S-TRANSFORM WITH FREQUENCY DEPENDENT KAISER WINDOW

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ABSTRACT

A time-frequency signal analysis tool, known as S-transform, can suffer from poor energy concentration in the time-frequency domain. In this paper, a frequency dependent Kaiser window is presented for improving the energy concentration of the S-transform. The new window is analyzed using a set of test signals. The results indicate that the proposed scheme can significantly improve the energy concentration in the time-frequency domain in comparison with the standard S-transform.

***Index Terms*—** Time-frequency analysis, window function, Kaiser window, Gauss window, concentration measure.

1. INTRODUCTION

The S-transform is a conceptual hybrid of short-time Fourier analysis and wavelet analysis. It employs variable window length and by using the Fourier kernel, the phase information provided by the S-transform is referenced to the time origin. Hence, it provides supplementary information about spectra which is not available from locally referenced phase obtained by the continuous wavelet transform [1]. The S-transform has already been used in several fields [2]-[11].

Even though the S-transform is a valuable tool for analysis of signals in many applications, still in some cases, it suffers from poor energy concentration in the time-frequency domain. Several improvements of the time-frequency representation of the S-transform have been reported. A generalized S-transform, proposed in [6] [7], provides greater control of the window function, and the algorithm also allows nonsymmetric windows to be used. Few window functions are considered, including two forms of exponential functions, amplitude modulation and phase modulation by cosine functions [6], and a bi-Gaussian window [8]. However, in all existing literature, none have considered how the proposed windows affect the autoterm [12] of the S-transform. It should be noted that it is desired to have a similar autoterm behaviour as in the standard S-transform in order to maintain the desirable properties of the S-transform, such as, sharp time resolution at higher frequencies or good frequency resolution at lower frequencies.

The main contribution of this paper is to implement a frequency dependent Kaiser window for the S-transform. The frequency dependence of the window is achieved through a window parameter α, and the variation of this parameter is proposed to provide diminished leakage of the signal components. The proposed scheme is tested using a set of synthetic test signals. Using these signals, the S-transform with the frequency dependent Kaiser window is evaluated and compared with the standard S-transform. The results have shown that the proposed window improves the energy concentration as compared with the Gaussian window.

This paper is organized as follows. In Section 2, the concept of the S-transform is introduced, along with the detailed theoretical development of the proposed algorithm. Section 3 evaluates the performance of the proposed scheme using test signals. Conclusions are drawn in Section 4 followed by a list of references.

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2. THE PROPOSED SCHEME

2.1. Standard S-transform

The standard S-transform of a function $x(t)$ is given by a convolution integral in [1] as:

$$S_x(t,f) = \int_{-\infty}^{+\infty} x(\tau)w(\tau-t,f)\exp(-j2\pi f\tau)d\tau \qquad (1)$$

with a constraint $\int_{-\infty}^{+\infty} w(t-\tau,f)d\tau = 1 \ \forall f$. The window function used in the S-transform is actually a scalable Gaussian function defined as

$$w(t,f) = \frac{1}{\sigma(f)\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\sigma^2(f)}\right) \qquad (2)$$

and the advantage of the S-transform over the short-time Fourier transform (STFT) is that the standard deviation $\sigma(f)$ is a function of frequency, f, defined as

$$\sigma(f) = \frac{1}{|f|}. \qquad (3)$$

Then, the window function is a function of time and frequency. As the time domain width of the window is dictated by the frequency, it can easily be seen that the window is wider in the time domain for the lower frequencies, and narrower for the higher frequencies. In other words, the window provides good localization in the frequency domain for the low frequencies, while it provides good localization in time domain for higher frequencies.

2.2. S-transform with frequency dependent Kaiser windows

A simple improvement to the time-frequency concentration of the S-transform can be obtained by a frequency dependent Kaiser window defined as:

$$w_K(t,f) = \frac{I_0\left(\alpha(f)\sqrt{1-t^2}\right)}{I_0\left(\alpha(f)\right)} \qquad (4)$$

where $I_0(\cdot)$ is the zeroth-order Bessel function of the first kind, and $\alpha(f)$ is a frequency dependent parameter. In order to determine the optimal variation of α, two requirements should be met:

- The frequency dependent window should provide similar time-frequency tilings as the Gaussian S-transform.
- The variation of the window parameter α should be chosen such that the autoterm with the proposed window provides almost identical widths in the time and frequency domains, where these widths are defined in [13]. Additionally, the autoterm should have a narrow mainlobe and sidelobes with relatively smaller magnitude.

Let's consider the first requirement. A frequency dependent Kaiser window should achieve good frequency resolution at low frequencies, and sharp time resolution at higher frequencies. Hence, it is important to understand the behaviour of the Kaiser window for various values of the parameter α in order to satisfy this requirement. This is displayed in Fig. 1.

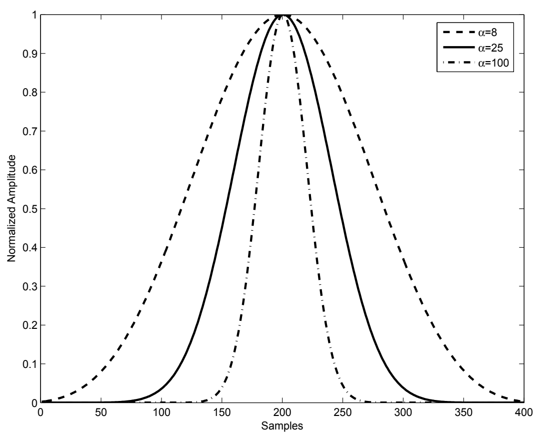

Fig. 1. Comparison of the Kaiser windows for various values of α.

It is clear from Fig. 1 that as the value of α increases, the window becomes narrower in the time domain. Therefore, for a frequency dependent Kaiser window to provide the same time-frequency tilings as the frequency dependent Gaussian window, the variation of α must be proportional to the frequency:

$$\alpha(f) \sim \beta f \tag{5}$$

where β is a constant, which is to be used in satisfying the second criterion.

The second requirement states that the proposed window should produce an autoterm with similar widths in the time and frequency domains to those produced with the Gaussian window, since the Gaussian functions are the only solution that minimizes the duration-bandwidth product posed by Heisenberg uncertainty principle [12]. Therefore, a further understanding of the autoterm obtained by the S-transform is needed. Let's consider a signal, $x(t) = A(t)\exp(j\phi(t))$, assuming that the signal satisfies $\left|A^{(1)}(t)/A(t)\right| \ll \left|\phi^{(1)}(t)\right|$[1]. If the following substitution $u = \tau - t$ is made in (1), then the squared magnitude of the S-transform of $x(t)$ is given by:

$$\begin{aligned} |S_x(t,f)|^2 &= \left| \int_{-\infty}^{+\infty} x(u+t) w(u,f) \exp(-j2\pi f u) du \right|^2 \\ &= \left| \int_{-\infty}^{+\infty} A(u+t) w(u,f) \exp(jf\mu(u)) du \right|^2 \end{aligned} \tag{6}$$

[1] $f^{(k)}(t)$ represents a k^{th} derivative of $f(t)$ with respect to t.

and $\mu(\bullet)$ is given by

$$\mu(u) = \frac{\phi(u+t)}{f} - 2\pi u. \tag{7}$$

Furthermore, let's assume that the product $A(\tau+t)w(\tau,f)$ satisfies the following condition:

$$A(u+t)w(u,f) \approx A(u)w(u,f). \tag{8}$$

Using the assumption made about the signal, that is, the phase of the signal varies faster than its amplitude, by applying the stationary phase method [14], the S-transform of the signal at some stationary point of the phase function, u_o, is then given by:

$$\begin{aligned} |S_x(t,f)|^2 &\cong \left| A(u_o) w(u_o,f) \sqrt{\frac{2\pi}{f\mu^{(2)}(u_o)}} \right. \\ &\qquad \left. \times \exp\left(j\frac{\pi}{4}\right) \exp\left[jf\mu(u_o)\right] \right|^2 \\ &\cong \frac{2\pi}{f\mu^{(2)}(u_o)} A^2(u_o) w^2(u_o,f). \end{aligned} \tag{9}$$

It is important to mention that the stationary point, u_o, is chosen such that

$$\mu^{(1)}(u_o) = 0. \tag{10}$$

Since (9) is a signal dependent expression, a linear FM signal, $x(t) = \exp\left(j\frac{a}{2}t^2\right)$, is considered as an example. Then, the S-transform is given by:

$$|S_x(t,f)|^2 \cong \frac{2\pi}{a} w^2\left(\frac{2\pi f + at}{a}, f\right). \tag{11}$$

Now, let's examine how the Gaussian and Kaiser windows affect the autoterm for the S-transform. Let's use $f = 3$ Hz and $a = 60\pi$. Also, for the Kaiser window, the effects of $\beta = \{1, \pi, 2\pi, 3\pi\}$ are considered as well. Fig. 2 represents the results of such analysis. The graphs on the left side represent $|S_x(t,f)|^2$, while the graphs on the right represent $10\log_{10}|S_x(t,f)|^2$. The results depict an interesting phenomenon. As the value of β is increased past π, i.e.

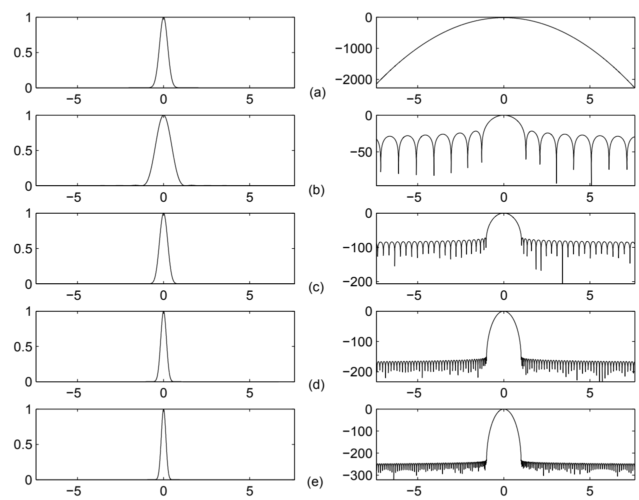

Fig. 2. Autoterm of the S-transform for $a = 60\pi$: (a) Gauss window; (b) Kaiser window with $\alpha(f) = f$; (c) Kaiser window with $\alpha(f) = \pi f$; (d) Kaiser window with $\alpha(f) = 2\pi f$; (e) Kaiser window with $\alpha(f) = 3\pi f$.

$\beta > \pi$, the width of the mainlobe remains approximately the same. Having almost the same width of the mainlobe, a further understanding of the widths of the autoterms in the time (Δt) and frequency (Δf) domains [13] is required for the Gaussian and Kaiser windows, in order to choose a proper value of β. The frequency and time widths of the autoterm obtained by the Gaussian window are equal to 1122 and 0.0002226, respectively. The results for the Kaiser window are shown in Table 1.

Table 1. The frequency and time widths for the proposed window.

	$\beta = 1$	$\beta = \pi$	$\beta = 2\pi$	$\beta = 3\pi$
Δf	3716	1089	540	359
Δt	0.0000683	0.0002297	0.0004630	0.0006962

From these results it is clear that the Gaussian window and a Kaiser window with $\beta = \pi$ provide almost the same widths in time and frequency domain. Other values of β provide results which either represent an autoterm which is narrower in the time domain (e.g. $\beta = 1$), hence wider in the frequency domain; or narrower in the frequency domain (e.g. $\beta = 2\pi$ and $\beta = 3\pi$), hence wider in the time domain. Inherently, these values of β would cause smearing of the autoterm in one of the axis, and leakage of the signal components.

Therefore, a linear variation of the parameter $\alpha(f)$ is proposed as

$$\alpha(f) = \pi f \tag{12}$$

since it provides the autoterm with almost the same widths in the time and frequency domain calculated according to [13] as the Gaussian window. However, it can be seen from Fig. 2 that this window produces a narrower window in the logarithmic scale and significantly reduced side-lobes meaning that this window form can provide improved time-frequency representation.

Further, it should be mentioned that additional improvement in the energy concentration of the S-transform with the frequency dependent Kaiser window may be achieved if an adaptive algorithm is derived for automatic evaluation of the parameter $\alpha(f)$. However, such adaptive scheme would inherently introduce additional computational burden.

3. PERFORMANCE ANALYSIS

In this section, the performance of the proposed scheme is examined using a set of synthetic test signals. The goal is to examine how the S-transform with the proposed window performs in comparison to the standard S-transform. The proposed algorithm is also compared to the short-time Fourier transform (STFT), in order to illustrate the enhanced performance of the S-transform with the proposed window.

Let's consider the following signal:

$$x_1(t) = \cos(20\pi t + 20\pi t^2) + \sin(150\pi t + 13\cos(4\pi t)) \tag{13}$$

where $0 \leq t < 1$, and the signal does not exist outside the given interval. For the STFT, a Gaussian window is also used in analysis, and its standard deviation is set to 0.01. The STFT provides relatively good time-frequency concentration for the sinusoidally modulated component, however, the chirp has very poor concentration. The S-transform with the Gaussian window provides good concentration of both components, however, some spectral leakage is evident from Fig. 3(c). Improvement to the time-frequency concentration is easily noticed if the signal is analyzed with the S-transform based on the frequency dependent Kaiser window. The concentration of the components is improved in comparison to both, the STFT and the S-transform with the Gaussian window.

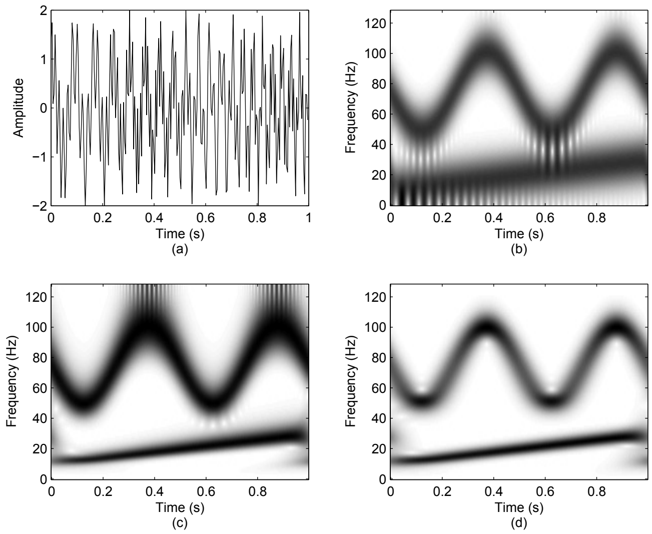

Fig. 3. Time-frequency analysis of $x_1(t)$: (a) Time-domain representation; (b) STFT; (c) S-transform with the Gaussian window; (d) S-transform with the proposed Kaiser window.

A slightly more complicated example is one with two crossing components defined as:

$$x_2(t) = \cos(20\pi \log(10t + 1)) + \cos(20\pi t + 90\pi t^2) \tag{14}$$

where $x_2(t) = 0$ outside $0 \leq t < 1$. The complexity of this signal lies in several facts. First, the components are crossing, and secondly, as frequency of the hyperbolic component decreases, the frequency of the linear chirp increases. Therefore, very often, it is difficult to provide good concentration for both. In the analysis, a Gaussian window is used for the STFT, and its standard deviation is set to 0.05. The time-frequency representation of the signal given by the

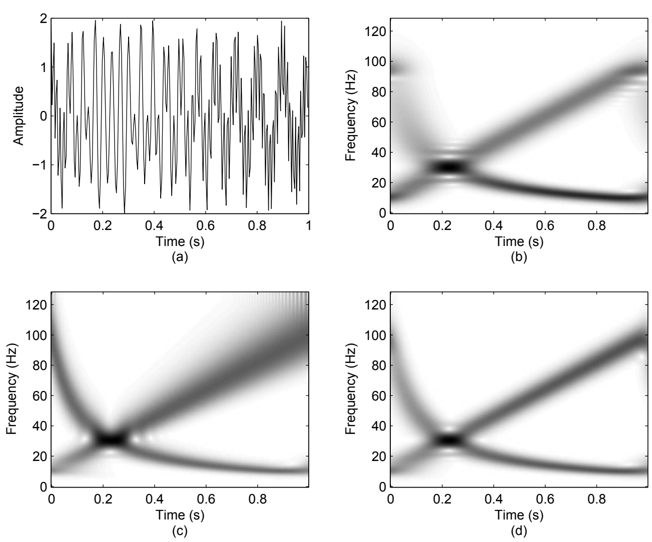

Fig. 4. Time-frequency analysis of $x_1(t)$: (a)Time-domain representation; (b) STFT; (c) S-transform with the Gaussian window; (d) S-transform with the proposed Kaiser window.

STFT, as depicted in Fig. 4(b), shows that the higher frequencies of the hyperbolic component are completely smeared, while the linear chirp has constant concentration at all frequencies. The S-transform with the Gaussian window provides good concentration of the hyperbolic component, however, the concentration of the linear chirp deteriorates at higher frequencies as shown in Fig. 4(c). The improvement to the concentration of the signal in the time-frequency domain is again noticed with the S-transform based on the proposed window. The hyperbolic component has a similar concentration as in the case of the S-transform with the Gaussian window, while the linear chirp is significantly better concentrated than the standard S-transform.

Another important aspect in a signal analysis is instantaneous frequency (IF) estimation [15]. In order to examine the behaviour of the proposed window for the S-transform, let's consider the following signal:

$$x_3(t) = \sin(100\pi t + 4\pi\cos(4\pi t)) \qquad (15)$$

where $x_3(t)$ is only defined in the interval given by $0 \leq t < 1$. The signal is contaminated with a white Gaussian noise, whose variance σ^2 is varied between 0 and 1 in steps of 0.1. The IF is estimated from the peaks of the magnitude of the time-frequency transform [15], and the mean square error (MSE) of the estimator is evaluated for the S-transform with the frequency dependent Gaussian and Kaiser windows, and also for a newly introduced window width optimized S-transform (WWOST) [16]. The error is defined as a difference between the true value of the IF and the estimated value. The MSE values represent an average of 1000 realizations.

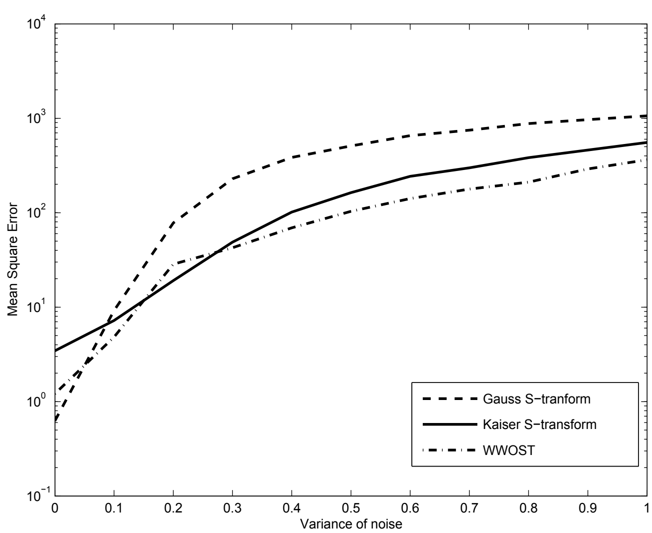

Fig. 5. MSE for the instantaneous frequency estimator based on the S-transform with the frequency dependent Gaussian window (dashed line), the frequency dependent Kaiser window (solid line) and the WWOST (dash dotted line).

The results in Fig. 5 demonstrate the behaviour of the instantaneous frequency estimator based on the S-transform. It is clear that the S-transform with the frequency dependent Kaiser window in most cases produces smaller MSE in comparison to the standard S-transform, and hence, more accurate estimation of the instantaneous frequency. The behaviour of the IF estimator for small values of variance, i.e., $\sigma = \{0, 0.1\}$, is the subject of a further investigation. It should also be mentioned that the Kaiser based S-transform estimator yields slightly higher MSE than the WWOST based estimator, but does not have the computation cost associated with the WWOST.

4. CONCLUSION

In this paper, a frequency dependent Kaiser window is introduced as an analysis window in the S-transform. The frequency dependence of the window is achieved through the window parameter α such that the autoterm with the Kaiser window has almost the same widths in the time and the frequency domain as the Gaussian window, while providing narrower mainlobe than the Gaussian window. The results of numerical analysis have shown that the proposed window can enhance the energy concentration of the signals in comparison to the standard S-transform. It is also shown that, in some cases, the proposed window is capable of achieving higher concentration than other standard methods, such as STFT. Furthermore, the instantaneous frequency estimator based on the S-transform with frequency dependent Gaussian and Kaiser windows are compared, and the results have indicated that the time-frequency representation with the proposed window provides a smaller MSE over range of noise levels.

5. REFERENCES

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