

SEQUENTIAL MCMC ESTIMATION OF NONLINEAR INSTANTANEOUS FREQUENCY

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ABSTRACT

Instantaneous frequency (IF) estimation of signals with nonlinear phase is challenging, especially for online processing. In this paper, we propose IF estimation using sequential Bayesian techniques, by combining the particle filtering method with the Markov chain Monte Carlo (MCMC) method. Using this approach, a nonlinear IF of unknown closed form is approximated as a linear combination of the IFs of non-overlapping waveforms with polynomial phase. Simultaneously applying parameter estimation and model selection, the new technique is extended to the IF estimation of multicomponent signals. Using simulations, the performance of this sequential MCMC approach is demonstrated and compared with an existing IF estimation technique using the Wigner distribution.

Index Terms— Frequency estimation, Bayes theorem, particle filter, Markov chain Monte Carlo

1. INTRODUCTION

Many naturally occurring signals have time-varying (TV) spectral characteristics. Whereas sinusoids have a single frequency over all time, linear frequency-modulated (FM) chirps (used in radar and sonar) have frequencies that vary linearly with time, and shallow-water acoustic waves undergo dispersive (nonlinear) frequency modulations.

For a single component signal,

$$s(t) = A(t)e^{j2\pi c\varphi(t)}, \quad (1)$$

with amplitude $A(t)$, phase $\varphi(t)$ and FM rate c , the frequency at a particular time can be described by the IF:

$$\zeta_s(t) = c \frac{d}{dt} \varphi(t). \quad (2)$$

In many real-life applications such as radar, sonar, underwater acoustics and structural health monitoring, the IF can be a powerful tool in estimating important parameters of the signal. For example, it was shown in [1] that varying $\zeta_s(t)$

from a linear to a hyperbolic function resulted in better tracking estimation. Furthermore, the ability to estimate the IF of a signal propagating through an unknown environment may provide information useful in characterizing the environment. As a result, it is important to be able to accurately estimate the IF to obtain the signal's spectral variation with time.

In this paper, we begin with a brief overview of the current IF estimation methods and their shortcomings in Section 2. In Section 3, we propose a new IF estimation method based on the use of sequential Bayesian techniques. Specifically, we combine a particle filter with MCMC to estimate static parameters of the IF of windowed signal segments. We extend this to signals with multiple components using model selection. Some simulation results that demonstrate the effectiveness of the new approach are presented in Section 4.

2. IF ESTIMATION METHODS

Parametric and non-parametric approaches to IF estimation are commonplace in the literature. Non-parametric IF estimation techniques do not assume a mathematical model for a signal and instead use time-frequency (TF) signal representations to describe the IF in the TF plane. Conversely, parametric IF estimation methods assume a specific signal model, as in (1), and then proceed to estimate the IF.

TF analysis is a well-known processing tool that has been applied to the IF estimation of signals as a non-parametric approach as it provides a natural means to display the signal spectrum at every time instant. The Wigner distribution (WD), $W_s(t, f) = \int s(t + \tau/2)s^*(t - \tau/2)e^{-j2\pi f\tau}d\tau$, of a signal $s(t)$ is a very popular time-frequency representation (TFR) that can provide high resolution along the signal's IF. In particular, it can be shown that the IF in (2) is $\zeta_s(t) = \int f W_s(t, f)df$. However, for multiple component signals, $\sum_l A_l(t)e^{j2\pi c\varphi_l(t)}$, with nonlinear phase $\varphi_l(t)$, the IF in (2) no longer provides the matched TV spectrum since (2) is not a linear representation and the WD is a quadratic TFR that suffers from cross terms for multicomponent signals.

The IF can also be estimated by extracting ridges (or peaks) of a TFR and then applying a peak detection algorithm to estimate the IF, $\hat{\zeta}_s(t) = \arg\{\max_{f \in \mathcal{F}} \{T_s(t, f; h)\}\}$, where

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$T_s(t, f; h)$ is a quadratic TFR that depends on a smoothing window $h(t)$ and \mathcal{F} contains frequency values for which the TFR is non-zero. An algorithm that used a smoothed WD with an adaptive window width determination to improve the estimation performance was used in [2]. Although this can improve the performance of the WD for multicomponent signals, the smoothing may smear the estimated IF values. IF estimation of seismic data was used in conjunction with the reassignment method in [3]. A drawback of the original reassignment method was a loss of performance in low signal-to-noise ratio (SNR) environments. To mitigate noise-sensitivity, a multi-taper TF reassignment method was proposed in [4].

IF estimation methods have also been developed based on parametric and often statistical models to describe the signal. For example, Newton's method was used in [5] to determine the maximum of the log-likelihood function, but this only worked for the joint estimation of frequency and FM rate of a signal in white Gaussian noise. In [6], a Bayesian approach for IF estimation was proposed by using the MCMC method with parametric processes when prior knowledge on the statistical properties of the processes was available.

3. SEQUENTIAL MCMC IF ESTIMATION

The previously mentioned methods do not perform well when (i) the IF is nonlinear, (ii) a closed form expression for the signal is unavailable, or (iii) the signal is multicomponent. Moreover, the estimates may need to be computed offline; in a wide variety of applications, estimates are required to be updated online and executed sequentially. Finally, the implementation method may require storage of intermediate data leading to excessive storage costs and increased computational complexity as additional measurements are computed.

In this work, we propose a sequential Bayesian IF estimator that combines a particle filter with MCMC methods for a non-parametric signal model. Without assuming the existence of a closed form expression for the phase $\varphi(t)$, we window the signal into non-overlapping segments with each segment modeled with polynomial phase with unknown coefficients. Thus, we approximate the IF $\zeta_s(t)$ as a linear combination of the derivatives of the polynomial functions. The static coefficients are estimated sequentially for each window: a particle filter allows for online processing and the MCMC increases estimation accuracy and reduces storage cost and computational complexity. Since the IF in each window can be approximated by polynomial, the problem is reduced to determine the polynomial coefficients. As expected, the larger the degree, the better the estimation accuracy. However, there is a trade-off between accuracy and computational complexity.

3.1. Estimation of Static Parameters

As particle filtering was designed for dynamic parameter estimation, it was shown to fail when used to estimate static

parameters [7]. Following the method discussed in [8], we estimate the parameter vector in each window by combining sequential importance sampling particle filtering with MCMC methods. Next, we provide the main steps of this sequential MCMC (SMCMC) approach.

Suppose at time step k , the particles and their corresponding weights, $(\mathbf{x}^1, w_k^1), (\mathbf{x}^2, w_k^2), \dots, (\mathbf{x}^{N_s}, w_k^{N_s})$, are used to represent the conditional probability density function $p(\mathbf{x}|\mathbf{Z}_k)$, where N_s is the number of particles, \mathbf{x} is the static parameter vector to be estimated, \mathbf{z}_k is the observation vector at time step k and $\mathbf{Z}_k = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k]$. The SMCMC method updates the weight for each particle using:

$$w_{k+m}^i \propto p(\mathbf{z}_{k+1}, \mathbf{z}_{k+2}, \dots, \mathbf{z}_{k+m} | \mathbf{x}^i, \mathbf{z}_k) w_k^i, \quad i = 1, \dots, N_s$$

where m is the batch size. Next, a rejuvenation test is performed using the Kullback-Leibler distance measure [8]. Specifically, if severe degeneracy occurs [8], indicated by the test, the MCMC method is applied using the independent Metropolis-Hastings (IMH) algorithm [9] with the Gaussian distribution $\mathcal{N}(\mathbf{x} | \mu_{\mathbf{x}}, \Sigma_{\mathbf{x}})$ as the proposal density, where $\mu_{\mathbf{x}} = \sum_{i=1}^{N_s} w_{k+m}^i \mathbf{x}^i$ and $\Sigma_{\mathbf{x}} = \sum_{i=1}^{N_s} w_{k+m}^i (\mathbf{x}^i - \mu_{\mathbf{x}})(\mathbf{x}^i - \mu_{\mathbf{x}})^T$. New particles and weights are then obtained by sampling from this Gaussian distribution, representing $p(\mathbf{x} | \mathbf{Z}_{k+m})$.

When \mathbf{x} can originate from different types of signal structures $\{H_1, H_2, \dots, H_M\}$, model selection is used simultaneously [8]. Let the parameter vector for model j be $\mathbf{x}^{(j)}$, then, the resulting distribution can be modified by:

$$p(\mathbf{x} | \mathbf{Z}_k) = \sum_{j=1}^M P(H_j | \mathbf{Z}_k) p(\mathbf{x}^{(j)} | \mathbf{Z}_k, H_j), \quad (3)$$

where $P(H_j | \mathbf{Z}_k)$ is the probability of model H_j given measurements \mathbf{Z}_k , and $p(\mathbf{x}^{(j)} | \mathbf{Z}_k, H_j)$ is obtained using SMCMC given model H_j . The model probabilities are updated sequentially, and after a few iterations, only one model has probability 1.

3.2. IF Estimation of Single Component Signals

The IF of a single component signal (that can be decomposed into basis functions which are non-overlapping in time) is given by the linear combination of the IFs of these basis functions. Specifically, assume that $s(t)$ can be decomposed as:

$$s(t) = \sum_{l=1}^L A_l(t) e^{j2\pi\varphi_l(t)} p_l(t),$$

where $p_l(t) = u(t - (l-1)T) - u(t - lT)$ is a rectangular window and $u(t)$ is the unit step. Then we can show that the IF of $s(t)$ is given by $\zeta_s(t) = \sum_{l=1}^L \frac{d}{dt} \varphi_l(t) p_l(t)$. Therefore, given a waveform with an unknown nonlinear phase $\varphi(t)$, we propose to approximate its IF $\zeta_s(t)$ using a linear

combination of IFs of FM waveforms with polynomial phase $\varphi_l(t) = \sum_{n=0}^N a_{n,l} t^n$ and duration T . Specifically,

$$\zeta_s(t) \approx \sum_{l=1}^L \sum_{n=1}^N n a_{n,l} t^{n-1} p_l(t). \quad (4)$$

With this assumption, the estimation of the IF becomes the estimation of a set of unknown static parameter vectors $\mathbf{x}_l = [a_{1,l}, a_{2,l}, \dots, a_{N,l}]$, $l = 1, \dots, L$ using the SMC MC method described in Section 3.1. In each window l , a new set of particles is used to estimate the parameter vector \mathbf{x}_l . To reduce approximation errors, we consider short duration windows $p_l(t)$; the window length, however, has to be long enough to reduce errors in estimating the polynomial coefficients.

3.3. IF Estimation of Multiple Component Signals

For a multicomponent signal, the IF definition in (2) is most often not appropriate as it will result in a single curve [10]. Instead, we need to obtain a representation of the IF of a multicomponent signal as the sum of the IFs of its single components; thus, the IF of a multicomponent signal will consist of multiple curves provided that we can decompose a multiple component signal into single components. If we assume that the number, Q , of components of a multicomponent signal is known, and that the signal can be decomposed as:

$$s(t) = \sum_{q=1}^Q \sum_{l=1}^L A_l^{(q)}(t) e^{j2\pi\varphi_l^{(q)}(t)} p_l(t),$$

where $A_l^{(q)}(t)$ and $\varphi_l^{(q)}(t)$ are the amplitude and phase of the q th component in the l th window, respectively, then the IF of this component is approximately:

$$\zeta_l^{(q)}(t) \approx \sum_{n=1}^N n a_{n,l}^{(q)} t^{n-1}, \quad (5)$$

where $a_{n,l}^{(q)}$ is the n th coefficient of component q in the l th window. Therefore, in the l th window, Q parameter vectors, $\mathbf{x}_l^{(q)} = [a_{1,l}^{(q)}, \dots, a_{N,l}^{(q)}]$, $q = 1, \dots, Q$, are estimated using SMC MC with a new set of particles for each vector.

Our approach can be extended to the IF estimation of multiple component signals when the number of components is unknown. In this case, we assume that there are several models corresponding to the number of components present in the l th window. For each model, Q_l FM waveforms ($Q_l \in [1, 2, \dots, Q]$) are used to approximate the IFs of the Q_l components in the l th window with Q_l sets of polynomial coefficients $\mathbf{x}_l^{(q)} = [a_{1,l}^{(q)}, \dots, a_{N,l}^{(q)}]$, $q = 1, \dots, Q_l$ as the parameter vectors to be estimated. The number of components Q_l , which may vary in different windows, is determined using model selection. Specifically, if the maximum number of

components is known to be Q , there are totally $Q + 1$ models to choose from:

$$\begin{aligned} H_0 : z_l(t) &= v_l(t) \\ H_1 : z_l(t) &= A_l(t) e^{j2\pi\varphi_l(t)} + v_l(t) \\ &\vdots \\ H_Q : z_l(t) &= \sum_{q=1}^Q A_l^{(q)}(t) e^{j2\pi\varphi_l^{(q)}(t)} + v_l(t). \end{aligned}$$

Then the state vector is estimated using (3). After a few steps, the model H_{Q_l} is selected if the model probability $P(H_{Q_l}|Z_k)$ increases to 1. The Q_l signal component parameter vectors in the l th window are estimated simultaneously.

4. SIMULATIONS

With the order $N = 2$ in (4), linear approximation works best when there are no sudden changes in the IF and if the window length is small. We demonstrate the estimation performance by comparing the true and estimated IF of a 9 dB SNR noisy single component signal in Fig. 1 (a) with window length 500 samples. The time averaged root mean square error (RMSE) for varying SNR is shown in Fig. 1 (b).

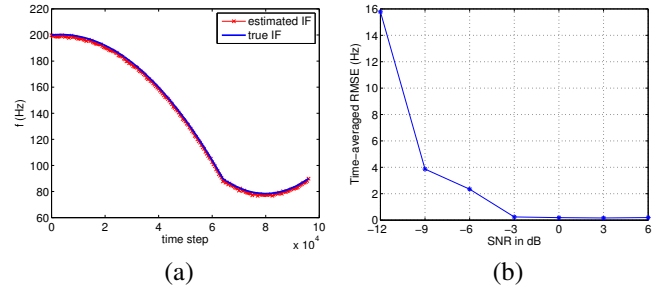


Fig. 1. IF estimation using linear approximation: (a) estimated compared with true IF, (b) RMSE vs SNR.

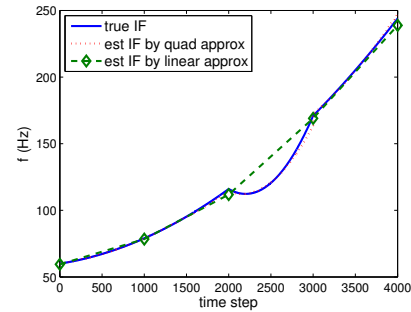


Fig. 2. Comparison of IF estimation of a single component signal using quadratic and linear approximation.

Quadratic approximation (polynomial with order $N = 3$) is performed in Fig. 2, and compared with linear approximation. We used a 9 dB SNR signal and 1000 samples window length. It can be seen that for this case, where there is a sudden change in the IF, the quadratic approximation is much

better than the linear one, resulting in a smaller estimation error. The averaged RMSE is 4.13 Hz for the linear approximation and 0.80 Hz for the quadratic. Note that the window length was chosen large in this example for comparison. The larger the window length, the worse the linear approximation.

Next, we demonstrate the use of SMCMC with model selection for multicomponent signal IF estimation. Fig. 3 shows the estimation of two component signals overlapping in time as they approach each other in frequency. The SNR is 9 dB and the window length is 500. The performance demonstrates the advantage of our method even when the two components are very close. Note that the IF estimation in Fig. 3 (b) is very difficult for most IF estimation methods.

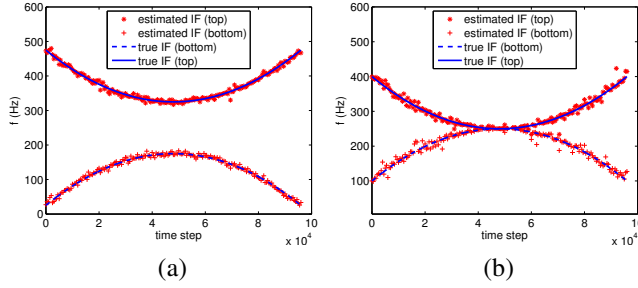


Fig. 3. SMCMC IF estimation; signal components are (a) not overlapping and (b) overlapping in frequency.

We compare the performance of our approach with the WD IF estimation technique. For comparison, the method in [11] is extended to short-time processing by executing the WD IF estimation on windowed data segments of length 512 samples with 5 Monte Carlo simulations. Specifically, we estimate linear FM parameters of the WD of windowed segments by integrating over all possible lines in the TF plane [11]. The simulation result is shown in Fig. 4 for 9 dB SNR. The WD suffers from cross terms when it is applied to multiple component signals. If two curves are close in the TF plane, the WD IF estimation method can pick the cross term location as the signal component; this is not the case with the SMCMC. Note that to implement the WD estimation, the number of signal components has to be known; only the maximum number of components is assumed known for the SMCMC. Also, the computational complexity and storage cost are reduced dramatically by our approach. The new method takes less than half of the CPU time and a quarter of the storage cost when compared to the short-time WD approach.

5. CONCLUSION

In this paper, we proposed an approach for the IF estimation of both single and multiple component signals with nonlinear phase that may not exist in closed form. Our approach approximates the IF of a signal by a linear combination of the IFs of windowed FM waveforms with polynomial phase.

Although we have shown results comparing our method to the short-time WD, we plan to consider adaptive windows in the SMCMC (to improve performance in lower SNRs) and to compare our method with the multi-taper reassignment in [4].

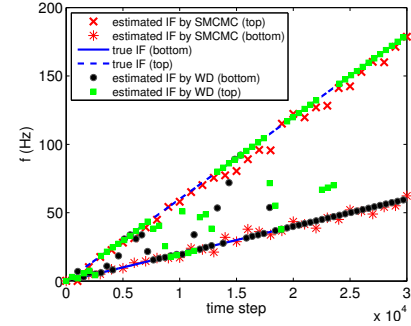


Fig. 4. IF estimation of two crossing linear FM chirps using linear approximation SMCMC (stars and crosses), compared with short-time WD (dots and squares).

6. REFERENCES

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