MODIFIED INVERSE *S* TRANSFORM FOR FILTERING IN TIME-FREQUENCY SPECTRUM

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ABSTRACT

The *S* transform is a powerful linear time-frequency representation, and it is especially useful for filter in the time-frequency spectrum. To improve the time resolution of the filter applications, Schimmel and Gallart have proposed a new method for inverse *S* transform. This new method performs much better than the original in time localization with time-frequency filters. However, there exists some distortion of the inverse transformed signal that it will be enhanced in the positive but suppressed in the negative frequency part. In this paper we will demonstrate the distortion and propose a modified method to derive the correct inverse transformed signal and provide satisfactory time localization in filtering.

***Index Terms*—** time-frequency representation, filter, *S* transform

1. INTRODUCTION

Time-frequency representation is a powerful tool for signal analysis, since it can investigate the signal varying in time and frequency domain simultaneously. A lot of methods have been proposed, and among them two of the most well-known may be Wigner distribution and Short Time Fourier Transform (STFT) [1]. The Wigner distribution provides excellent resolution but suffers the crosstalk problem because it is a bilinear transform. On the other hand, STFT has no crosstalk but poorer resolution. It is linear with a sliding window, whose width is fixed and determines the time-frequency resolution.

To improve the resolution of STFT, and get adaptive resolution with varying frequency, as wavelet transform, the *S* transform was proposed by Stockwell [2]. The main difference between *S* transform and STFT is that the width of window varies with frequency, and therefore it provides progressive resolution. The *S* transform adopts the Gaussian window, whose area under the window is equal to one, and therefore the sum of the *S* spectrum in the time direction becomes the spectrum of the original signal. This property provides an efficient algorithm of inverse *S* transform.

Generally speaking, most filters are designed in the frequency domain and have the same effect to the whole time series. However, in the time-frequency representation, there is much more flexibility in filter design. Considering the linear *S* transform, for example, the filter may consist of data adaptive weighting window with higher values for signal and lower for noise. By multiplying the window in the *S* domain, we can derive the filtered signal after inverse transformation. This kind of filter has been employed in many medical and seismic signal processing, like Zhu et al. [3] and Pinnegar et al. [4]. They identified the noise regions and down-weighted them to remove the artifacts.

To improve the time resolution in filters, Schimmel and Gallart [5] proposed a new algorithm of inverse *S* transform, which provides a much better time resolution than the original one in filters. However, there exists some distortion in this method. It will be enhanced the positive and suppressed the negative low frequency part of the signal.

In this paper, we illustrate this problem and propose a modified inverse *S* transform algorithm that solves this problem and provides satisfactory time resolution in filtering.

2. *S* TRANSFORM WITH TWO INVERSE ALGORITHMS

2.1. Generalized S transform and inverse algorithm

The *S* transform, derived by Stockwell et al. [2], of a time series $u(t)$ is

$$S(\tau, f) = \int_{-\infty}^{\infty} u(t) w(\tau - t, f) e^{-i2\pi ft} dt \tag{1}$$

where the $w(\tau,f)$ is the Gaussian window

$$w(\tau - t, f) = \frac{|f|}{k\sqrt{2\pi}} e^{\frac{-f^2(\tau-t)^2}{2k^2}}, \quad k > 0 \tag{2}$$

τ and t are the time and f is the frequency variables. τ is the center of the Gaussian window, and the width is controlled by f and k, which also determines the time-frequency resolution.

The similarity between *S* transform and STFT is that they are both derived from the Fourier transform of the time series multiplied by a time-shift window. However, unlike STFT, the width of window varies with frequency in *S* transform.

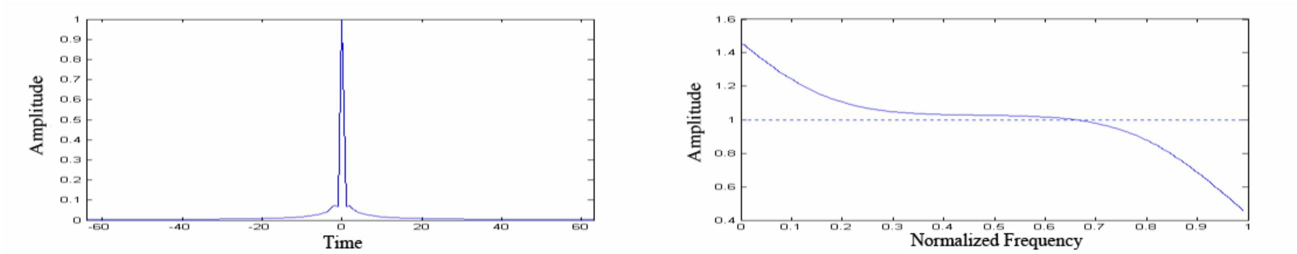

Fig. 1. The discrete series $m(kT)$ in (19) and its spectrum. The spectrum of the delta function is also plotted in dotted line for comparing. One can find that the positive low frequency is enhanced and the negative low frequency is suppressed.

To simplify the computation, Stockwell *et al.* also proposed another form of *S* transform [2]

$$S(\tau,f)=\int_{-\infty}^{\infty}U(\alpha+f)e^{-\frac{2\pi^2k^2\alpha^2}{f^2}}e^{i2\pi\alpha\tau}d\alpha \tag{3}$$

$U(\alpha)$ is the Fourier transform of $u(t)$. Therefore, the *S* transform can be calculated by the inverse Fourier transform.

The inverse transform algorithm is simple since the Gaussian window satisfy the condition

$$\int_{-\infty}^{\infty}w(\tau-t,f)d\tau=1 \tag{4}$$

and therefore the integral of S spectrum over time becomes the frequency spectrum

$$\int_{-\infty}^{\infty}S(\tau,f)d\tau=\int_{-\infty}^{\infty}u(t)\left[\int_{-\infty}^{\infty}w(\tau-t,f)d\tau\right]e^{-i2\pi ft}dt=U(f) \tag{5}$$

It means that the S spectrum is invertible and gives an easy and efficient inverse transform algorithm

$$u_1(t)=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}S(\tau,f)d\tau\right]e^{i2\pi ft}df \tag{6}$$

Filtering in time-frequency representation, such as *S* domain, can be considered as multiplying the spectrum $S(\tau,f)$ with a weighting function $F(\tau,f)$, that is assigned high values to signals and low ones to noise. Consequently the filtered output time series is

$$u_{filter1}(t)=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}S(\tau,f)F(\tau,f)d\tau\right]e^{i2\pi ft}df \tag{7}$$

The imposed time localization of the filter may not correctly translate to output time series, because integration over time eliminate the time localization of the filter.

2.2. Schimmel's inverse *S* transform

To keep the time localization property of the filter, another inverse transform algorithm was proposed by Schimmel *et al.* [5]. They found the time-time representation of the windowed time series $u(t)$

$$x(\tau,t)=u(t)e^{\frac{-f^2(\tau-t)^2}{2k^2}},\quad k>0 \tag{8}$$

which reduces to $u(t)=x(t,t)$ at $\tau=t$. They claimed that the Fourier transform of (8) is the main element of S transform

$$x(\tau,f)=\int_{-\infty}^{\infty}u(t)e^{\frac{-f^2(\tau-t)^2}{2k^2}}e^{-i2\pi ft}dt=\frac{k\sqrt{2\pi}}{|f|}S(\tau,f) \tag{9}$$

Therefore the original time series can be obtained by

$$u_2(t)=x(t,t)=\int_{-\infty}^{\infty}x(t,f)e^{i2\pi ft}df=k\sqrt{2\pi}\int_{-\infty}^{\infty}\frac{S(t,f)}{|f|}e^{i2\pi ft}df \tag{10}$$

In analogy to (7), the output time series after filtering is

$$u_{filter2}(t)=k\sqrt{2\pi}\int_{-\infty}^{\infty}\frac{S(t,f)F(t,f)}{|f|}e^{i2\pi ft}df \tag{11}$$

The major difference between (6) and (10) is that in (10) it integrates over frequency. For every instant, the inverse transformed time signal can be obtained just by the mapping spectrum information.

3. THE MODIFIED INVERSE *S* TRANSFORM

However, we found that the retrieved series by (10) is not identical to the original.

$$\begin{aligned}u_2(t)&=k\sqrt{2\pi}\int_{-\infty}^{\infty}\frac{S(t,f)}{|f|}e^{i2\pi ft}df\\&=\int_{-\infty}^{\infty}u(s)\int_{-\infty}^{\infty}e^{\frac{-f^2(t-s)^2}{2}}e^{i2\pi f(t-s)}df\,ds\\&=\int_{-\infty}^{\infty}u(s)m(t-s)ds=u(t)\otimes m(t)\end{aligned} \tag{12}$$

where $\otimes$ is the convolution and

$$m(t)=\int_{-\infty}^{\infty}e^{\frac{-f^2t^2}{2}}e^{i2\pi ft}df \tag{13}$$

It shows that the retrieved signal is the convolution of the original signal $u(t)$ and $m(t)$. If $m(t)$ is the delta function, then the retrieved signal will be identical to the original. However, as demonstrated in Fig. 1, $m(t)$ just approximates the delta function, and the spectrum of $m(t)$ reveals that this algorithm will enhance the positive but suppress the negative low frequency components.

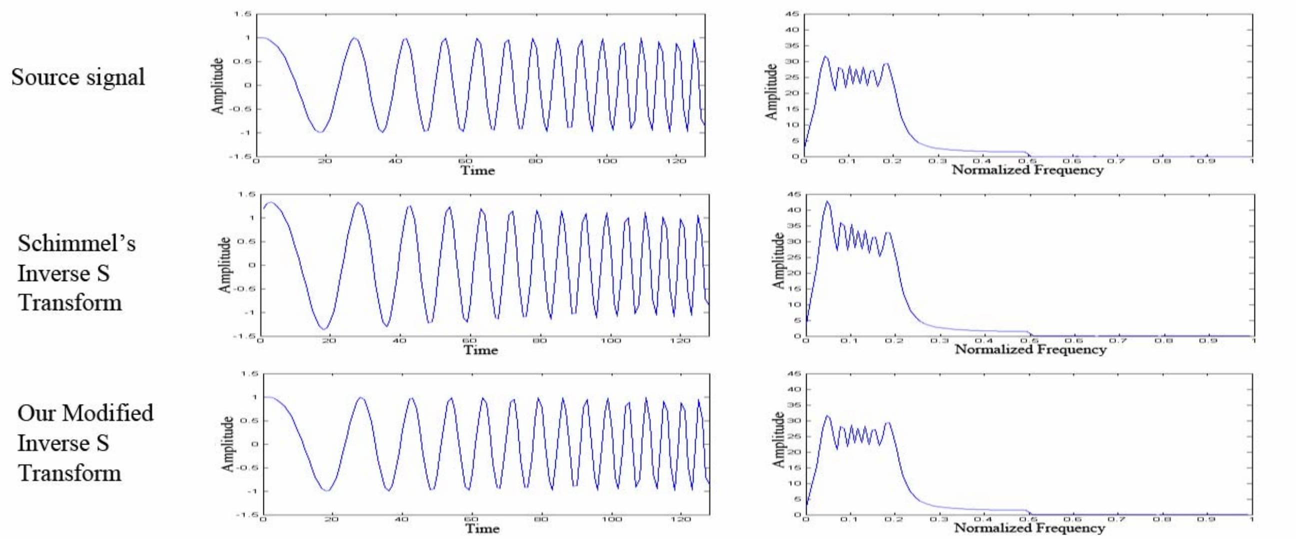

Fig. 2. The source chirp series in our example along with the retrieved series by Schimmel's and our inverse algorithms. Distortion of Schimmels method in low frequency component is obvious. We perform Hilbert transform before S transform to make the signal analytic and avoid self-aliasing problem. Therefore the energy concentrates in the positive spectrum.

To fix this problem and derive the correct inverse transformed signal, one should deconvolute the retrieved signal in (10) with $m(t)$.

$$u_3(t) = IFT\left\{\frac{FT[u_2(t)]}{FT[m(t)]}\right\}$$

(14)

where $m(t)$ is signal independent.

For filtering in time-frequency spectrum as in (11), our algorithm also reduces the distortion

$$u_{filter3}(t) = IFT\left\{\frac{FT[u_{filter2}(t)]}{FT[m(t)]}\right\} \quad (15)$$

4. DISCRETE IMPLEMENTATION ALGORITHM

Letting $f \rightarrow n/NT$, $\tau \rightarrow jT$ and $t \rightarrow kT$ where T is the sampling interval and j and $n = 0, \ldots, N\text{-}1$, the discrete S transform [2] of time series $u[kT]$ is

$$S\left[jT, \frac{n}{NT}\right] = \sum_{l=-N/2}^{N/2-1} U\left[\frac{l+n}{NT}\right] e^{-\frac{2\pi^2 l^2}{n^2}} e^{\frac{i2\pi lj}{N}}, \quad n \neq 0 \quad (16)$$

$$S[jT, 0] = \frac{1}{N}\sum_{k=0}^{N-1} u(kT) \quad (17)$$

the discrete algorithm of Schimmel's method is

$$u_2(kT) = S[kT, 0] + \sum_{n=1}^{N-1} \frac{\sqrt{2\pi}}{|n|} S\left[kT, \frac{n}{NT}\right] e^{i2\pi nk} \quad (18)$$

let $m[kT]$, $k = -N/2, \ldots, N/2-1$, denote the periodic series corresponding to $m(t)$ in (13)

$$m(kT) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-\frac{n^2 k^2}{2N^2}} e^{\frac{i2\pi nk}{N}} \quad (19)$$

The final inverse transformed discrete series are

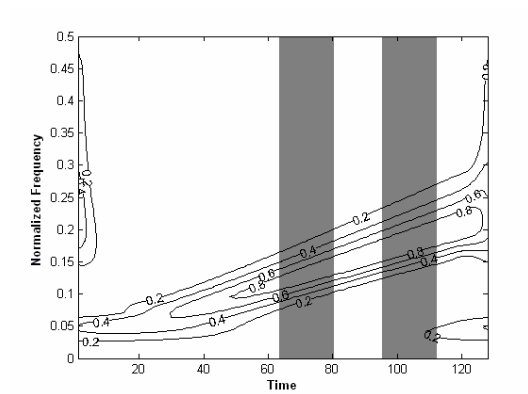

Fig. 3. S transform of the chirp series in Fig. 2. The gray regions are windows whose values are zero.

$$u_3(kT) = IFFT\left\{\frac{FFT[u_2(kT)]}{FFT[m(kT)]}\right\} \quad (20)$$

$$u_{filter3}(kT) = IFFT\left\{\frac{FFT[u_{filter2}(kT)]}{FFT[m(kT)]}\right\} \quad (21)$$

5. EXAMPLES

5.1. Example 1: Distortion of Schimmel's method

Considering a chirp series (assume $N = 128$ and $k = 0\sim N\text{-}1$)

$$c(k) = \cos(2\pi(2 + k/10)k/N)$$

In order to compare the performance of three inverse algorithms, we apply Hilbert transform, S transform and then inverse S transform by Stockwell's original method, Schimmel's new method, and our modified method. The Hilbert transform is performed before S transform to make the

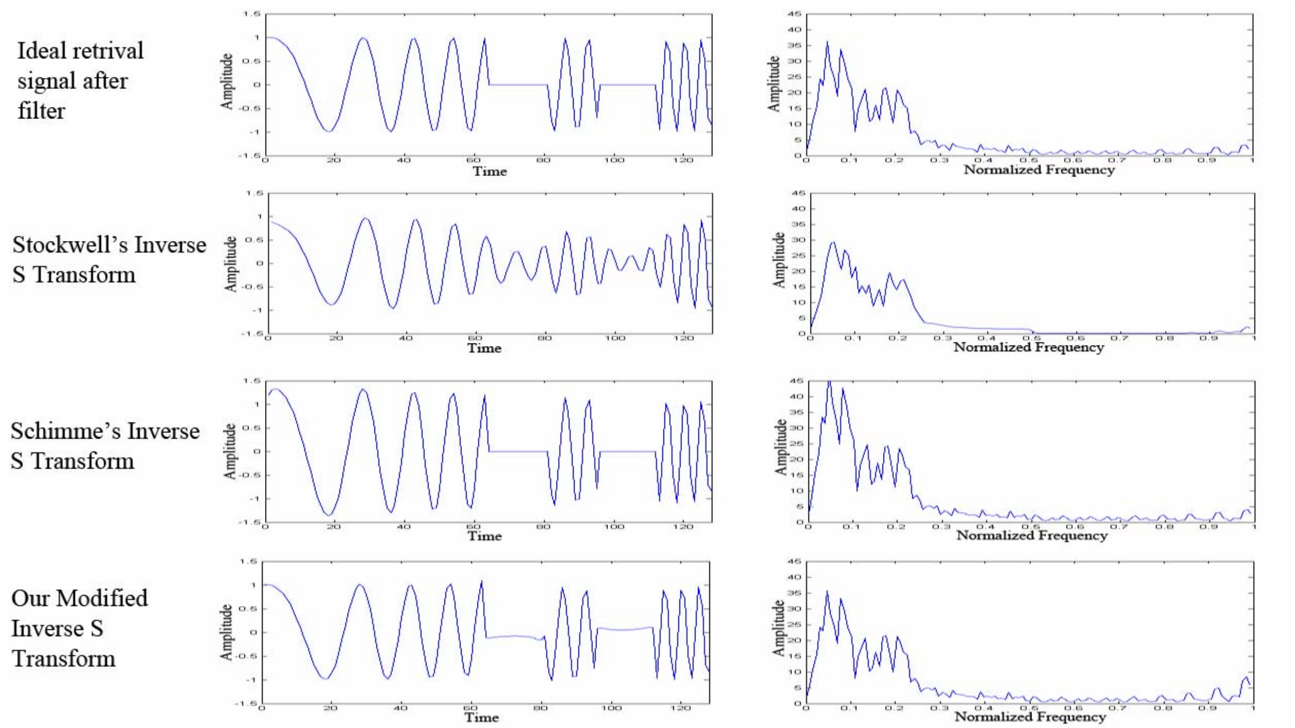

Fig. 4. The expected ideal retrieved series along with the actual result from Stockwell's, Schimmel's and our method.

series analytic and avoid the self-aliasing problem. The source series is plotted in Fig. 2 along with the result of inverse algorithms by Schimmel's and our method. The result of Stockwell's method does not present here since it is identical to the original chirp series. If we define the mean-square-error as

$$MSE = \sum_{n=0}^{N-1}\left[\hat{s}(n) - s(n)\right]^2 / N$$

the MSE of Schimmel's method is 3.02E-002, and the MSE of our method is 6.93E-030 in this example.

5.2. Example 2: Modified inverse *S* transform in filtering

Considering the same chirp series, whose time-frequency representation by S transform is plotted in Fig. 3. The gray regions in Fig. 3 are the windows whose values are zero. The chirp is multiplied by the windows in the time-frequency domain. The expected ideal retrieved series after filtering is plotted in Fig. 4 along with the actual retrieved series by Stockwell's, Schimmel's and our modified method. It is obviously that the time localization of Stockwell's inverse transform method is very poor. Schimmel's method performs excellent time resolution but also brings some distortion. Our method fixes the distortion of Schimmel's and provides much better time localization than Stockwell's. In this example, the MSE of Stockwells method is 0.0311, Schimmel's method is 0.0265, and our method is 0.0030 respectively.

6. CONCLUSION

The inverse S transform algorithm is discussed and a new method is proposed here to complete the one by Schimmel *et al.* [5]. It derives the correct back transformed signal and also conserves satisfactory time localization property in filtering. Besides, it is computationally efficient and can be implemented by fast Fourier transform.

7. REFERENCES

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