right] \left[\begin{array}{c} 1 \\ z^{-1} \\ \vdots \\ z^{-(L_0-1)} \end{array} \right]. \quad (8)$$

From Eqs. (7) and (8), the subband structure models exactly the FIR system $S(z)$ when

$$\widehat{\mathbf{G}}(z)^T \mathbf{H}_p(z) = \left[\begin{array}{cccc} S_0(z) & S_1(z) & \cdots & S_{L_0-1}(z) \end{array} \right]. \quad (9)$$

Pos-multiplying both sides of the Eq. (9) by a matrix $\mathbf{F}_p(z)$ such that $\mathbf{H}_p(z)\mathbf{F}_p(z) = z^{-\Delta p}\mathbf{I}$, with $\mathbf{I}$ the identity matrix and Δp a positive integer, we obtain the following relation among the coefficients of the sparse subfilters and the coefficients of the unknown system:

$$\widehat{\mathbf{G}}(z) z^{-\Delta p} = \left[\begin{array}{cccc} S_0(z) & S_1(z) & \cdots & S_{L_0-1}(z) \end{array} \right] \mathbf{F}_p(z). \quad (10)$$

The matrix $\mathbf{F}_p(z)$ corresponds to the type-2 polyphase matrix of the synthesis filters that result in perfect reconstruction [6].

The optimal k-th subfilter $G_k^*(z)$ is then given by

$$G_k^*(z) = \sum_{i=0}^{l_k-1} z^{-i} G_{k,i}(z^{l_k}), \quad (11)$$

where $G_{k,i}(z)$ are related to $\widehat{G}_k(z)$ through Eqs. (4) and (5).

4. STEADY-STATE MEAN-SQUARE ERROR

In this section we present a mean-square analysis, which takes into account the error caused by the assumption of non-overlaping analysis filters in the algorithm derivation.

In order to simplify the notation, we define the $N_{G_i} \times 1$ vectors with the delayed subband signals:

$$\widehat{\mathbf{X}}_{i,j}(m) = \mathbf{X}_{i,j}(m - \frac{\Delta_i}{L_{i,j}}), \quad (12)$$

$$\widehat{\mathbf{X}}_{i,j}^{\downarrow}(m) = \mathbf{X}_{i,j}^{\downarrow}(m - \frac{\Delta_i}{L_{i,j}}). \quad (13)$$

For the general case (not assuming that non-adjacent analysis filters do not overlap), the desired signal at the k-th subband can be expressed in terms of the optimal coefficients $\mathbf{G}_k^*$ as

$$\begin{aligned} D_k(m_k' - \Delta_{D_k}) &= \sum_{i=0}^{k-1} \widehat{\mathbf{X}}_{i,k}^{\downarrow}(m_k')^T \mathbf{G}_i^* \quad (14) \\ &+ \sum_{i=k}^{M-1} \widehat{\mathbf{X}}_{i,k}(\frac{L_k}{L_i} m_k')^T \mathbf{G}_i^* + V_k(m_k'). \end{aligned}$$

Denoting as $\mathbf{D}(m)$ the vector formed by the desired signals of the M subbands at times $l_k m - \Delta_{D_k}$, it can be written as

$$\mathbf{D}(m) = X(m)\mathbf{G}^* + \mathbf{V}(m), \quad (15)$$

where $\mathbf{G}^*$ contains the optimal subfilter coefficients, $\mathbf{V}(m)$ contains the M subband residual modeling errors at time $l_k m$, and $X(m)$ is the matrix with the input subband signals whose k-th row is given by

$$\begin{aligned} [X(m)]_k &= [\ \widehat{\mathbf{X}}_{0,k}^{\downarrow}(l_k m)^T \ \cdots \ \widehat{\mathbf{X}}_{k-1,k}^{\downarrow}(l_k m)^T \ \widehat{\mathbf{X}}_{k,k}(l_k m)^T \\ & \widehat{\mathbf{X}}_{k+1,k}(2 l_k m)^T \ \cdots \ \widehat{\mathbf{X}}_{M-1,k}(2^{M-1-k} l_k m)^T \ ]. \quad (16) \end{aligned}$$

Considering only overlaping of adjacent subbands, the vector formed by the M subband output signals of the non-uniform structure at time $l_k m$ can be written as

$$\mathbf{Y}(m) = \widehat{X}(m)\mathbf{G}(m), \quad (17)$$

where $\widehat{X}(m)$ is the matrix with the input signals of the adaptive filters, that is,

$$\begin{aligned} [\widehat{X}(m)]_k &= [\ \mathbf{0}^T \ \cdots \ \mathbf{0}^T \ \widehat{\mathbf{X}}_{k-1,k}^{\downarrow}(l_k m)^T \ \widehat{\mathbf{X}}_{k,k}(l_k m)^T \\ & \widehat{\mathbf{X}}_{k+1,k}(2 l_k m)^T \ \ \mathbf{0}^T \ \cdots \ \mathbf{0}^T \ ]. \quad (18) \end{aligned}$$

Thus, the vector formed by the subband error signals is given by

$$\begin{aligned} \mathbf{E}(m) &= \mathbf{D}(m) - \mathbf{Y}(m) \\ &= X(m)\mathbf{G}^* - \widehat{X}(m)\mathbf{G}(m) + \mathbf{V}(m). \quad (19) \end{aligned}$$

For lossless filter banks, the total MSE is

$$\begin{aligned} \xi(m) &= \sum_{k=0}^{M-1} \frac{1}{l_k} \sum_{q=0}^{l_k-1} E[E_k^2(m_k')] \\ &\approx \sum_{k=0}^{M-1} E[E_k^2(l_k m)] = E[\mathbf{E}(m)\mathbf{E}(m)^T]. \quad (20) \end{aligned}$$

Substituting Eq. (19) in the above equation, assuming that the residual errors have zero mean and that, after convergence, $E[\mathbf{G}(\infty)] \approx \mathbf{G}^*$, we obtain

$$\xi_{ss} = \xi(\infty) \approx {\mathbf{G}^*}^T \mathbf{\Phi} \mathbf{G}^* + \sigma_v^2, \quad (21)$$

where

$$\mathbf{\Phi} = E[(X(m) - \widehat{X}(m))(X(m) - \widehat{X}(m))^T] \quad (22)$$

and σ_v^2 is the variance of the measurement noise. The k-th row of the matrix $X(m) - \widehat{X}(m)$, given by

$$\begin{aligned} & [X(m) - \widehat{X}(m)]_k = [\ \widehat{\mathbf{X}}_{0,k}^{\downarrow}(l_k m)^T \ \cdots \ \widehat{\mathbf{X}}_{k-2,k}^{\downarrow}(l_k m)^T \\ & \mathbf{0}^T \ \cdots \ \mathbf{0}^T \ \ \widehat{\mathbf{X}}_{k+2,k}(2 l_k m)^T \ \cdots \ \widehat{\mathbf{X}}_{M-1,k}(2^{M-1-k} l_k m)^T], \end{aligned} \quad (23)$$

contains the components of the k-th subband input signal in the bands that are non-adjacent to the k-th subband. Therefore, the matrix $\mathbf{\Phi}$ is formed by the correlation matrices of non-adjacent subband signals. Such matrices can be written in terms of the coefficients of the analysis filters $H_k(z)$ and of the input autocorrelation function, since

$$E\left[\widehat{\mathbf{X}}_{i,j}(m_j')\widehat{\mathbf{X}}_{r,s}(m_s')^T\right] = \widehat{\mathbf{H}}_{i,j}\mathbf{R}_{xx}\widehat{\mathbf{H}}_{r,s}^T, \quad (24)$$

and

$$E[\widehat{\mathbf{X}}_{i,j}(m_j')\widehat{\mathbf{X}}_{r,s}^{\downarrow}(m_s')^T] = \widehat{\mathbf{H}}_{i,j}\mathbf{R}_{xx}(\widehat{\mathbf{H}}_{r,s}^{\downarrow})^T. \quad (25)$$

In the above equations, $\mathbf{R}_{xx}$ is the input auto-correlation matrix, $\widehat{\mathbf{H}}_{i,j}$ contains the coefficients of $H_i(z)H_j(z)$ shifted to the right by

L_j positions from one row to the next one, with the first non-zero element of the first row at position $\Delta_i + 1$, and $\widehat{\mathbf{H}}^{\downarrow}_{i,j}$ is similar to $\widehat{\mathbf{H}}_{i,j}$ but with a shift of L_i positions from one row to the next one.

For selective analysis filters, the adaptive coefficients will converge to approximately the optimal coefficients of the sparse structure. Hence, the vector $\mathbf{G}^*$ will contain approximately the coefficients of the filters $G^*_k(z)$ of Eq. (11).

From (21), the steady-state MSE of the non-uniform subband structure will be, in general, larger than the measurement noise variance σ_v^2, because of the residual aliasing not canceled in the simplified structure. The corresponding increase in the steady-state MSE is related to the stopband attenuation of the analysis filters, and can be estimated from the following simplified expression:

$$\xi_{ss} \approx \lambda_{mean}(\mathbf{\Phi})|\mathbf{G}^*|^2 + \sigma_v^2, \qquad (26)$$

where $\lambda_{mean}(\mathbf{\Phi})$ is the mean of the eigenvalues of the matrix $\mathbf{\Phi}$. Observe that such simplified estimate for the excess in the MSE does not require knowledge of the optimal coefficients, but only of the squared norm of the corresponding vector, which can be easily estimated. It is also a function of the coefficients of the analysis filters and of the input autocorrelation sequence.

5. EXPERIMENTAL RESULTS

In this section, we compare the theoretical and experimental steady-state mean-square errors obtained in the identification of an FIR system of length $N_S = 128$. The coefficients of the unknown system were obtained randomly (white-noise with uniform distribution). The simulations were performed with both white and colored input signals. The colored input signal was generated by passing a gaussian white noise through a first-order IIR filter having its pole at $z = 0.9$. The proposed non-uniform subband structure was simulated with $M = 4$ subbands, employing 3-level octave-band tree structured filter banks with prototype filters [7] of order N_{H_3}= 15, 31 and 63. The theoretical estimates of the steady-state MSE (ξ_{ss} in Eqs. (21) and (26)) as well as the experimental MSE evolution obtained with the three prototype filters are shown in Figs. 4 and 5 for white and colored noise inputs, respectively. From these figures,

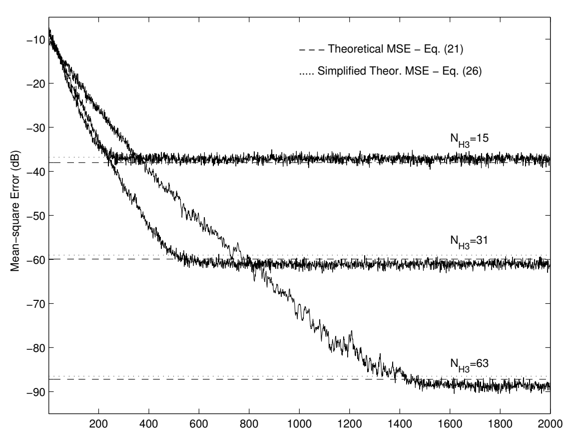

Fig. 4. MSE evolution and theoretical steady-state MSEs for white input.

we observe that the theoretical steady-state mean-square values are in good agreement with the experimental results.

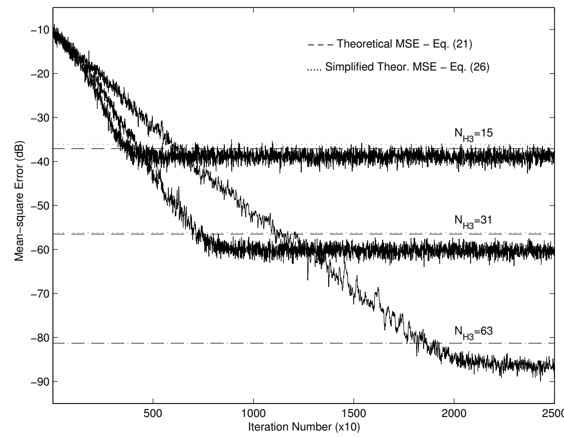

Fig. 5. MSE evolution and theoretical steady-state MSEs for colored input.

6. CONCLUSION

In this paper, we derived a critically sampled adaptive subband structure that employs a non-uniform filter bank to decompose the input signal. The adaptation is performed at the lowest sampling rate, using an LMS-type algorithm with step-size normalization. A convergence analysis of the proposed adaptive algorithm was presented, from which the steady-state MSE, with the residual aliasing in the subband structures taken into account, could be estimated. Simulation results confirmed the theoretical analysis results.

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