UNIVERSAL CONSTANT REBALANCED PORTFOLIOS WITH SWITCHING

Suleyman S. Kozat

IBM Research
Yorktown Heights, NY
Email: kozat@us.ibm.com

Andrew C. Singer

University of Illinois at Urbana Champaign
Urbana, IL 61801, USA
Email: acsinger@uiuc.edu

ABSTRACT

In this paper, we consider online (sequential) portfolio selection in a competitive algorithm framework. We construct a sequential algorithm for portfolio investment that asymptotically achieves the wealth of the best piecewise constant rebalanced portfolio tuned to the underlying individual sequence of price relative vectors. Without knowledge of the investment duration, the algorithm can perform as well as the best investment algorithm that can choose both the partitioning of the sequence of the price relative vectors as well as the best constant rebalanced portfolio within each segment based on knowledge of the sequence of price relative vectors in advance. We use a transition diagram similar to that in [1] to compete with an exponential number of switching investment strategies, using only linear complexity in the data length for combination. The regret with respect to the best piecewise constant strategy is at most $O(\ln(n))$ in the exponent, where n is the investment duration. This method is also extended in [2] to switching among a finite collection of candidate algorithms, including the case where such transitions are represented by an arbitrary side-information sequence.

Index Terms— Adaptive signal processing, Bayes procedures, Finance

1. INTRODUCTION

In this paper, we define a competitive framework for portfolio selection algorithms for a market with a finite number of stocks to trade. The behavior of a market with m stocks is modeled by a sequence of price relative vectors $\boldsymbol{x}^n = \boldsymbol{x}[1], \ldots, \boldsymbol{x}[n]$, where $\boldsymbol{x}[t] \in R_+^m$. The jth entry $x_j[t]$ of a price relative vector $\boldsymbol{x}[t]$ represents the ratio of closing to opening price of the jth stock for the tth trading day. An investment at day t is represented by the portfolio vector $\boldsymbol{b}[t]$, $\boldsymbol{b}[t] \in R_+^m$ and $\sum_{j=1}^m b_j[t] = 1$ for all t. Each entry $b_j[t]$ corresponds to the portion of the wealth invested in the stock $x_t[j]$ at day t. The wealth achieved after n trading periods is given by $\prod_{t=1}^n \boldsymbol{b}^T[t]\boldsymbol{x}[t]$. For our competitive framework, the performance measure for a candidate portfolio selection algorithm is defined with respect to the performance of the best algorithm from a class of competing algorithms. As an example, Cover [3] presented a portfolio selection algorithm which achieves the sequentially accumulated wealth of the best constant rebalanced portfolio from the class of all constant rebalanced portfolios for any sequence of price relatives, to first-order in the exponent. We call such algorithms that asymptotically achieve the performance of the single-best algorithm from a given class of algorithms (for any sequence of price relatives) “static” universal algorithms, since the competition class contains a fixed set of algorithms, and performance is compared with the best, fixed element of the class.

In this paper, we extend the results for static algorithms to a framework where the underlying competition class includes the ability to switch (in time) among the various static elements. We investigate this problem, when each competing algorithm can divide the sequence of price relatives into arbitrary segments, say k of them, and fit each contiguous segment with the best portfolio assignment algorithm from a given class of static algorithms for that segment, such as a fixed constant rebalanced portfolio. For k such transitions, there exist $k + 1$ segments. The total wealth growth of a class member with such a partition is the product of the wealth growth of all fixed static algorithms associated with each segment. The best partition is the one which gives the maximum total wealth. We seek to outperform all such switching algorithms, simultaneously for any number of possible switches, k. A natural restriction is $k < n$. Unlike [3, 4, 5], here we try to exploit the time-varying nature of the best choice of algorithm for any given sequence of price relatives, since the choice of the best portfolio from a class of static portfolios can change over time. Rather than trying to find the best partitioning of the data or even the best number of transitions, our objective is simply to achieve the performance of the best partition directly, for all k.

For a given sequence of price relative vectors $\boldsymbol{x}^n$, a competing portfolio selection algorithm with a transition path $\mathcal{T}_{k,n}$ with k transitions, represented by $(t_1, \ldots, t_k)$, partitions $\boldsymbol{x}^n$ into $k + 1$ segments such that $\boldsymbol{x}^n$ is represented by the concatenation of

$$\{\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t_1-1]\}\{\boldsymbol{x}[t_1], \ldots, \boldsymbol{x}[t_2-1]\} \ldots \{\boldsymbol{x}[t_k], \ldots, \boldsymbol{x}[n]\}.$$

Given n and k, there exist $\binom{n-1}{k}$ such possible transition paths

$\mathcal{T}_{k,n}$. Given the past values of the desired price relatives $\boldsymbol{x}[t]$, $t = 1, \ldots, n-1$, a competing algorithm assigns a portfolio vector $\boldsymbol{b}_i$ in each segment as

$$\hat{\boldsymbol{b}}[t] = \boldsymbol{b}_i$$

where $t_{i-1} \leq t < t_i$, $i = 1, \ldots, k+1$. For simplicity we assume $t_0 = 1$ and $t_{k+1} = n+1$. Here, the competing class contains all constant portfolios in each segment that have the same $\boldsymbol{b}_i$ for each sample of the sequence $\boldsymbol{x}[t]$ for $t = t_{i-1}, \ldots, t_i - 1$, where each $\boldsymbol{b}_i$ can be selected independently for each region $i = 1, \ldots, k+1$. In determining the best algorithm in the competing class, we attempt to outperform all such portfolios, including the one that has been selected by choosing the transition path $\mathcal{T}_{k,n}$ and the constant portfolio vectors $\boldsymbol{b}_i$ in each segment based on observing the entire sequence $\boldsymbol{x}^n$ in advance, simultaneously, for all k. As such we try to minimize the following regret:

$$R_{\boldsymbol{b}}[n] \stackrel{\triangle}{=} \sup_{\boldsymbol{x}^n} \frac{\sup_{\substack{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k+1} \in R_+^m \\ t_1, \ldots, t_k \in \{2, \ldots, n\}}} \prod_{i=1}^{k+1} \prod_{t=t_{i-1}}^{t_i - 1} \boldsymbol{b}_i^T \boldsymbol{x}[t]}{\prod_{t=1}^{n} \hat{\boldsymbol{b}}^T[t] \boldsymbol{x}[t]}$$

where $\hat{\boldsymbol{b}}[t]$ is a sequential portfolio assignment at time t, i.e., $\hat{\boldsymbol{b}}[t]$ may be a function of $\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t-1]$ but does not depend on the future, $\mathcal{T}_{k,n}$ is any transition path representing $(t_1, \ldots, t_k)$ with an arbitrary number of transitions k. We will show that we can construct a sequential portfolio selection algorithm for which this regret is at most $(k+1)(m-1)\ln(n)/2 + k\ln(n) + O(k+1)$ in the exponent for any $\mathcal{T}_{k,n}$, k or n and with no prior knowledge of $\mathcal{T}_{k,n}$, k or n. We recognize the term $(k+1)(m-1)\ln(n)/2$ as the parameter regret or additional loss due to the estimation of the best constant portfolio in each of the $k+1$ separate regions and the term $k\ln(n)$ as the transition path regret due to not knowing the best transitions times.

Universal algorithms that can compete against constant rebalanced portfolios or against a set of finite portfolio selection algorithms have studied by a number of authors [5, 3, 4]. Competing against a portfolio selection algorithm that can switch among a finite number of M static strategies, e.g., $\boldsymbol{b}_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ is all zeros except a single component for some pure strategy, (not against a constant rebalanced portfolio which is the arbitrary linear combination of these M pure strategies) is studied in [6]. In [6], the authors provide two sequential portfolio selection algorithms. The first algorithm needs a switching rate parameter which can only be optimized after observing the whole $\boldsymbol{x}^n$. The second algorithm is an extension of this first algorithm and uses a time dependent switching rate which does not need a priori optimization. This second algorithm also has a better performance bound. However, even with this restrictions on $\boldsymbol{b}_i$'s, they are unable to provide an update directly on the portfolio vectors for the second algorithm. Here, we can provide the direct portfolio updates that can achieve the final wealth of the best constant rebalanced portfolios in each segment. Although, we use Cover's universal algorithm in our derivations for the corresponding bounds, the methods we use are generic. The algorithms we introduce can easily employ other algorithms such as [5] instead of [3], or other algorithms that are sequentially universal with respect to the static class of constant rebalanced portfolios. The additional complexity of our algorithms over the complexity of the static algorithms used in the construction is linear in data size n.

The organization of the paper is as follows. In Section 1, we provide the main theorem of this paper as an upper bound on the performance of the universal portfolio selection algorithm. The construction of the algorithm and an outline of the proof of the theorem are given in Section 3. Detailed proofs and extension of this algorithm to switching with an arbitrary side-information sequence are provided in [2].

2. UPPER BOUNDS

The main results of this section are the upper bounds contained in Theorem 1. The corresponding universal and strongly sequential portfolio selection algorithm is constructed at the end of the proof in Section 3.

For a given sequence of price relative vectors $\boldsymbol{x}^n$, a competing portfolio selection algorithm with a transition path $\mathcal{T}_{k,n}$ with k transitions, represented by $(t_1, \ldots, t_k)$, divides $\boldsymbol{x}^n$ into $k+1$ segments such that $\boldsymbol{x}^n$ is represented by the concatenation of

$$\{\boldsymbol{x}[1], \ldots, \boldsymbol{x}[t_1 - 1]\}\{\boldsymbol{x}[t_1], \ldots, \boldsymbol{x}[t_2 - 1]\} \ldots \{\boldsymbol{x}[t_k], \ldots, \boldsymbol{x}[n]\}.$$

Given the past values of the desired price relatives $\boldsymbol{x}[t]$, $t = 1, \ldots, n-1$, a competing algorithm assigns a portfolio vector $\boldsymbol{b}_i$ in each segment as $\hat{\boldsymbol{b}}[t] = \boldsymbol{b}_i$ where $t_{i-1} \leq t < t_i$, $i = 1, \ldots, k+1$. For simplicity we assume $t_0 = 1$ and $t_{k+1} = n+1$. For this setting we have the following theorem.

Theorem 1 : *Let $\boldsymbol{x}^n = \boldsymbol{x}[1], \ldots, \boldsymbol{x}[n]$ be an arbitrary sequence of price relative vectors such that $\boldsymbol{x}[t] \in R_+^m$ for all n and some components of $\boldsymbol{x}[t]$ can be zero. Then we can construct sequential portfolios $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[n]$ complexity linear in $O(n^{m+1})$ per trading period such that for any $\epsilon > 0$ and all k*

$$R_{\boldsymbol{b}}[n] = \frac{\sup_{\substack{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k+1} \in R_+^m \\ t_1, \ldots, t_k \in \{2, \ldots, n\}}} \prod_{i=1}^{k+1} \prod_{t=t_{i-1}}^{t_i - 1} \boldsymbol{b}_i^T \boldsymbol{x}[t]}{\prod_{t=1}^{n} \tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}^T[t] \boldsymbol{x}[t]} \quad (1)$$

satisfies

$$\ln \frac{R_{\boldsymbol{b}}[n]}{n} \leq \frac{(k+1)(m-1)}{2} \frac{\ln(n)}{n} + 2(k+\epsilon)\frac{\ln(n)}{n} + O\left(\frac{k}{n}\right) \quad (2)$$

and

$$\ln \frac{R_{\boldsymbol{b}}[n]}{n} \leq \frac{(k+1)(m-1)}{2} \frac{\ln(n)}{n} + 2\left((k+1)\epsilon + k\right) \frac{\ln(n/k)}{n} + O\left(\frac{k}{n}\right) \tag{3}$$

for any $\mathcal{T}_{k,n}$ representing transition path $(t_1, \ldots, t_k)$ and any k, such that $\tilde{\boldsymbol{b}}_{\boldsymbol{b},u}[t]$ does not depend on $\mathcal{T}_{k,n}$, k or n.

The upper bound in Equation (2) is better (tighter) when the number of transitions is small, i.e., $k \ll n$. If the number of transitions is closer to $O(n)$, then the upper bound in Equation (3) is better. Theorem 1 states that the average regret of the universal portfolio $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ is within $O((k+1)n^{-1}\ln(n))$ of the best batch piecewise constant rebalanced portfolios with k transitions (tuned to the underlying sequence), uniformly, for every sequence of price relatives $\boldsymbol{x}^n$.

3. PROOF AND IMPLEMENTATION

Outline of Proof of Theorem 1: The proof of the theorem uses ideas from sequential probability assignment. For each possible transition path $\mathcal{T}_{k,n}$ representing $(t_1, \ldots, t_k)$ with k transitions and data length n, we consider a family of portfolios, each with its own set of constant vectors $\boldsymbol{B}_k = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{k+1}]^T$ where each $\boldsymbol{b}_i$ represents a constant portfolio vector for the ith region. Without loss of generality we assume that $t_0 = 1$ and $t_{k+1} = n+1$. For each pairing of $\mathcal{T}_{k,n}$ and $\boldsymbol{B}_k$, a measure of the sequential wealth achieved by the corresponding competition algorithm is constructed: $W(\boldsymbol{x}^n \mid \boldsymbol{B}_k, \mathcal{T}_{k,n}) \triangleq \prod_{i=1}^{k+1} \prod_{t=t_{i-1}}^{t_i - 1} \boldsymbol{b}_i^T \boldsymbol{x}[t]$. Given any $\mathcal{T}_{k,n}$, the competing algorithm with best constant portfolios in each region assigns to $\boldsymbol{x}^n$ the largest wealth, i.e., $W^*(\boldsymbol{x}^n \mid \mathcal{T}_{k,n}) \triangleq \sup_{\boldsymbol{B}_k} W(\boldsymbol{x}^n \mid \boldsymbol{B}_k, \mathcal{T}_{k,n})$. Maximizing $W^*(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})$ over all $\mathcal{T}_{k,n}$ (with k transitions) yields $W^*(\boldsymbol{x}^n \mid \mathcal{T}^*_{k,n}) \triangleq \sup_{\mathcal{T}_{k,n}} W^*(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})$

Here, $W^*(\boldsymbol{x}^n \mid \mathcal{T}^*_{k,n}) = W(\boldsymbol{x}^n \mid \boldsymbol{B}_k{}^*, \mathcal{T}_{k,n}{}^*)$ corresponds to the wealth achieved by the best portfolio in the competition class with k transitions. Our goal is to demonstrate a sequential algorithm which achieves $W(\boldsymbol{x}^n \mid \boldsymbol{B}_k{}^*, \mathcal{T}_{k,n}{}^*)$ given any k and n, and without a priori knowledge of k or n. We will accomplish this result using a double mixture approach. First we will demonstrate an algorithm achieving the performance of the competing algorithm with the best constant portfolios in each region given any $\mathcal{T}_{q,n}$, i.e., $W^*(\boldsymbol{x}^n \mid \mathcal{T}_{q,n})$. Then we will show that a proper weighted combination of all such algorithms over all $\mathcal{T}_{q,n}$, $q = 1, \ldots, n$, can be used to find a sequential algorithm that will achieve $W^*(\boldsymbol{x}^n \mid \mathcal{T}^*_{k,n})$ for any k.

For any given $\mathcal{T}_{k,n}$, the wealth achieved by the algorithm with the best constant rebalanced portfolios in each region, $W^*(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})$, can be asymptotically obtained by using the sequential portfolio assignment algorithm [3] (which is universal with respect to the class of all constant rebalanced portfolios), independently for each segment, i.e., apply $\tilde{\boldsymbol{b}}_{t_{i-1}}[t]$ between time t_{i-1} up to t_i, where $\tilde{\boldsymbol{b}}_{t_{i-1}}[t]$ is the Cover's algorithm with an mth order Dirichlet distribution, i.e., $\boldsymbol{\mu}(\boldsymbol{b}) = D(1/2, \ldots, 1/2)$ that uses the price relative sequence starting from time t_{i-1} up to time t, $t < t_i$. In each segment this universal algorithm achieves the performance of the best constant rebalanced portfolio for that region, hence,

$$\ln(\tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})) \geq \ln\{W^*(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})\} - \frac{(k+1)(m-1)}{2}\ln(n) + O(1). \tag{4}$$

Hence given $\mathcal{T}_{k,n}$, using $\tilde{\boldsymbol{b}}_{t_{i-1}}[t]$ in each segment defines a sequential algorithm that asymptotically achieves the performance of the algorithm with the best constant rebalanced portfolio for each segment. For all $\mathcal{T}_{k,n}$ and k, we construct a similar sequential algorithm yielding a total of 2^{n-1} such sequential portfolio assignment algorithms.

We then define a weighted mixture of the wealth achieved by all such sequential predictors over all possible $\mathcal{T}_{k,n}$ and k

$$\tilde{W}_u(\boldsymbol{x}^n) \triangleq \sum_{k=0}^{n-1} \sum_{\mathcal{T}_{k,n}} P(\mathcal{T}_{k,n}) \tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}_{k,n}), \tag{5}$$

with a suitable prior over the partitions $\mathcal{T}_{k,n}$, $P(\mathcal{T}_{k,n})$. For any transition path $\mathcal{T}_{k,n}$, the weighting (or the assigned probability $P(\mathcal{T}_{k,n})$ to each transition path) would be nonnegative and would satisfy $\sum_{k=0}^{n-1} \sum_{\mathcal{T}_{k,n}} P(\mathcal{T}_{k,n}) = 1$. Now we have a total wealth achieved by the class of all possible constant portfolios and for all possible transition paths. By Equation (5), we can conclude that this achieved wealth satisfies $\ln \tilde{W}_u(\boldsymbol{x}^n) \geq \ln P(\mathcal{T}_{k,n}) + \ln \tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})$, for any transition path $\mathcal{T}_{k,n}$, since $\tilde{W}_u(\boldsymbol{x}^n) \geq P(\mathcal{T}_{k,n})\tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}_{k,n})$.

Clearly, the assigned probability or weight for $\mathcal{T}_{k,n}$ directly contributes to the regret as $\ln(P(\mathcal{T}_{k,n}))$ over the best batch algorithm given any path. Hence, it is desirable that the weight of the "best path" be assigned as large as possible. This weight assignment should also be constructed so that the overall weighting and the resulting portfolio assignment algorithm can be sequentially computable.

We will use three different weighting methods, first one is a form of estimated probability for $P(\mathcal{T}_{k,n})$ with k transitions, i.e, the Krichevsky-Trofimov (KT) weighting used in [1, 7]. The other two are from [8] and they yield tighter upper bounds for $\ln(P(\mathcal{T}_{k,n}))$ than the KT estimate. For these there weighting methods we have

$$\ln P(\mathcal{T}_{k,n}) \leq \frac{3k+1}{2}\ln(n/k) + O(k) \tag{6}$$

$$\text{and } \ln P(\mathcal{T}_{k,n}) \leq (k+\epsilon)\ln(n) + O(k) \tag{7}$$

$$\text{and } \ln P(\mathcal{T}_{k,n}) \leq (k+(k+1)\epsilon)\ln(n/k) + O(k) \tag{8}$$

for all $\epsilon > 0$. Other weighting methods used in [1, 8] (such as the reduced state, quadratic complexity probability assignment) can be extended to yield prediction algorithms using the same methodology that we introduce in this paper.

We now have a method of selecting portfolios that achieves, to first order in the exponent, the same wealth as that achieved by the best batch constant portfolios, for any partition $\mathcal{T}_{k,n}$. In this sense, $\tilde{W}_u(\boldsymbol{x}^n)$ is a "universal" portfolio selection method. It still remains to find a sequential algorithm whose wealth assignment is as large as $\tilde{W}_u(\boldsymbol{x}^n)$, the wealth achieved by all sequential algorithms represented in Equation (4) weighted by the corresponding $P(\mathcal{T}_{k,n})$, $k = 1, \ldots, n$.

We are now ready to find the actual universal portfolio assignment algorithm. By definition $\tilde{W}_u(\boldsymbol{x}^n) = \prod_{t=1}^{n} \frac{\tilde{W}_u(\boldsymbol{x}^t)}{\tilde{W}_u(\boldsymbol{x}^{t-1})}$. If we look each term in the product closely, we observe that

$$\frac{\tilde{W}_u(\boldsymbol{x}^n)}{\tilde{W}_u(\boldsymbol{x}^{n-1})} = \tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}^T[t]\boldsymbol{x}[t].$$

for a strongly sequential portfolio assignment algorithm $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$. Hence, $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ is the required portfolio vector at each time t. Nevertheless, in this form the sequential algorithm $\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[t]$ requires 2^n different sequential algorithms to run in parallel on the sequence of price relatives. We will now demonstrate that this sequential portfolio assignment algorithm can be calculated efficiently by using a linear transition diagram, similar to that used in [1] (after assigning appropriate weights to each branch).

At each time n, we divide the set of all possible paths $\mathcal{T}_{k,n}$, $k = 1, \ldots, n$ into n disjoint sets. We label each set by a state variable s_n representing the most recent transition of a corresponding path within the period $1 \leq t \leq n$ such that for any $\mathcal{T}_{k,n}$, $s_n = t_k$. Given n, there can be at most n states $s_n = 1, \ldots, n$. At time n, all transition paths with the same last transition instant, $t_k = s$, are represented by the state $s_n = s$. We then define $W_n(s_n = s, \boldsymbol{x}^n)$ as the achieved wealth of all sequential algorithms at state s_n at time n. Here, $W_n(s_n = s, \boldsymbol{x}^n)$ is the weighted sum of all the wealth achieved by the sequential algorithms as given in [1] whose transition paths ended up at $s_n = s$; i.e., for all paths $\mathcal{T}^{'}$ such that the last transition was at $s_n = s$

$$W_n(s_n = s, \boldsymbol{x}^n) \stackrel{\triangle}{=} \sum_{\mathcal{T}^{'}:s_n=s} P(\mathcal{T}^{'})\tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}^{'}).$$

Since the states partition the set of paths $\mathcal{T}_{k,n}$, $\tilde{W}_u(\boldsymbol{x}^n) = \sum_{\mathcal{T}} P(\mathcal{T})\tilde{W}(\boldsymbol{x}^n \mid \mathcal{T}) = \sum_{s_n=1}^{n} W_n(s_n, \boldsymbol{x}^n)$. To obtain a closed form expression for $\frac{\tilde{W}_u(\boldsymbol{x}^n)}{\tilde{W}_u(\boldsymbol{x}^{n-1})}$, we show that $W_n(s_n = s, \boldsymbol{x}^n)$ can be calculated recursively by using the linear transition diagram as in [1]. As such, state s_n represents the most recent transition within the period $1 \leq t \leq n$. After some algebra, it can be shown that [2], the final universal algorithm is given as

$$\tilde{\boldsymbol{b}}_{u,\boldsymbol{b}}[n] = \sum_{s_{n-1}=1}^{n-1} \mu(s_{n-1})$$
$$\left\{ P_{tr}(s_n = s_{n-1} \mid s_{n-1})\tilde{\boldsymbol{b}}_{s_{n-1}}[n] + P_{tr}(s_n = n \mid s_{n-1})\frac{\boldsymbol{1}}{m} \right\}.$$

where $\boldsymbol{1}$ is a vector of size $(m \times 1)$ of all ones, $\tilde{\boldsymbol{b}}_{s_{n-1}}[n]$ is Cover's algorithm that started at time s_{n-1} and the weights $\mu(s_{n-1})$ are defined as

$$\mu(s_{n-1} = j) \stackrel{\triangle}{=} \frac{W_{n-1}(s_{n-1} = j, \boldsymbol{x}^{n-1})}{\sum_{s_{n-1}=1}^{n-1} W_{n-1}(s_{n-1}, \boldsymbol{x}^{n-1})}$$

and are a form of performance-weighting for the states s_{n-1}. $P_{tr}(s_n \mid s_{n-1})$ are the transition probabilities from state s_{n-1} to state s_n that are needed to sequentially calculate the path weights in Equation (6) (or Equation (7) or (8)) where $P_{tr}(s_n \mid s_{n-1})$ will be different depending on the weighting used for $P(\mathcal{T}_{k,n})$. ■.

4. CONCLUSION

In this paper, we considered online (sequential) portfolio selection in a competitive algorithm framework. We constructed a sequential algorithm for portfolio investment that asymptotically achieves the wealth of the best piecewise constant rebalanced portfolio tuned to the underlying individual sequence of price relative vectors. We demonstrated that the regret of this algorithm over the performance of the best piecewise constant rebalanced portfolio selection algorithm is at most $(k+1)(m-1)\ln(n)/2 + k\ln(n) + O(k+1)$ in the exponent for any $\mathcal{T}_{k,n}$, k or n and with no prior knowledge of $\mathcal{T}_{k,n}$, k or n. The additional complexity of our algorithm over the complexity of the static algorithms used in the construction is linear in data size n.

5. REFERENCES

[1] F. M. J. Willems, "Coding for a binary independent piecewise-identically-distributed source," *IEEE Trans. on Info. Theory*, vol. 42, pp. 2210–2217, 1996.

[2] S. S. Kozat and A. C. Singer, "Universal portfolios with switching and side-information," *IEEE Trans. on Sig. Proc.*, to be submitted, 2006.

[3] T. Cover and E. Ordentlich, "Universal portfolios with side-information," *IEEE Trans. on Info. Theory*, vol. 42, no. 2, pp. 348–363, 1996.

[4] V. Vovk and C. Watkins, "Universal portfolio selection," in *COLT*, 1998, pp. 12–23.

[5] D. P. Helmbold, R. E. Schapire, Y. Singer, and M. K. Warmuth, "Online portfolio selection using multiplicative updates," *Mathematical Finance*, vol. 8, no. 4, pp. 325–347, 1998.

[6] Y. Singer, "Swithcing portfolios," in *Proc. of Conf. on Uncertainty in AI*, 1998, pp. 1498–1519.

[7] R. E. Krichevsky and V. K. Trofimov, "The performance of universal encoding," *IEEE Trans. on Info. Theory*, vol. 27, pp. 190–207, 1981.

[8] G. I. Shamir and N. Merhav, "Low-complexity sequential lossless coding for piecewise-stationary memoryless sources," *IEEE Trans. on Info. Theory*, vol. 45, no. 5, pp. 1498–1519, 1999.