GALTON WATSON FRACTAL SIGNALS

G. Decrouez, P-O. Amblard, J-M. Brossier

LIS/ENSIEG
Laboratoire des Images et des signaux
BP 46, 38402 Saint Martin d'Hères, France

O.D. Jones

The University of Melbourne
Department of Mathematics and Statistics
Victoria 3010, Australia

ABSTRACT

Iterated Function Systems (IFS) is a relevant model to produce fractal functions, whether deterministic (with strict self-similarity) or random (self-similar up to probability distribution). The basic idea of such a construction is to start with an initial function and then compress, dilate and translate it such that by doing so over and over again, we end up with a self-similar signal. This construction relies on a construction tree which has always been deterministic in the litterature for signals. Here we introduce new fractals, called Galton Watson fractals, as fixed points of IFS with a random underlying construction tree and deterministic operators. We give a proof of the existence and uniqueness of a fixed point at the random and distribution level.

Index Terms— Fractals, Galton Watson Trees, Iterated Function Systems, Random fixed points, Self-Similarity.

1. INTRODUCTION

Since the discovery of the relevancy of fractal and $1/f$ processes to model natural phenomena, the fractal formalism has received increasing interests during the last 20 years, with applications to a wide variety of fields such as finance, turbulence, meteorology, image compression, network traffic [1, 2, 3]. Iterated Function Systems (IFS) are a simple class of fractal sets, first described rigorously by Hutchinson [4]. IFS were later adapted to functions and measures and were randomised in various ways [5]. In what follows we will define a new class of random IFS for functions, and prove existence and uniqueness using the approach of Hutchinson and Rüschendorf [5].

Roughly speaking, a (deterministic) IFS recursively apply a contractive operator T (random or not) on an initial function f_0. Functions considered in signal processing are usually finite energy signals and we will denote this space by $L_2(\mathbb{X})$ for deterministic signals and $\mathbb{L}_2(\mathbb{X})$ for random signals so that we can assume $f_0 \in L_2(\mathbb{X})$. The completeness of the metric space where the fractal lives assures the existence and uniqueness of a fixed point f^*, thanks to the well known Banach fixed point theorem. In other words, if we denote by T^n the n-th iterate of T, one has:

$$T^n f_0 \to f^* \text{ as } n \to +\infty \tag{1}$$

where f^* is the only function which satisfies $f = Tf$. At each iteration, functions are stretched, compressed, translated by means of the contractive operator T. We assume that we can decompose this operator into a set of M simpler operators $\phi_i : \mathbb{R} \times \mathbb{X} \to \mathbb{R}$, $i \in \{1, ..., M\}$. Each ϕ_i will have its own way of deforming the signal, and the resulting signal will lie in a subinterval of $\mathbb{X}$ (hence the compression). Mathematically, this can be written as

$$(Tf)(x) = \sum_{i=1}^{M} \phi_i[f(\varrho_i^{-1}(x)), \varrho_i^{-1}(x)]\mathbf{1}_{\varrho_i(\mathbb{X})}(x) \tag{2}$$

for any $x \in \mathbb{X}$. The ϱ_i's, $\varrho_i : \mathbb{X} \to \mathbb{X}$, partition the interval $\mathbb{X}$ into disjoint subintervals and $\mathbf{1}_{\varrho_i(\mathbb{X})}$ is the indicator function of the interval $\varrho_i(\mathbb{X})$. In [5] this construction is randomised by letting the set of M operators $(\phi_1, \dots, \phi_M)$ be a random variable, but M remains fixed. The function f must also be randomised, and in the right hand side of (2) we replace $\phi_i[f(\varrho_i^{-1}(x)), \varrho_i^{-1}(x)]$ by $\phi_i[f^{(i)}(\varrho_i^{-1}(x)), \varrho_i^{-1}(x)]$ where the $f^{(i)}$ are i.i.d. copies of f. If $f \overset{d}{=} Tf$ then we say f is a random fractal function satisfying the random IFS. $\overset{d}{=}$ stands for equality in distribution.

In this study we randomise (2) in a different manner: instead of fixing M we allow it to be random. We will assume that for each value j of M there is a unique set ϕ^j of j operators $(\phi_{j,1}, \dots, \phi_{j,j})$. We can allow ϕ^j to be random but to simplify our discourse we not do so here.

IFS for sets and measures allowing random M have been considered by Falconer [6], Mauldin and Williams [7], but not for functions. More recently, Barnsley *et. al.* have generalised IFS to so called V-variable superfractals, however they also keep M fixed [8].

When we iterate T we get a tree-structure describing the recursive application of the ϕ_i. If M is fixed then this is an M-ary tree. In figure (1) we depict the underlying construction tree of a deterministic IFS with 2 maps. However if M is random then assuming it is chosen independently each time T is applied, the tree is a Galton-Watson tree. In the next section, we will briefly describe Galton Watson trees before describing our new random IFS in section 3.

2. GALTON WATSON TREES

A Galton Watson tree is a tree with a random number of branches at each node where the offspring distribution is independent and identically distributed at each node. A node can be identified by means of its label. If we denote by $\emptyset$ the root node, then the first generation of children will be denoted by i where $1 \leqslant i \leqslant \nu_\emptyset$ and $\nu_\emptyset$ is the number of children at $\emptyset$. Then the second generation will be labelled ij, $1 \leqslant j \leqslant \nu_i$, and so on. More generally, a node is an element of $U = \bigcup_{n \geqslant 0} \mathbb{N}^{*n}$ and a branch is a couple (u, uj) where $u \in U$ and $j \in \mathbb{N}^*$. The length of a node $u = i_1 \dots i_n$, $|u|$, is n.

By definition a tree is a set of nodes, that is each tree ω is a subset of U: $\omega \subset U$. However, a subset of U must meet further requirements in order to be a tree: (i) The root node $\emptyset$ belongs to the tree. (ii) If a node $i = i_1 \dots i_n$ of ω has length n, then every shorter node $i_1 \dots i_k$, $k \leqslant n$ belongs to the tree as well. (iii) If the

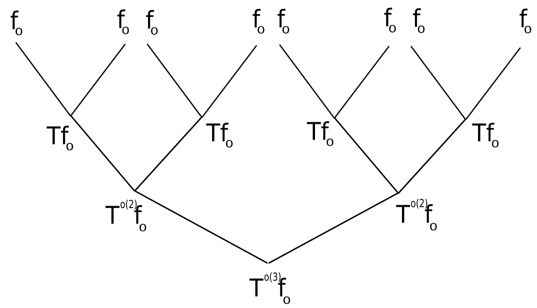

Fig. 1. Underlying tree structure of fixed points of IFS

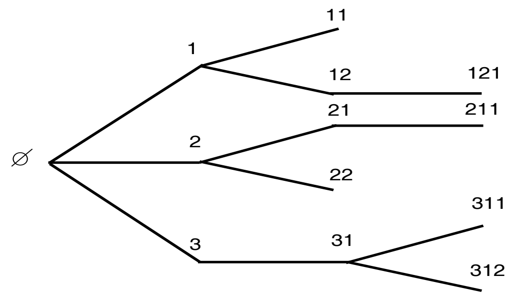

Fig. 2. Example of a Galton Watson tree ω

node labelled u belongs to ω, then uj is also in ω if j is a child of u. Formally, these three conditions can be written as follows:

- $\emptyset \in \omega$
- $\forall v \in U \quad uv \in \omega \Longrightarrow u \in \omega$
- If $u \in \omega$ then $uj \in \omega \iff 1 \leqslant j \leqslant \nu_u(\omega)$ where $\nu_u(\omega)$ represents the number of children at node u for the tree ω.

To illustrate, we present in figure (2) the tree $\omega = \{\emptyset, 1, 2, 3, 11, 12, 121, 21, 22, 211, 31, 311, 312\}$.

Let Ω be the space of trees, and $u \in U$. Then, define:

$$\Omega_u = \{\omega \mid u \in \omega\} \tag{3}$$

Ω_u is a subset of Ω whose elements are trees containing the node u. In particular, $\Omega_\emptyset = \Omega$. Clearly, ν_u defines a map from Ω_u to $\mathbb{N}$ noting that ν_u is not defined over the whole space Ω and represents the number of children (in $\mathbb{N}$) at a given node u of $\omega \in \Omega_u$. Note that if $j \in \mathbb{N}^*$ then there is no change of notations: Ω_{uj} is the space of trees containing the node uj. Formally,

$$\Omega_{uj} = \Omega_u \cap \{\nu_u \geqslant j\} \tag{4}$$

We endow Ω with the σ-algebra $\mathcal{A}$ defined by

$$\mathcal{A} = \sigma(\Omega_u \mid u \in U) \tag{5}$$

Then we endow the subspaces Ω_u with the σ-algebras $\Omega_u \cap \mathcal{A}$ so that ν_u are measurable. Next we define another function which will be relevant in the remainder. Let $T_u(\omega)$ be the tree $\{v \mid v \in U \text{ and } uv \in \omega\}$. In other words, if $\omega \in \Omega$ then $T_u(\omega)$ is the subtree of ω rooted at u. Then T_u is a map from $\Omega_u \to \Omega$. One can check that T_u are also Ω_u-measureable functions.

Next, we endow the space $(\Omega, \mathcal{A})$ with a probability measure. We do this so that given the tree up to generation n, the number of children of each generation node are i.i.d.. One have the following result [9]

Proposition 1 *For each probability $q = (q_j, j \in \mathbb{N})$ on $\mathbb{N}$, there exists a unique probability measure P_q on $(\Omega, \mathcal{A})$ which gives to the random variable $\nu_\emptyset$ the law q and for which, conditionnally on the event $\nu_\emptyset = j$, the random variables T_i, $1 \leqslant i \leqslant j$ are independent and identically distributed with distribution P_q.*

$(\Omega, \mathcal{A}, P_q)$ is the space of Galton Watson trees.

3. GALTON WATSON SIGNALS

3.1. Definition

We are concerned with the definition of a new fractal construction when the underlying tree structure is no longer deterministic. Indeed, we cannot apply the same random operator at each node of the tree as the number of offspring is random. Instead, the operator applied depends (only) on the number of offspring at a given node and is always the same after conditioning on the number of children. The randomness in the construction comes therefore from the non-deterministic tree structure. Consider the space of p-integrable functions on a compact subset $\mathbb{X}$ of the real line:

$$L_p(\mathbb{X}) = \{f : \mathbb{X} \to \mathbb{R} | \int_{\mathbb{X}} |f(t)|^p dt < +\infty\} \tag{6}$$

Let $(\Sigma, \mathcal{F}, P)$ be any probability space and consider the more general space of p-integrable random functions:

$$\mathbb{L}_p = \{f : \Sigma \to L_p(\mathbb{X}),\ t \in \mathbb{X} \mid \mathbb{E}\Big[\int_{\mathbb{X}} |f(t)|^p dt\Big] < +\infty\} \tag{7}$$

endowed with the metric

$$d_p^* : \forall (f,g) \in \mathbb{L}_p\ d_p^*(f,g) = \mathbb{E}^{\frac{1}{p}}\left[d_p^p(f,g)\right] \tag{8}$$

where $d_p(f,g) = (\int_{\mathbb{X}} |f(t) - g(t)|^p dt)^{\frac{1}{p}}$ is the usual L_p metric. Set $p = 2$ to work with finite energy signals. We will write f_σ for the value of f at some $\sigma \in \Sigma$. That is $f_\sigma \in L_p(\mathbb{X})$. In the remainder we will be particularly interested in the probability space $(\Sigma, \mathcal{F}, P) = (\Omega, \mathcal{A}, P_q)$. The operator T is then defined on $\mathbb{L}_p(\mathbb{X})$ by:

$$(Tf)(x) = \sum_{j=1}^{\nu_\emptyset} \phi_{\nu_\emptyset,j}[f^{(j)}(\varrho_{\nu_\emptyset,j}^{-1}(x)), \varrho_{\nu_\emptyset,j}^{-1}(x)]\mathbf{1}_{\varrho_{\nu_\emptyset,j}(\mathbb{X})}(x) \tag{9}$$

where $f^{(j)}$ are i.i.d. copies of f and the $\varrho_{\nu_\emptyset,j}$ partition $\mathbb{X}$ into disjoint subintervals. We can consider for example uniform partitions: $\varrho_{\nu_\emptyset,j}(t) = \dfrac{t}{\nu_\emptyset} + \dfrac{j-1}{\nu_\emptyset}$, $t \in \mathbb{X}$, $1 \leqslant j \leqslant \nu_\emptyset$. The contraction factor of $\varrho_{\nu_\emptyset,j}$ is denoted by $r_{\nu_\emptyset,j}$. $\phi_{\nu_\emptyset,j}$ are 2-variable maps Lipschitz in their first variable, with Lipschitz constant $K_{\nu_\emptyset,j}$. Also note that and that $\nu_\emptyset$ is a random variable with probability distribution q.

Definition 1 *$(\{\phi_{i,j}, \varrho_{i,j}\}; q)$ for $i \geqslant 1$ and $1 \leqslant j \leqslant i$ is a random function scaling system.*

We say that a function is statistically self-similar for an IFS if it satisfies $f = Tf$ in distribution. The major result in the next part is to show the existence and uniqueness at random (a.s. convergence) and distribution level of a function f which satisfies the random function scaling system.

3.2. Existence and Uniqueness of a fixed point

THEOREM 1 *Let $(\Omega, \mathcal{A}, P_q)$ be the space of Galton Watson trees. Consider $(\{\phi_{i,j}, \varrho_{i,j}\}; q)$ a random function scaling system, $i \geqslant 1$ and $1 \leqslant j \leqslant i$. Let $1 < p < +\infty$. If $\lambda = \sum_{i\geqslant 1}\sum_{j=1}^{i} q_i r_{i,j} K_{i,j}^p < 1$ and $\sum_{i\geqslant 1}\sum_{j=1}^{i} r_{i,j} \int |\phi_{i,j}(0,x)|^p dx < +\infty$ then for any $f_0 \in \mathbb{L}_p(\mathbb{X})$, there exists a unique random function f^* which satisfies $f^* = Tf^*$ and such that*

$$d_p^*(T^n f_0, f^*) \leqslant \frac{\lambda^{\frac{n}{p}}}{1-\lambda^{\frac{1}{p}}} d_p^*(f_0, Tf_0) \tag{10}$$

which tends to 0 *as* $n \to +\infty$.

This theorem states that the IFS converges to a random fixed point starting from any initial function f_0 under certain conditions. The fixed point exhibits self-similarity up to probability distribution. The proof is in two steps. The first thing to check is that the operator $T : \mathbb{L}_p \to \mathbb{L}_p$. Secondly, we need to show that T is contractive in the complete metric space $(\mathbb{L}_p, d_p^*)$. The Banach fixed point will assure the existence and uniqueness of a limit function at the random level.

We first define $f^{(j)}$ and $g^{(j)}$. To prove theorem 1 we need to construct i.i.d. copies of the random function f. This can be achieved using the homogeneity property of Galton Watson trees: $f_\omega^{(j)} = f_{T_j(\omega)}$. Since by proposition 1 the variables T_j are independent and identically distributed with distribution P_q, the $f^{(j)}$ functions are also i.i.d.

Step 1: Let $f \in \mathbb{L}_p$. We show $\mathbb{E}\int_{\mathbb{X}} |(Tf)(x)|^p dx < +\infty$, that is $Tf \in \mathbb{L}_p$ To do so, first notice that in the expression of Tf, the indicator function partitions $\mathbb{X}$ into disjoint subintervals, so that the absolute value of the sum equals the sum of absolute values. Furthermore, using $\mathbb{E}(.) = \mathbb{E}[\mathbb{E}(.|\nu_\emptyset)]$ (tower property of expectation) and contractive properties of ϱ_{ij}, it is straightforward to check that $\mathbb{E}\int_{\mathbb{X}} |(Tf)(x)|^p dx$ is always smaller than

$$\mathbb{E}\sum_{j=1}^{\nu_\emptyset} r_{\nu_\emptyset,j}\mathbb{E}[\int_{\mathbb{X}} |\phi_{\nu_\emptyset,j}[f^{(j)}(y),y]|^p dy|\nu_\emptyset] \tag{11}$$

On the right hand side, we have set $y = \varrho_{\nu_\emptyset,j}^{-1}(x)$ and we have majored the Jacobian of the tranformation by $r_{\nu_\emptyset,j}$, the Lipschitz factor of $\varrho_{\nu_\emptyset,j}$. Note that $\mathbb{E}\int_{\mathbb{X}} |\phi_{\nu_\emptyset,j}[f^{(j)}(y),y]|^p dy$ can also be written $d_p^{*p}(\phi_{\nu_\emptyset,j}[f^{(j)}, Id], 0)$ where Id stands for the identity function and 0 the zero function. Combining the triangle inequality of distance and the fact that for any positive x and y: $(x+y)^p \leqslant 2^p(x^p + y^p)$, (11) is majored by:

$$2^p\mathbb{E}\sum_{j=1}^{\nu_\emptyset} r_{\nu_\emptyset,j} d_p^{*p}(\phi_{\nu_\emptyset,j}[f^{(j)}, Id], \phi_{\nu_\emptyset,j}[0, Id]) + 2^p\mathbb{E}\sum_{j=1}^{\nu_\emptyset} r_{\nu_\emptyset,j} d_p^{*p}(\phi_{\nu_\emptyset,j}[0, Id], 0) \tag{12}$$

The first part is bounded since $f \in \mathbb{L}_p$. The second part can be written $\sum_{i\geqslant 1}\sum_{j=1}^{i} q_i r_{i,j} \int |\phi_{i,j}(0,x)|^p dx$ and is bounded by assumption.

Step 2: Let f and g in $\mathbb{L}_p$. Then,

$$\begin{aligned} d_p^{*p}(Tf, Tg) &= \mathbb{E}\, d_p^p(Tf, Tg) \\ &= \mathbb{E}\int \Big|(Tf)(x) - (Tg)(x)\Big|^p dx \end{aligned}$$

By replacing $(Tf)(x)$ and $(Tg)(x)$ by their own expression, the distance becomes:

$$\begin{aligned} = \quad & \mathbb{E}\int \Big|\sum_{j=1}^{\nu_\emptyset} \phi_{\nu_\emptyset,j}[f^{(j)}(\varrho_{\nu_\emptyset,j}^{-1}(x)), \varrho_{\nu_\emptyset,j}^{-1}(x))]\mathbf{1}_{\varrho_{\nu_\emptyset,j}(\mathbb{X})}(x) \\ & - \sum_{j=1}^{\nu_\emptyset} \phi_{\nu_\emptyset,j}[g^{(j)}(\varrho_{\nu_\emptyset,j}^{-1}(x)), \varrho_{\nu_\emptyset,j}^{-1}(x))]\mathbf{1}_{\varrho_{\nu_\emptyset,j}(\mathbb{X})}(x)\Big|^p dx \end{aligned}$$

Now, using similar arguments as in Step 1 (tower property of expectation and Lipschitz property of $\varrho_{\nu_\emptyset,j}$), the distance between Tf and Tg is smaller than

$$\mathbb{E}\sum_{j=1}^{\nu_\emptyset} r_{\nu_\emptyset,j}\mathbb{E}\Big[\int_{\mathbb{X}} \Big|\phi_{\nu_\emptyset,j}[f_\omega^{(j)}(y)), y] - \phi_{\nu_\emptyset,j}[g_\omega^{(j)}(y), y]\Big|^p dy|\nu_\emptyset\Big]$$

where we have made the same change of variable $y = \varrho_{\nu_\emptyset,j}^{-1}(x)$. By furthermore exploiting the Lipschitz property of $\phi_{\nu_\emptyset,j}$ and the i.i.d distributions of $f^{(j)}(y)$ and $g^{(j)}(y)$ $\forall j$, we obtain the inequality

$$d_p^{*p}(Tf, Tg) \leqslant \lambda d_p^{*p}(f, g) \tag{13}$$

where $\lambda = \sum_{i\geqslant 1}\sum_{j=1}^{i} q_i r_{i,j} K_{i,j}^p$. Since by assumption λ is smaller than 1, the operator T is contractive in the complete metric space $(\mathbb{L}_p, d_p^*)$ and therefore admits a unique fixed point f^* (at the random level) by the Banach fixed point theorem. Clearly:

$$d_p^{*p}(T^n f_0, f^*) \leqslant \lambda d_p^{*p}(T^{n-1} f_0, f^*) \tag{14}$$

which lead to

$$d_p^*(T^n f_0, f^*) \leqslant \lambda^{\frac{n}{p}} d_p^*(f_0, f^*) \tag{15}$$

Now using triangle inequality:

$$d_p^*(f_0, f^*) \leqslant d_p^*(f_0, Tf_0) + \lambda^{\frac{1}{p}} d_p^*(f_0, f^*) \tag{16}$$

so that:

$$d_p^*(T^n f_0, f^*) \leqslant \frac{\lambda^{\frac{n}{p}}}{1-\lambda^{\frac{1}{p}}} d_p^*(f_0, Tf_0) \tag{17}$$

which concludes the proof of the theorem.

Remark: It follows directly from this result that for all f in $\mathbb{L}_p(\mathbb{X})$, $T^n f \to f^*$ almost surely as $n \to +\infty$. To see this, suppose that the converse is true, that is suppose that for some $f \in \mathbb{L}_p(\mathbb{X})$ there exists $A \in \mathcal{A}$ such that $P_q(A) > 0$ and for all $\omega \in A$, $T^n f_\omega \not\to f^*$. Given A we can find $\epsilon > 0$ and $B_\epsilon \subset A$ such that $P_q(B_\epsilon) > 0$ and $\liminf d_p(T^n f_\omega, f_\omega^*) \geqslant \epsilon$ for all $\omega \in B_\epsilon$. Then it follows that $\liminf d_p^*(T^n f, f^*) \geqslant \epsilon P_q(B_\epsilon)^{\frac{1}{p}} > 0$, a contradiction.

Next, note that this equality at the random level is also true at the distribution level. However, equality in distribution does not implies equality at the random level. The following result can be proven in the same way as Hutchinson and Ruschendorf [5].

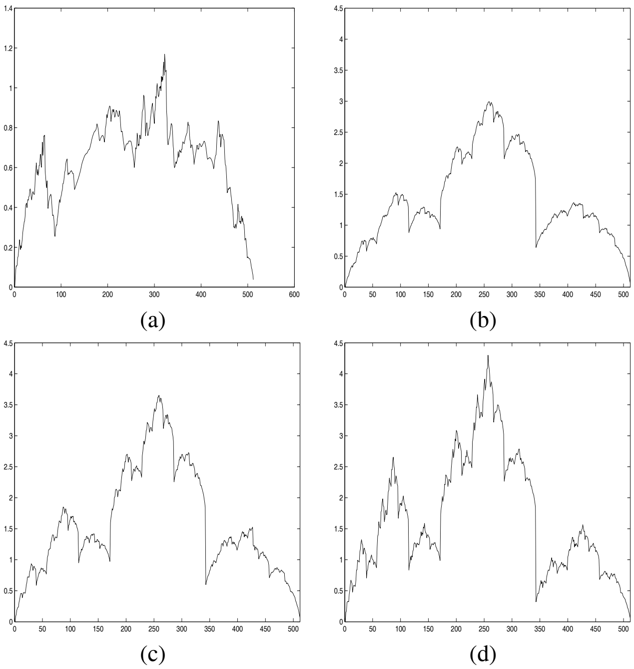

Fig. 3. A Snapshot of the random fixed point (a) and its mean (b). The $\phi_{i,j}$ take the spacial form $\phi_{i,j}(u,v) = s_i u + \zeta_{i,j}(v)$ with $s_1 = 0.6$, $s_2 = 0.7$, $s_3 = 0.3$, $\zeta_{1,1}(t) = t(1-t)$, $\zeta_{2,1}(t) = t^3$, $\zeta_{2,2}(t) = 1-t^2$, $\zeta_{3,1}(t) = t$, $\zeta_{3,2}(t) = (t+1)(2-t)$ and $\zeta_{3,3}(t) = t(1-t)^3$. The probability generating vector is $(0.2, 0.3, 0.5)$ in (a) and (b), $(0.2, 0.2, 0.6)$ in (c) and $(0.2, 0.1, 0.7)$ in (d).

COROLLARY 1 *f^* is the unique fixed point of this IFS at the distribution level.*

The idea is to define a new space consisting of probability distributions of elements of $\mathbb{L}_p$ and a new metric over this space which leads to a complete metric space. One can prove then that the operator T seen at the distribution level is contractive in this space and therefore admits a unique fixed point.

To illustrate our new construction, we present in figure (3) a snapshot of the random fixed point of a particular IFS. We also consider 2 other random function scaling system by varying the probability generating vector and we plot their estimated mean using 100 realisations of the random fixed point. If we assume that we are working with finite energy signals, then $\lambda = \sum_{i\geqslant 1}\sum_{j=1}^{i} q_i r_{i,j} K_{i,j}^2$. The mean is plotted for different probability distributions q. Looking at the shapes of the mean, it seems that there is a continuous dependency of the moment of first order of the fixed point with respect to the distribution q. IFS parameters are given in the figure caption, and are such that the operator T satisfies for example $\lambda = 0.48 < 1$ in (b). Because of the convergence of the Galton Watson IFS, one can derive a formal solution of the equation $f \overset{d}{=} Tf$ by iterating T endlessly. To do so, consider $i = i_1 \dots i_n$.

$$f = \lim_{n\to+\infty} \sum_{i\in\omega,|i|=n} \phi_{\nu_\emptyset,i_1} \circ \dots \circ \phi_{\nu_{i_1,\dots i_{n-1}},i_n} \circ f \circ \dots$$
$$\dots \circ \varrho^{-1}_{\nu_{i_1,\dots i_{n-1}},i_n} \circ \dots \circ \varrho^{-1}_{\nu_\emptyset,i_1} \qquad (18)$$

where we have only considered functions ϕ with only 1 variable for simplicity. From this formal solution, one clearly see that the randomness directly appers in the IFS parameters.

4. CONCLUSION AND PERSPECTIVES

This new approach generalizes the IFS based construction of fractal signals proposed by Hutchison and Ruschendorf in [5]. We proved the existence and uniqueness of a self-similar function when we allow a random construction tree and deterministic operators. Moreover, this construction does not force the number of offsprings to be bounded.

The fixed points obtained all have a very erratic behaviour and we speculate that they might not have a density. An efficient tool to caracterize such irregular objects is their multifractal spectrum introduced first by Frisch and Parisi [10] in the context of turbulence, and adapted to random processes and functions. The motivation to think of a multifractal spectrum for such fractal signals is due to its cascade construction. Cascade processes are indeed known to exhibit multifractal properties and results for random measures are known [11].

Finally, his construction can be further extended when we consider a random construction tree with random operators. In this case the space of Galton Watson trees need to be extended in order to endow each branch of a tree with a new operator.

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