A NEW INTERPRETATION OF BILINEAR TIME-FREQUENCY DISTRIBUTIONS

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ABSTRACT

Wigner's theorem states that there exists no bilinear time-frequency distribution (TFD) that has correct marginals and is nonnegative everywhere. This means that any attempt to interpret a bilinear TFD as an energy or power distribution must be fraught with problems. In this paper, an alternative perspective is proposed, which allows a local interpretation at a point in the time-frequency plane. This approach is based on analyzing the properties of a chirping ellipse that, at a given time instant, gives the best local approximation of the signal from a given frequency. This chirping ellipse is described in terms of its mean shape, orientation, and direction of polarization (counterclockwise or clockwise). A time-frequency coherence measures the quality of the approximation that this ellipse presents. The ellipse parameters and the time-frequency coherence can be expressed in terms of the Rihaczek TFD.

Index Terms— Time-frequency analysis, rotary component method, polarization analysis.

1. INTRODUCTION

Wigner's theorem [1] says that there can be no bilinear time-frequency distribution (TFD) that has correct marginals and is nonnegative everywhere. Despite this fundamental limitation, it is quite common in the time-frequency literature to refer to bilinear TFDs as energy or power distributions. The most popular bilinear TFD that is covariant to shifts in time and frequency is the Wigner-Ville distribution. The Wigner-Ville TFD has several attractive properties [2] but its main perceived advantage seems to be the fact that it is real. However, since the Wigner-Ville TFD generally takes on negative values except for a special class of signals, it does not allow a *local* interpretation as the energy or power in the vicinity of some time instant t and frequency f.

In this paper, we propose an alternative perspective in bilinear time-frequency analysis, which does not try to interpret a TFD as an energy or power distribution. Our treatise is based on the Rihaczek TFD (R-TFD) [3], which is also bilinear and covariant to shifts in time and frequency, but less popular than the Wigner-Ville TFD. The advantage of the R-TFD is that it presents an *inner product* between the time-domain signal at given time t and its frequency-domain representation at given frequency f [4]. As such, it determines a time-varying Wiener filter for estimating the signal at time t from phasors rotating with frequency $\pm f$. In this paper, we build upon this result from [4] to develop a powerful geometric interpretation of the R-TFD. Our approach is inspired by the *rotary component method* from meteorology and oceanography [5] and *polarization analysis* from geophysics and optics [6].

2. RIHACZEK DISTRIBUTION

In this paper, we analyze a zero-mean real nonstationary random signal $x(t)$ via its corresponding analytic signal $s(t) = x(t) + jH\{x(t)\}$, where $H\{\cdot\}$ denotes the Hilbert transform. We first review the R-TFD [3] as a means to describe the second-order statistics of a nonstationary signal.

The signal $s(t)$ is assumed to be harmonizable. Its Cramér–Loève spectral representation is then given by the mean-square convergent integral

$$s(t) = \int_0^\infty dS(f)\, e^{j2\pi f t}.$$

For a general nonstationary random process, the increment process $dS(f)$ is nonorthogonal and improper [7] with *(Hermitian) spectral correlation* $R_{ss^*}(\nu, f)$, defined by

$$E\{dS(f+\nu)dS^*(f)\} = R_{ss^*}(\nu, f)\, d\nu\, df, \qquad (1)$$

and *complementary spectral correlation* [8] $R_{ss}(\nu, -f)$, defined by

$$E\{dS(-f+\nu)dS(f)\} = R_{ss}(\nu, -f)\, d\nu\, df. \qquad (2)$$

This representation can require the use of Dirac delta functions in $R_{ss^*}(\nu, f)$ and $R_{ss}(\nu, -f)$; in particular, in the stationary case. If the complementary spectral correlation vanishes, $s(t)$ is called *proper*.

In (1) and (2), f is a global frequency variable and ν is a local frequency offset. Since $s(t)$ is analytic, $R_{ss^*}(\nu, f)$ is zero for $f < \max(0, -\nu)$, and $R_{ss}(\nu, -f)$ is zero for $f < 0$ or $f > \nu$. Therefore, throughout this paper, f will be assumed to be *nonnegative*, $f \geq 0$.

Since $s(t)$ is harmonizable, so are its temporal correlation and complementary temporal correlation,

$$\begin{aligned} r_{ss^*}(t,\tau) &= E\{s(t)s^*(t-\tau)\} \\ &= \int_0^\infty \int_{-f}^\infty R_{ss^*}(\nu, f) e^{j2\pi(\nu t + f\tau)}\, d\nu\, df \qquad (3) \\ r_{ss}(t,\tau) &= E\{s(t)s(t-\tau)\} \\ &= \int_0^\infty \int_{f}^\infty R_{ss}(\nu, -f) e^{j2\pi(\nu t - f\tau)}\, d\nu\, df \qquad (4) \end{aligned}$$

In (3) and (4), t is a global time variable and τ is a local time lag. In order to obtain a characterization in terms of global time t and global frequency f, we either Fourier-transform the temporal correlation and complementary correlation on local τ, or inverse Fourier-transform the spectral correlation and complementary correlation on local ν. This yields the Hermitian Rihaczek time-frequency distribution (HR-TFD) [3] and the complementary Rihaczek time-frequency distribution (CR-TFD) [8],

$$\begin{aligned} V_{ss^*}(t,f) &= \int_{-\infty}^\infty r_{ss^*}(t,\tau) e^{-j2\pi f\tau} d\tau \\ &= \int_{-f}^\infty R_{ss^*}(\nu, f) e^{j2\pi\nu t} d\nu \qquad (5) \\ V_{ss}(t,-f) &= \int_{-\infty}^\infty r_{ss}(t,\tau) e^{j2\pi f\tau} d\tau \\ &= \int_{f}^\infty R_{ss}(\nu, -f) e^{j2\pi\nu t} d\nu \qquad (6) \end{aligned}$$

Together, the HR-TFD and CR-TFD comprise the R-TFD.

The R-TFD is generally complex, yet time- and frequency-marginals of the HR-TFD are nonnegative [2]. The time-marginal is the *instantaneous power* at time t,

$$\int_0^\infty V_{ss^*}(t,f)\, df = r_{ss^*}(t,0) \geq 0.$$

The frequency-marginal is the *energy spectral density* (ESD) at frequency f,

$$\int_{-\infty}^\infty V_{ss^*}(t,f)\, dt = R_{ss^*}(0,f) \geq 0.$$

The nonnegativity of the the HR-TFD marginals has lead many to interpret the HR-TFD and other TFDs in Cohen's class as energy or power distributions. As the HR-TFD itself takes on complex values, such an interpretation seems dissatisfactory because it does not work locally in the vicinity of some point (t, f) in the time-frequency plane. The purpose of this paper is to offer an alternative perspective, which is based on the key insight that the R-TFD is an inner product [4]. This can be seen by explicitly computing the HR-TFD (5) and CR-TFD (6),

$$V_{ss^*}(t,f)\, df = E\{s(t)(dS(f)e^{j2\pi ft})^*\}, \qquad (7)$$

$$V_{ss}(t,-f)\, df = E\{s(t)(dS^*(f)e^{-j2\pi ft})^*\}. \qquad (8)$$

This shows that the HR-TFD is the Hilbert space inner product between the random variable $s(t)$, at fixed time instant t, and the *counterclockwise* rotating phasor $dS(f)e^{j2\pi ft}$, at fixed frequency f. The CR-TFD is the Hilbert space inner product between $s(t)$ and the *clockwise* rotating phasor $dS^*(f)e^{-j2\pi ft}$.

3. POLARIZATION ELLIPSE

It was found in [4] that because of (7) and (8), the HR-TFD determines the Wiener filter for estimating $s(t)$ from the counterclockwise rotating phasor $dS(f)e^{j2\pi ft}$, for fixed time t and frequency f. Similarly, the CR-TFD determines the Wiener filter for estimating $s(t)$ from the clockwise rotating phasor $dS^*(f)e^{-j2\pi ft}$. We now consider estimating $s(t)$ from both counterclockwise and clockwise rotating phasors as

$$\hat{s}_f(t) = W_1(t,f)dS(f)e^{j2\pi ft} + W_2(t,-f)dS^*(f)e^{-j2\pi ft}. \quad (9)$$

Such an estimate is called *widely linear* [9] because it is linear-conjugate linear in $dS(f)e^{j2\pi ft}$. With the short-hand notation

$$d\mathbf{Z}(t,f) = \begin{bmatrix} dS(f)e^{j2\pi ft} \\ dS^*(f)e^{-j2\pi ft} \end{bmatrix},$$

the Wiener filter is obtained as

$$\begin{bmatrix} W_1(t,f) \\ W_2(t,-f) \end{bmatrix}^T = E\left[s(t)d\mathbf{Z}^H(t,f)\right] E\left[d\mathbf{Z}(t,f)d\mathbf{Z}^H(t,f)\right]^\dagger,$$

where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudo-inverse (or generalized inverse). The use of the pseudo-inverse is necessary because the matrix $E\left[d\mathbf{Z}(t,f)d\mathbf{Z}^H(t,f)\right]$ is singular if $R^2_{ss^*}(0,f) = |R_{ss}(2f,-f)|^2$. In the singular case, $dS^*(f) = \alpha dS(f)$, $|\alpha| = 1$, so that a widely linear estimator offers no advantage over a strictly linear estimator, which means we may let $W_2(t,-f) = 0$. For $R^2_{ss^*}(0,f) \neq |R_{ss}(2f,-f)|^2$, the matrix $E\left[d\mathbf{Z}(t,f)d\mathbf{Z}^H(t,f)\right]$ is nonsingular and its pseudo-inverse is the regular matrix inverse. As a result, we obtain

$$W_1(t,f) = \begin{cases} \frac{V_{ss^*}R_{ss^*} - V_{ss}R^*_{ss}e^{-2j2\pi ft}}{(R^2_{ss^*} - |R_{ss}|^2)\, df} & R^2_{ss^*} \neq |R_{ss}|^2 \\ \frac{V_{ss^*}}{R_{ss^*}\, df} & R^2_{ss^*} = |R_{ss}|^2 \end{cases}$$

$$W_2(t,-f) = \begin{cases} \frac{V_{ss}R_{ss^*} - V_{ss^*}R_{ss}e^{2j2\pi ft}}{(R^2_{ss^*} - |R_{ss}|^2)\, df} & R^2_{ss^*} \neq |R_{ss}|^2 \\ 0 & R^2_{ss^*} = |R_{ss}|^2 \end{cases}$$

For notational convenience, we have dropped the function arguments. It is understood that V_{ss^*} stands for $V_{ss^*}(t,f)$, V_{ss} for $V_{ss}(t,-f)$, R_{ss^*} for $R_{ss^*}(0,f)$, and R_{ss} for $R_{ss}(2f,-f)$. Now let

$$\begin{aligned} U(t,f) &= W_1(t,f)dS(f) \\ U(t,-f) &= W_2(t,-f)dS^*(f). \end{aligned}$$

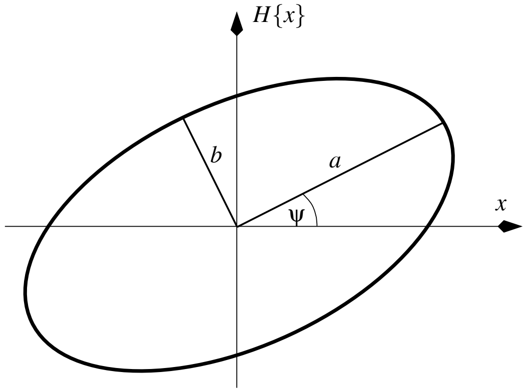

Fig. 1. Ellipse traced out in the complex time-domain plane.

For fixed t_0 and f, but varying t,

$$u_{t_0,f}(t) = U(t_0, f)e^{j2\pi ft} + U(t_0, -f)e^{-j2\pi ft} \tag{10}$$

describes a random ellipse in the complex time-domain plane [10]. This ellipse provides the widely linear minimum mean squared error approximation of $s(t)$ at $t = t_0$, and one ellipse exists for every time instant t_0 and every frequency f.

As depicted in Fig. 1, each sample function of $u_{t_0,f}(t)$ describes an ellipse whose major axis is tilted by ψ. The lengths of the major and minor axis are $2a$ and $2b$, respectively. In order to characterize the (ensemble) *mean* properties of the random ellipse $u_{t_0,f}(t)$, we introduce the energy spectral density (ESD) matrix

$$\begin{aligned}\mathbf{J}_{ss}(t,f) &= \begin{bmatrix} J_{ss^*}(t,f) & J_{ss}(t,f) \\ J^*_{ss}(t,f) & J_{ss^*}(t,-f)\end{bmatrix} \\ &= E\begin{bmatrix} U(t,f) \\ U^*(t,-f)\end{bmatrix}\begin{bmatrix} U^*(t,f) & U(t,-f)\end{bmatrix}. \end{aligned} \tag{11}$$

Here, $J_{ss^*}(t_0, f) = E\,|U(t_0, f)|^2$ is the ESD and $J_{ss}(t_0, f) = E\,[U(t_0, f)U(t_0, -f)]$ is the complementary ESD (C-ESD) of the ellipse $u_{t_0,f}(t)$ given by (10).

It can be shown [5, 10] that, for fixed t_0 and f, the mean orientation of the ellipse is given by the phase of the C-ESD,

$$\tan 2\overline{\psi}_{t_0,f} = \frac{\operatorname{Im} J_{ss}(t_0, f)}{\operatorname{Re} J_{ss}(t_0, f)}.$$

We will also introduce another angle $\overline{\chi}_{t_0,f}$ $(-\pi/4 \le \overline{\chi}_{t_0,f} \le \pi/4)$ following [6, 10],

$$\sin 2\overline{\chi}_{t_0,f} = \frac{J_{ss^*}(t_0, f) - J_{ss^*}(t_0, -f)}{J_{ss^*}(t_0, f) + J_{ss^*}(t_0, -f)}. \tag{12}$$

It can be shown [6] that

$$\tan \overline{\chi}_{t_0,f} = \pm\frac{\overline{b}_{t_0,f}}{\overline{a}_{t_0,f}},$$

where $\overline{a}_{t_0,f}$ and $\overline{b}_{t_0,f}$ denote the mean lengths of the major and minor axis of $u_{t_0,f}(t)$. If $\overline{\chi}_{t_0,f} > 0$, then sample functions of $u_{t_0,f}(t)$ trace out the ellipse in *counterclockwise* direction. This situation is referred to as *counterclockwise* (or left-handed) *polarized.* On the other hand, if $\overline{\chi}_{t_0,f} < 0$, then sample functions of $u_{t_0,f}(t)$ trace out the ellipse in *clockwise* direction, which is called *clockwise* (or right-handed) *polarized.* Hence, the angle $\overline{\chi}_{t_0,f}$ specifies the mean shape and polarization of the ellipse.

There are special cases where the ellipse degenerates into a circle or straight line. If $\overline{\chi}_{t_0,f} = \pi/4$, then the lengths of the major and minor axis are equal and $u_{t_0,f}(t)$ describes a circle in counterclockwise direction, i.e., it is *counterclockwise circularly polarized.* Similarly, if $\overline{\chi}_{t_0,f} = -\pi/4$, $u_{t_0,f}(t)$ is *clockwise circularly polarized.* Finally, if $\overline{\chi}_{t_0,f} = 0$, then the length of the minor axis is 0, and $u_{t_0,f}(t)$ describes a line, i.e., it is *linearly polarized.*

So far, we have constructed a different ellipse for every time instant t_0. An alternative point of view, which we will pursue, is that the approximation

$$u(t,f) = \hat{s}_f(t) = U(t,f)e^{j2\pi ft} + U(t,-f)e^{-j2\pi ft} \tag{13}$$

represents a *time-varying "chirping ellipse,"* characterized by time-frequency dependent mean orientation $\overline{\psi}(t,f)$ and shape and polarization $\overline{\chi}(t,f)$. We will continue to refer to (13) as an ellipse, while keeping in mind that it describes a strict ellipse only for time-independent $U(t,f)$ and $U(t,-f)$.

We will now evaluate the mean ellipse properties (orientation, shape, and polarization) by explicitly computing the ESD matrix (11)

$$\begin{aligned}\mathbf{J}_{ss}(t,f) &= E\begin{bmatrix} W_1(t,f)dS(f) \\ W_2^*(t,-f)dS(f)\end{bmatrix}\begin{bmatrix} W_1^*(t,f)dS^*(f) \\ W_2(t,-f)dS^*(f)\end{bmatrix}^T \\ &= R_{ss^*}(0,f)\,(df)^2 \\ &\quad\times\begin{bmatrix} |W_1(t,f)|^2 & W_1(t,f)W_2(t,-f) \\ W_1^*(t,f)W_2^*(t,-f) & |W_2(t,-f)|^2\end{bmatrix}\end{aligned}$$

It is particularly interesting to examine the ellipse shape and polarization, which, for $R^2_{ss^*} \neq |R_{ss}|^2$, can be done through the angle $\overline{\chi}(t,f)$ defined by (12),

$$\sin 2\overline{\chi}(t,f) = \frac{(|V_{ss^*}|^2 - |V_{ss}|^2)(R^2_{ss^*} - |R_{ss}|^2)}{(|V_{ss^*}|^2 + |V_{ss}|^2)(R^2_{ss^*} + |R_{ss}|^2) - 4\operatorname{Re}\{V_{ss^*}V^*_{ss}R_{ss^*}R_{ss}e^{2j2\pi ft}\}}.$$

If $s(t)$ is proper at time t and frequency f, $V_{ss} = 0$ and $R_{ss} = 0$, then $\overline{\chi}(t,f) = \pi/4$. This says that a proper analytic signal is (completely) *counterclockwise circularly polarized.* On the other hand, if $R^2_{ss^*} = |R_{ss}|^2$, then $dS^*(f) = \alpha dS(f)$, $|\alpha| = 1$, and it follows that $|V_{ss^*}|^2 = |V_{ss}|^2$. In this case, the signal $s(t)$ can be regarded as maximally improper at (t,f). A maximally improper analytic signal has $\overline{\chi}(t,f) = 0$ and is therefore *linearly polarized.*

While $|R_{ss}|^2 \leq R^2_{ss^*}$, the magnitude of the HR-TFD does not provide an upper bound on the magnitude of the CR-TFD, i.e., $|V_{ss}|^2 \nleq |V_{ss^*}|^2$. Moreover, $|V_{ss^*}|^2 = |V_{ss}|^2$ does not imply $R^2_{ss^*} = |R_{ss}|^2$. Therefore, it is possible that $s(t)$ is *clockwise polarized* at (t, f), i.e., $\overline{\chi}(t, f) < 0$, provided that the signal is "sufficiently improper" at (t, f). This result may seem surprising, considering that an analytic signal is synthesized from *counterclockwise* phasors only.

In order to judge the quality of the approximation that the ellipse $u(t, f)$ presents, we calculate the mean squared error at time t as

$$E\,|s(t) - \hat{s}_f(t)|^2 = r_{ss^*}(t,0)(1 - |\rho(t, f)|^2).$$

Here we have introduced the *time-frequency coherence*

$$\begin{aligned}|\rho(t, f)|^2 &= \frac{E\left[s(t)d\mathbf{Z}^H(t, f)\right]\, E\left[d\mathbf{Z}(t, f)d\mathbf{Z}^H(t, f)\right]^\dagger}{r_{ss^*}(t,0)} \\ &\quad \times E\left[d\mathbf{Z}(t, f)s^*(t)\right] \\ &= \frac{R_{ss^*}(|V_{ss^*}|^2 + |V_{ss}|^2) - 2\,\mathrm{Re}\left\{V_{ss^*}V^*_{ss}R_{ss}e^{2j2\pi ft}\right\}}{r_{ss^*}(R^2_{ss^*} - |R_{ss}|^2)}.\end{aligned}$$

The final expression is valid for $R^2_{ss^*} \neq |R_{ss}|^2$, and r_{ss^*} is understood to mean $r_{ss^*}(t,0)$. The time-frequency coherence satisfies $0 \leq |\rho(t, f)|^2 \leq 1$. If $|\rho(t, f)|^2 = 1$, the ellipse $u(t, f)$ is a perfect approximation of $s(t)$ at time t. If $|\rho(t, f)|^2 = 0$, it is not possible to construct a widely linear estimate of $s(t)$ at time t from frequency f, and no ellipse $u(t, f)$ exists at (t, f).

In both the proper case, which is characterized by $V_{ss} = 0$ and $R_{ss} = 0$, and the most improper case, characterized by $R^2_{ss^*} = |R_{ss}|^2$, the time-frequency coherence becomes

$$|\rho(t, f)|^2 = \frac{|V_{ss^*}|^2}{r_{ss^*}R_{ss^*}} \tag{14}$$

because in these cases $s(t)$ can be estimated from counterclockwise rotating phasors only. In other words, the optimum widely linear estimator is strictly linear, which means $W_2(t, -f) = 0$ in (9). The optimum strictly linear estimator and the corresponding time-frequency coherence (14) have been previously considered in [4].

4. DISCUSSION AND CONCLUSIONS

We have presented an alternative way of interpreting bilinear time-frequency distributions. This approach is based on analyzing the properties of a chirping ellipse that, at a given time instant t, gives the best local approximation of the signal from a given frequency f. This ellipse is characterized by its mean shape, orientation, and direction of polarization. We have also defined a time-frequency coherence that determines the quality of the approximation that this ellipse presents at (t, f). This perspective allows a local interpretation at a point (t, f) in the time-frequency plane, which is not possible from the classical point of view that considers time-frequency representations as energy or power distributions.

The ellipse parameters and time-frequency coherence can all be expressed in terms of the R-TFD, but both the HR-TFD and the CR-TFD are required. The CR-TFD is necessary even though this paper restricted attention to the analysis of a real signal via its corresponding analytic signal. The shape and orientation of the ellipse as well as the direction of polarization depend on the CR-TFD. Ignorance of the CR-TFD would imply that all analytic signals be counterclockwise circularly polarized. In reality, however, the ellipse may take on any shape between the two extreme cases of line and circle and may turn either counterclockwise or clockwise.

The discussion can be extended to general complex signals that are not analytic. A significant difference of the general complex case is that nonanalytic signals can have varying degrees of polarization. This is reported in the journal version [10] of this conference paper.

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