ALGORITHM EXTENSION OF CUBIC PHASE FUNCTION FOR ESTIMATING QUADRATIC FM SIGNAL

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ABSTRACT

In this paper, an extended algorithm for parameter estimation of quadratic FM signal is derived by exploring the time diversity in the cubic phase (CP) function. The performance of the proposed algorithm is analyzed in terms of estimate bias and variance, and compared with other methods. Although the proposed algorithm employs a fourth-order nonlinearity which results in higher threshold SNR, it provides a number of advantages, such as low mean-square error (MSE) for the estimates at high SNR and simply extension for multicomponent signals. Extension to cubic FM signal is also discussed. The theoretical analysis is verified by the simulation results.

Index Terms— Parameter estimation, FM signal, statistical signal processing.

1. INTRODUCTION

In the signal processing literature, considerable attention has been paid to parameter estimation of the frequency-modulated (FM) signal from noisy observations. The FM signal can be found in a number of applications such as radar, sonar, geophysics, and biomedicine [1], [2]. This paper focuses on the quadratic FM signal and also discusses the cubic FM signal.

The most accurate way for analyzing the quadratic FM signal is the maximum likelihood (ML) estimation. It yields optimal results but requires a three-dimensional maximization, and thus it is computational exhausting. Moreover, if the objective function is not convex, the maximization is easy to converge to local maxima. To avoid the multidimensional search, the suboptimal approaches such as the high-order ambiguity function (HAF) [1], [3], integrated general ambiguity function (IGAF) [4], and product HAF (PHAF) [5], are proposed. Recently, a bilinear transform — the CP function was presented in [2] and [6]. For a quadratic FM signal defined as

$$s(n) = Ae^{j\phi(n)} = Ae^{j(a_0+a_1n+a_2n^2+a_3n^3)}, \quad -\frac{(N-1)}{2} \leq n \leq \frac{(N-1)}{2}, \quad (1)$$

where A, $\phi(n)$, and $\{a_i\}_{i=0}^{3}$ are the amplitude, phase and phase coefficients respectively, and N is odd, the CP function is presented as

$$\text{CP}(n, \Omega) = \int_0^{+\infty} s(n+\tau)s(n-\tau)e^{-j\Omega\tau^2}d\tau. \quad (2)$$

Substituting $s(n)$ in (2) with (1), the result is

$$s(n+\tau)s(n-\tau) = A^2e^{j2[\phi(n)+(a_2+3a_3n)\tau^2]}. \quad (3)$$

From (2) and (3), the CP function will peak at $2(a_2 + 3a_3n)$, which is the instantaneous frequency rate (IFR) of (1) [2], [6]. Once the IFR is obtained, the parameters, a_2 and a_3, can be estimated by selecting two different time positions and solving the resulting equations set. In this paper, we explore the time diversity in the CP function by using two special time positions, which are symmetric with respect to origin, i.e., one is n and another $-n$. Although this extension results in fourth-order nonlinearities, it offers the following advantages:

- low mean-square error (MSE) at high SNR for estimating a_3;
- simple extension to multicomponent signals.

This paper is organized as follows. In Section 2, we present the algorithm for estimating the quadratic FM signal in two steps. Section 3 derives the statistical results for the estimate using the first-order permutation analysis. In Section 4, extension to multicomponent case is considered. The simulation results are provided to validate the theoretical results in Section 5. Section 6 focuses on further development. Finally, conclusions are drawn in Section 7.

2. THE PROPOSED ALGORITHM

Motivated by the CP function, we further exploit the time diversity on the basis of the CP function, which forms a fourth-order nonlinear estimator — the Generalized Cubic Phase (GCP) function as

$$\mathrm{GCP}_s(n,\omega) = \int_0^{\infty} \psi(n,\tau) e^{-j\omega\tau^2} d\tau, \tag{4}$$

where $\psi(n,\tau) = s(n+\tau)s(n-\tau)s^*(-n+\tau)s^*(-n-\tau)$. Here, we assume n is positive.

Compared between (2) and (4), the significant difference is the employing nonlinear transform. The first one involves a bilinear transform and the later employs a fourth-order nonlinearities. In the next paragraph, we will realize the GCP function in two steps: the nonlinear transform and the quadratic phase filter.

2.1. The Fourth-order Nonlinear Transform $\psi(s)$

For an arbitrary signal with phase $\phi(n)$, assume that $\phi_1 = phase[s(n+\tau)]$, $\phi_2 = phase[s(n-\tau)]$, $\phi_3 = phase[s(-n+\tau)]$, $\phi_4 = phase[s(-n-\tau)]$, and $\phi_G = phase[s(n+\tau)s(n-\tau)s^*(-n+\tau)s^*(-n-\tau)]$, where $phase[.]$ denotes the phase extractor. Using the Taylor series expansion, order to M, we obtain

$$\phi_1 + \phi_2 = \sum_{l=0}^{M/2} \frac{2\phi^{(2l)}(n)\tau^{2l}}{(2l)!}; \tag{5}$$

$$\phi_3 + \phi_4 = \sum_{l=0}^{M/2} \frac{2\phi^{(2l)}(-n)\tau^{2l}}{(2l)!}; \tag{6}$$

$$\phi_G = \sum_{l=0}^{M/2} \frac{2[\phi^{(2l)}(n) - \phi^{(2l)}(-n)]\tau^{2l}}{(2l)!}. \tag{7}$$

Substituting $\phi(n)$ with $\sum_{i=0}^{P} a_i n^i$, where P is the phase order and letting $\eta(\phi) = \phi^{(2l)}(n) - \phi^{(2l)}(-n)$ yield

$$\eta(\phi) = \begin{cases} \sum\limits_{v=l}^{P/2-1} \frac{2a_{2v+1}n^{2v-2l+1}(2v+l)!}{(2v-2l+1)!} & P \text{ is even;} \\ \sum\limits_{v=l}^{(P-1)/2} \frac{2a_{2v+1}n^{2v-2l+1}(2v+l)!}{(2v-2l+1)!} & P \text{ is odd.} \end{cases} \tag{8}$$

Inserting (8) in (7) we obtain

$$\phi_G = \begin{cases} \sum\limits_{l=0}^{P/2} \sum\limits_{v=l}^{P/2-1} \frac{4a_{2v+1}n^{2v-2l+1}\tau^{2l}(2v+1)!}{(2l)!(2v-2l+1)!} & P \text{ is even;} \\ \sum\limits_{l=0}^{\frac{P-1}{2}} \sum\limits_{v=l}^{\frac{P-1}{2}} \frac{4a_{2v+1}n^{2v-2l+1}\tau^{2l}(2v+1)!}{(2l)!(2v-2l+1)!} & P \text{ is odd.} \end{cases} \tag{9}$$

For a quadratic or cubic FM signal, (9) yields

$$\phi_G = 4(a_1 n + a_3 n^3) + 12 a_3 n \tau^2, \tag{10}$$

It can be said that the multilinear transform is to convert the quadratic and cubic FM signals into a space that, at any given value of time sets, has a quadratic term in τ and another invariant to τ. In particular, the quadratic phase coefficient of the resulting signal is $12a_3 n$. Hence, once this coefficient is obtained, the a_3 can be then estimated.

2.2. The Quadratic Phase Filter

According to the phase in (10), a quadratic phase filter is applied to compensate the quadratic phase term in τ in (4). Using the identity [7]

$$\int_{-\infty}^{+\infty} e^{-j\tau t^2} dt = \sqrt{\frac{\pi}{\tau}} e^{-j(\pi/4)}, \qquad \tau > 0, \tag{11}$$

we obtain that

$$|\mathrm{GCP}_s(n,\omega)| = \frac{A^4}{2}\sqrt{\frac{\pi}{|12a_3 n - \omega|}}, \tag{12}$$

will be maximized if $\omega = 12a_3 n$. Thus, we can estimate a_3 if a distinct peak is detected. Using the nonlinear least squares, the estimate of a_3 is given as

$$\hat{a}_3 = \frac{\arg\max_\omega |\mathrm{GCP}_s(n,\omega)|}{12n}. \tag{13}$$

Directly computing (13) requires about $O(N^2)$ operations. Motivated by the fast implementation of the CP function, evaluation of the GCP function can be reduced to $O(N \log N)$ operations using the subband decomposition techniques [5].

3. STATISTICAL ANALYSIS OF THE ESTIMATES

This section analyzes the bias and variance of the a_3 estimate using the first-order permutation method [3], [8]. For brevity, we use the same notions as in [2]. First, we specify $g_N(\omega)$ and $\delta g_N(\omega)$ as

$$g_N(\omega) = \mathrm{GCP}_s(n,\omega) \text{ (in discrete form)}, \tag{14}$$

$$\delta g_N(\omega) = \sum_{m=0}^{(N-1)/2-n} z_{vs}(n,m) e^{-j\omega m^2}, \tag{15}$$

respectively, where z_{vs} approximates the interference terms containing not more than two noise factors. A number of intermediate results are given below:

$$g_N(\omega_0) \approx A^4 K (N/2 - n), \tag{16}$$

$$\frac{\partial g_N(\omega_0)}{\partial \omega} \approx -jA^4 K \frac{(N/2-n)^3}{3}, \tag{17}$$

$$\frac{\partial^2 g_N(\omega_0)}{\partial \omega^2} \approx -A^4 K \frac{(N/2-n)^5}{5}, \tag{18}$$

$$\delta g_N(\omega_0) \approx \sum_{m=0}^{N/2-n} z_{vs}(n,m) e^{-j\omega_0 m^2}, \tag{19}$$

$$\frac{\partial \delta g_N(\omega_0)}{\partial \omega} \approx -j \sum_{m=0}^{N/2-n} m^2 z_{vs}(n,m) e^{-j\omega_0 m^2}, \tag{20}$$

where $\omega_0 = 12a_3 n$ and $K = e^{j4(a_1 n + a_3 n^3)}$. Then we derive

$$\alpha \approx \frac{-8A^8 (N/2-n)^6}{45}, \tag{21}$$

$$\beta \approx -2A^4(N/2-n)[Im\{\Gamma\}], \tag{22}$$

where

$$\Gamma \approx K \sum_{m=0}^{N/2-n} (m^2 - \frac{(N/2-n)^2}{3}) z_{vs}^*(n,m) e^{j\omega_0 m^2}. \tag{23}$$

Using (21) and (22), the first-order approximation for $\delta\omega$ is

$$\delta\omega \approx -\frac{45 \cdot Im\{\Gamma\}}{4A^4(N/2-n)^5}. \tag{24}$$

Its expected value, the bias of the a_3 estimate, is approximately zero (according to the first-order approximation). Following (23), expectations $E\{\Gamma\Gamma^*\}$ and $E\{\Gamma\Gamma\}$ can be written as

$$E\{\Gamma\Gamma^*\} \approx \frac{8}{45} A^4 \delta^2 (2A^2 + 3\delta^2)(N/2-n)^5, \tag{25}$$

$$E\{\Gamma\Gamma\} \approx \frac{1}{90} A^6 \delta^2 (N/2-3n)\varphi(n,N)u(N/2-3n), \tag{26}$$

where $\varphi(n,N) = (2N^4 - 33nN^3 + 82n^2N^2 + 52n^3N - 8n^4)$ and $u(.)$ denotes the unit step function.

Combining (21), (22), (25), and (26) yields

$$E\{(\delta a_3)^2\} \approx \frac{45}{576 \cdot \text{SNR} \cdot n^2(N/2-n)^5} \cdot \Bigg[(2 + \frac{3}{\text{SNR}}) - \frac{(N/2-3n)\varphi(n,N)}{16(N/2-n)^5} u(N/2-3n)\Bigg],$$

where $\text{SNR} = A^2/\delta^2$. It is shown that the variance of $\hat{a}_3$ depends on the values of N, SNR and n. For a given N and SNR, numerical study shows that $n \approx 0.2291N$ gives the minimal variance of $\hat{a}_3$ at high SNR. When $n = 0.2291N$, the theoretical MSE of the a_3 estimate is

$$E\{(\delta a_3)^2\} \approx \frac{1582.5 + \frac{3060.7}{\text{SNR}}}{N^7 \text{SNR}}. \tag{27}$$

Table 1 summarizes the theoretical MSE for the a_3 estimate using different methods at high SNR and the Cramér-Rao lower bound (CRLB). From the listed in the table, the GCP function has the lowest MSE for the a_3 estimate at high SNR, which is about 32.54% and 43.18% lower than the CP function and the HAF, respectively. On the other hand, the fourth-order nonlinearities in the GCP function, which is higher than that of the CP function, give rise to higher value in the SNR^{-2} terms and thus result in greater MSE at low SNR. As to other methods, the PHAF is robust to the noise and cross-terms, but it may lead to a wrong estimation [9] in certain cases. The IGAF is statistically efficient for the quadratic FM signal, however, it requires a two-dimensional search and is hence computationally demanding [4]. The polynomial Wigner-Ville distribution (PWVD) is another method used for analyzing polynomial phase signals [10]. For the a_3 estimate, the GCP function outperforms the PWVD since the PWVD has sixth-order nonlinearities.

Table 1. Comparisons of the asymptotic MSE (AMSE) among different methods for the a_3 estimate at high SNR

	GCPF	CPF	HAF	CRLB
AMSE	$\frac{1582.5}{N^7\text{SNR}}$	$\frac{2038}{N^7\text{SNR}}$	$\frac{2187}{N^7\text{SNR}}$	$\frac{1400}{N^7\text{SNR}}$

4. MULTICOMPONENT CASE

It has been shown in [11] that for multicomponent signals the distinct cross-terms or spurious peaks occur producing problem with identification of parameters of signal components. The proposed algorithm can be simply extended for multicomponent case. To discern the auto-terms from the cross-terms or possible spurious peak, it is useful to make use of the time dependence. From (12), it is clear that the auto-term is linearly related to the time position, i.e. $\omega = 12a_3n$. However, the cross-terms have not this type of time dependence. By using the spectral scaling technique introduced in the PHAF [5], the product form of GCP functions is defined as

$$\text{PGCP}(\omega; n_1) = \prod_{l=1}^{L} \text{GCP}_s(n_l, \frac{n_1}{n_l}\omega). \tag{28}$$

It can be said in (28) that the spectral scaling ensures that the peaks are properly aligned at $\omega = 12a_3n_1$. Thus the later multiplication amplifies the auto-terms and weakens the cross-terms in the product.

5. SIMULATIONS

For straight comparisons with other methods, the tested signal is the same quadratic FM signal in [1] and [2]. The SNR is incremented in 1-dB interval between 0 and 20 dB, the sampling interval is 1, and the number of samples $N = 257$. The selected parameters are $A = 1$, $a_3 = \pi 10^{-5}$, $a_2 = -\pi 10^{-3}$, $a_1 = 0.3\pi$, and $a_0 = 0$. For each SNR, 1000 runs of the Monte Carlo simulations are performed.

Fig. 1 plots the measured, theoretical MSE of the a_3 estimate. It is evident that the theoretical MSE matches the theoretical results at SNR above 3dB, whereas a threshold effect can be observed at around 3dB. Fig. 2 highlights the measured MSE using different methods at high SNR, i.e., above 3 dB. The GCP function has lower MSE than the HAF method at almost all SNR and the CP function at SNR above 3dB. Thus, it can be said the derived simulation results agree with the theoretical analysis.

6. DISCUSSION

The GCP function can be further extended for cubic FM signal. From (10), the GCP function can estimate a_3 no matter whether a_4 exists. To analyze the cubic FM signal, we can first estimate a_3 using (10), then dechirp the sampled signal using the estimated a_3, and finally estimate the a_4 using the following equation as $\mathrm{GCP}_m(n,\omega) = \int_0^{+\infty} s_d(n+\tau)s_d(n-\tau)s_d^*(\tau)s_d^*(-\tau)e^{-j\omega\tau^2}d\tau$, where s_d represents the dechirped signal. Compared with other methods for cubic FM signal, the proposed algorithm in this paper involves fourth-order nonlinearities only that is lower than the sixth-order nonlinearities in the higher-order phase function (HPF) [2] and PWVD [10], and the eighth-order nonlinearities in the HAF-based method. Thus, it allows to estimate the cubic FM signal at low SNR.

7. CONCLUSION

An algorithm for parameter estimation of a noisy quadratic FM signal is proposed in this paper. It further explores the time diversity in the CP function and employs a fourth-order nonlinear transform. Statistical analysis shows that the variance of the a_3 estimate is only 13.04% higher than the CRLB at high SNR. The extensions to multicomponent case and cubic FM signal are also discussed. The simulation results have been provided and show adherence to the theoretical analysis. In the future, we will study the error propagation effort throughout the estimation both in theory and in simulation.

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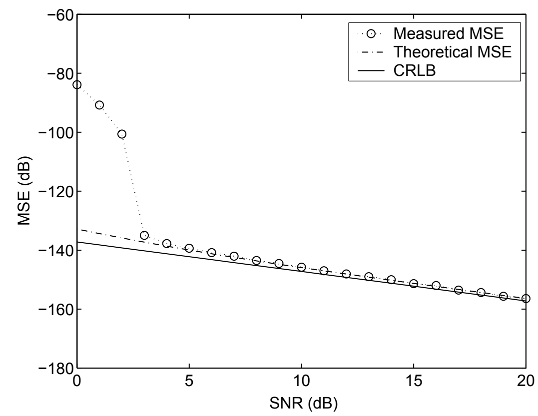
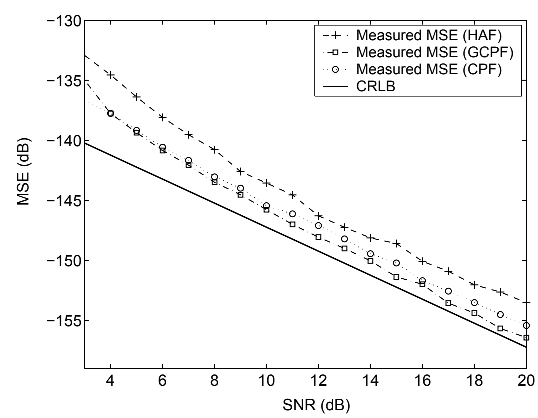


(a)



(b)

Fig. 1. (a) The theoretical results, measured MSE and the CRLB; (b) Comparisons of the measured MSE using different methods for high SNR