ML ESTIMATION OF POSITION IN A GNSS RECEIVER USING THE SAGE ALGORITHM

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ABSTRACT

In this paper, the Maximum Likelihood Estimator (MLE) of position in satellite based navigation systems is studied. Recent results have shown that this novel approach provides an interesting way of introducing prior information in the position estimation and that the estimator is consistent for large sample sizes. However, one of the main drawbacks of this approach is the lack of a computationally efficient optimization algorithm due to the high dimensionality and nonlinearity of the resulting cost function, since there is not a closed form solution for this estimator. The aim of this paper is to investigate the application of the Space-Alternating Generalized Expectation Maximization (SAGE) algorithm to the estimation of position. The SAGE algorithm is a low-complexity generalization of the EM (Expectation-Maximization) algorithm, which iteratively approximates the MLE. Computer simulation results are provided, comparing the performance obtained by the algorithm with the Cramér-Rao Bound.

Index Terms— Maximum likelihood estimator, Optimization methods, Position estimation.

1. INTRODUCTION

Global Navigation Satellite Systems (GNSS) is the general concept used to identify those systems that allow user position computation basing on a constellation of satellites. Specific GNSS systems are the well-known american GPS or the forthcoming european Galileo. Both systems rely on the same principle: the user computes its position from measured distances between the receiver and a set of in-view satellites. These distances are calculated estimating the propagation time that transmitted signals take from each satellite to the receiver and geometrically solving the position of the receiver by trilateralization [1]. Thus, the input for this estimation is the pseudorange of each satellite (i.e. the distance from the receiver to each satellite). In this paper, we take a novel approach to positioning by obtaining the Maximum Likelihood Estimator (MLE) of position, as opposite to synchronization–parameter based positioning where the synchronization parameters are first estimated and then the position is computed with these estimates. The approach herein proposed is seen to overcome many limitations of conventional GNSS receivers such as multipath propagation or signal blockages [2].

In a large variety of signal processing applications the Maximum Likelihood Estimator is the chosen approach for parameter estimation purposes. Unfortunately, in many cases there is not a closed-form solution for this estimator but a cost function to be optimized, as is the case of GNSS positioning. The cost function can be multi–dimensional and/or non–linear, which makes not feasible the use of gradient–based methods, such as the Steepest Descent or the Newton–Raphson algorithms. These algorithms diverge in the presence of high nonlinearities, thus alternative methods must be studied to deal with the optimization in a more suitable way. To this aim, the Space-Alternating Generalized Expectation Maximization (SAGE) algorithm has been investigated. The SAGE algorithm is a low–complexity generalization of the EM (Expectation Maximization) algorithm [3] which sequentially approximates the MLE [4]. A study of the computational cost required has been done and simulations are provided to compare the algorithm performance with the Cramér-Rao Bound in terms of positioning error.

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2. MAXIMUM LIKELIHOOD ESTIMATION OF POSITION IN GNSS

In a GNSS receiver, measurements are considered to be a superposition of plane waves corrupted by thermal noise and, possibly, interferences and multipath. The antenna receives M scaled, time–delayed and Doppler–shifted signals corresponding to each in-view satellite. The received complex baseband signal is modeled as

$$x(t) = \sum_{i=1}^{M} a_i s_i(t - \tau_i(\gamma)) \exp\{j2\pi f_{d_i}(\gamma)t\} + n(t) \qquad (1)$$

where $s_i(t)$ is the transmitted complex baseband low rate BPSK signal spreaded by the pseudorandom code of the i–th satellite, considered known, a_i is its complex amplitude, $\tau_i(\gamma)$ is the time-delay, $f_{d_i}(\gamma)$ the Doppler deviation and $n(t)$ is zero-mean additive white Gaussian noise (AWGN) of variance σ_n^2. The novelty relies on gathering all user motion parameters in a real vector γ, which can contain for instance position and velocity $\gamma = \left[\mathbf{p}^T, \mathbf{v}^T\right]^T$, and noticing that time–delays and Doppler shifts can be expressed as functions of γ from pseudorange and pseudorange rate expressions [1] [2].

If a receiver captures K snapshots, the model in equation (1) can be expressed as

$$\mathbf{x} = \mathbf{a}\mathbf{D}(\gamma) + \mathbf{n} \tag{2}$$

where

- $\mathbf{x} \in \mathbb{C}^{1\times K}$ is the observed signal vector,
- $\mathbf{a} \in \mathbb{C}^{1\times M}$ is a vector whose elements are the amplitudes of the M received signals $\mathbf{a} = [a_1 \quad \dots \quad a_M]$,
- $\mathbf{D}(\gamma) = [\mathbf{d}(t_0) \quad \dots \quad \mathbf{d}(t_{K-1})] \in \mathbb{C}^{M\times K}$, known as the basis–function matrix, being $\mathbf{d}(t) = [d_1 \quad \dots \quad d_M]^T \in \mathbb{C}^{M\times 1}$, where each component is defined by $d_i = s_i(t - \tau_i(\gamma))\exp\{j2\pi f_{d_i}(\gamma)t\}$ the delayed and Doppler shifted narrowband signal envelopes, and
- $\mathbf{n} \in \mathbb{C}^{1\times K}$ represents K snapshots of zero-mean AWGN with piecewise constant variance σ_n^2 during the observation interval.

We now consider the Maximum Likelihood Estimation (MLE) of signal parameters taking into account the measurement model presented in equation (2), parametrized by motion parameter vector γ. We first take into account that the MLE is equivalent to the solution obtained by a Least Squares (LS) criteria under the assumption of zero-mean AWGN. Neglecting additive and multiplicative constants, maximizing the likelihood function of measurement equation (2) is equivalent to minimizing the following Nonlinear Least Squares (NLLS) problem

$$\Lambda(\mathbf{a}, \gamma) = ||\mathbf{x} - \mathbf{a}\mathbf{D}(\gamma)||^2 \tag{3}$$

where the operator $||\cdot||$ denotes the L^2–norm of a vector. A straightforward gradient computation yields to the MLE of complex amplitudes:

$$\hat{\mathbf{a}}_{ML} = \mathbf{x}\mathbf{D}^H(\gamma)\left(\mathbf{D}(\gamma)\mathbf{D}^H(\gamma)\right)^{-1}\Big|_{\gamma=\hat{\gamma}_{ML}} \tag{4}$$

The ML estimation of considered parameters is then obtained by minimizing the nonlinear cost function resulting from the substitution of (4) in (3),

$$\hat{\gamma}_{ML} = \arg\min_{\gamma}\{\Lambda(\gamma)\} \tag{5}$$

where the resulting ML cost function can be defined in terms of a signal-subspace projection:

$$\begin{aligned}\Lambda(\gamma) &= \mathbf{x}\mathbf{x}^H - \mathbf{x}\mathbf{D}^H(\gamma)\left(\mathbf{D}(\gamma)\mathbf{D}^H(\gamma)\right)^{-1}\mathbf{D}(\gamma)\mathbf{x}^H \\ &= \mathbf{x}\left(\mathbf{I} - \mathbf{D}^H(\gamma)\left(\mathbf{D}(\gamma)\mathbf{D}^H(\gamma)\right)^{-1}\mathbf{D}(\gamma)\right)\mathbf{x}^H \\ &= \mathbf{x}\left(\mathbf{I} - \Pi(\gamma)\right)\mathbf{x}^H = \left|\left|\mathbf{x}\Pi^{\perp}(\gamma)\right|\right|^2 \end{aligned} \tag{6}$$

being $\Pi(\gamma)$ the projection matrix onto the subspace spanned by $\mathbf{D}^H(\gamma)$ and $\Pi^{\perp}(\gamma)$ is its orthogonal complement. In addition, we have taken into account that projection matrices are idempotent. Hence, the cost function is equivalent to the projection of data in the orthogonal complement of the signal subspace, defined by $\mathbf{D}(\gamma)$.

Whereas in the synchronization–parameter based positioning a two-dimensional optimization has to be performed for each tracked satellite [1], the position-dependent cost function takes into account signals coming from all satellites to obtain a position estimate, dealing with a single multivariate optimization problem for all the received satellites. For the sake of clarity and without loss of generality, we now consider that one of the coordinates (say z) and the velocity vector are known (or vary slowly with time and can be tracked by other means) so that we can plot the three-dimensional likelihood function, being $\gamma = [x, y]^T$. Figure 1 shows the cost function in equation (5) in a realistic GNSS scenario. The considered benchmark scenario is composed of 6 satellites forming a four–sided pyramid in a hemisphere with two satellites at zenith, being all satellites equally spaced. According to [1], this constellation geometry optimizes the Geometric Dilution Of Precision (GDOP), which is a parameter that gives information of the quality of the constellation geometry.

A gradient–like method can be used to iteratively minimize the cost function, such as the Steepest Descent or the Newton-Raphson algorithm. However, these methods diverge when the function to optimize has high nonlinearities and an increasing dimensionality, in the general multi-dimensional case γ could consist of the three–dimensional position, velocity and acceleration among other possible parameters. Hence, alternative methods must be studied to deal with the optimization in a more suitable and implementable way. Aiming at finding a computationally affordable algorithm to compute the MLE of position, Expectation-Maximization algorithms have been explored.

3. THE SAGE ALGORITHM

The Space-Alternating Generalized Expectation Maximization algorithm, SAGE for short, is a low–complexity generalization of the EM (Expectation Maximization) algorithm, which iteratively approximates the MLE [4]. The SAGE algorithm deals with high dimensional and non-linear cost functions, instead of optimizing directly the cost function it performs a sequence of optimization steps in spaces of lower di-

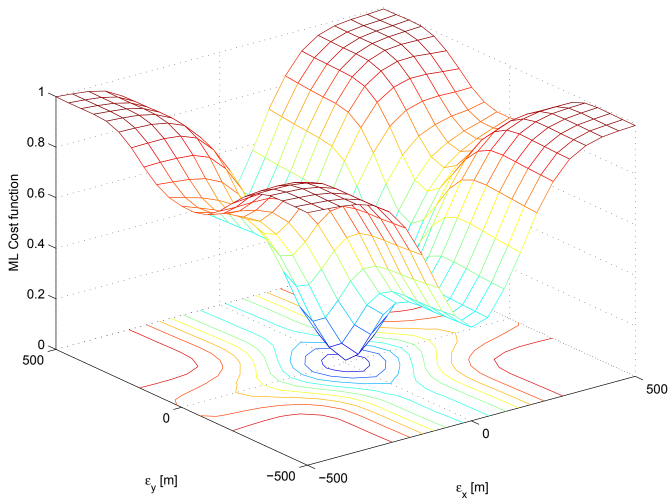

Fig. 1. The ML cost function in equation (5) for a realistic scenario ($M = 6$ satellites) as a function of different x and y coordinate errors, denoted as ε_x and ε_y respectively. In general, a single multi-dimensional optimization has to be solved.

mension and thereby reducing the problem complexity considerably. Basically, the SAGE algorithm sequentially estimates a reduced subset of the unknown parameter vector while keeping the others fixed.

Without loss of generality, we now consider the case of estimating the three-dimensional receiver position. Thus, the vector of unknown parameters considered is composed of the receiver coordinates:

$$\gamma = \mathbf{p} \triangleq \begin{bmatrix} x & y & z \end{bmatrix}^T \tag{7}$$

As aforementioned, instead of optimizing the cost function directly with respect to γ, the SAGE algorithm simplifies the problem into a number of decoupled optimization problems. Thus, performing single optimization steps with respect to a reduced set of elements of the vector of unknown parameters at a time. In our case $\Lambda(\gamma)$ is optimized with respect to a single parameter in each M-step. Hence, an iteration cycle (defined as the consecutive iteration steps for updating the whole vector γ) of the SAGE algorithm is expressed as

$$\begin{aligned} \hat{x}'' &= \arg\min_{x} \left\{ \Lambda\left(x; \hat{y}', \hat{z}'\right)\right\} \\ \hat{y}'' &= \arg\min_{y} \left\{ \Lambda\left(y; \hat{x}'', \hat{z}'\right)\right\} \\ \hat{z}'' &= \arg\min_{x} \left\{ \Lambda\left(z; \hat{x}'', \hat{y}''\right)\right\} \end{aligned} \tag{8}$$

where $(\cdot)'$ denotes previous parameter estimation and $(\cdot)''$ the updated estimate. The multivariate optimization problem in equation (5) is here splitted in several one-dimensional optimization problems, to be performed iteratively N_t times. These one-dimensional optimizations can be easily solved by varying the corresponding γ component and obtaining the optimum by gradient computation and interpolation [5] [6].

In the general case of an L–dimensional unknown parameters space γ, we can define

$$\tilde{\gamma}_{(i)} = \left[\gamma''_{(1)}, \ldots, \gamma''_{(i-1)}, \gamma'_{(i+1)}, \ldots, \gamma'_{(L)}\right]^T \tag{9}$$

being an $(L-1) \times 1$ vector where all considered parameters are fixed to its latest estimates except for the i–th element of γ on which the optimization is performed. Hence, the algorithm operates as:

Algorithm *MLE of position with the SAGE algorithm*
($*$ SAGE implementation of equation (5) $*$)
1. Initial Estimates: $\hat{\gamma}'$
2. $k = 1$
3. **for** $i \leftarrow 1$ **to** L
4. E-step: Compute $\Lambda(\gamma_{(i)}; \tilde{\gamma}_{(i)})$
5. M-step: $\hat{\gamma}''_{(i)} = \arg\min_{\gamma_{(i)}} \{\Lambda(\gamma_{(i)}; \tilde{\gamma}_{(i)})\}$
6. **end**
7. $k \leftarrow k+1$
8. **if** $k \leq N_t$
9. **then**
10. $\hat{\gamma}' \leftarrow \hat{\gamma}''$
11. go to line 3
12. **else**
13. Final Estimates: $\hat{\gamma}_{SAGE} \leftarrow \hat{\gamma}''$
14. **end**
15. Extract position information from the estimated $\hat{\gamma}_{SAGE}$ vector, $\hat{\mathbf{p}}$.

4. COMPUTATIONAL COST

As the most burdensome operation of the algorithm is to evaluate the cost function in equation (6), it is useful to define C_Λ as the computational effort needed to compute it. Hence, from the pseudo code description *MLE of position with the SAGE algorithm* it is apparent that the computational cost of the SAGE algorithm is

$$C_{SAGE} = N_t \cdot L \cdot N_e \cdot C_\Lambda \tag{10}$$

where N_e is the number of points used to characterize each one–dimensional problem, i.e. the number of samples of the cost function computed in the E-step.

For simulation purposes, two different set of algorithm parameters have been used. On the one hand, the proposed SAGE algorithm use typical values of $N_e = 20$ points and $N_t = 8$ iterations for estimating $L = 3$ parameters. Thus, the computational cost is $C_{SAGE} = 480 \cdot C_\Lambda$, as arises from equation (10). On the other hand, another set of values considered are $N_e = 20$ points and $N_t = 12$ iterations, meaning that $C_{SAGE} = 720 \cdot C_\Lambda$ computational resources are needed.

5. SIMULATION RESULTS

The simulated satellites, in the benchmark constellation described in Section 2, transmit C/A code GPS signals, whose chip rate is $f_c = 1.023$ MHz. The received signals are filtered with a 2 MHz bandwidth filter and a IF sampling frequency of $f_s = 5.714$ MHz is considered. 1 ms of data has been recorded and processed. The three-dimensional coordinates of the receiver compose the vector of unknown parameters γ, as considered in equation (7).

In Figure 2 the performance of the proposed SAGE algorithm is shown in terms of positioning Root Mean Square Error (RMSE), defined as

$$\xi = ||\mathbf{p} - \hat{\mathbf{p}}|| = \sqrt{(x-\hat{x})^2 + (y-\hat{y})^2 + (z-\hat{z})^2} \qquad (11)$$

being $\hat{\mathbf{p}}$ the estimated position vector and the operator $||\cdot||$ the L^2–norm of a vector. The considered carrier-to-noise density ratio (C/N_0) values are typical in GNSS links. In addition, the Cramér-Rao Bound (CRB) of variances has been computed in order to assess the optimal behavior of the SAGE algorithm. Taking into account different computational costs, as explained in Section 4, we have also considered two different initialization points. The initial ambiguity is identically defined for all coordinates, thus forming a sphere whose center is the position guess and its radius the considered uncertainty (defined as r_u). From Figure 2, it is apparent that the SAGE algorithm depends on the error committed in the initialization point. Accordingly, the lower the ambiguity radius, the less the computational burden needed to achieve the CRB. In particular, for $r_u = 1$ meter, the algorithm attains the CRB even when relaxing the computational requirements. The SAGE algorithm can claim to properly approximate the ML solution when the CRB is attained, which is known to yield the lower variance, i.e. the CRB. In addition, the stabilization of the error to a certain value is mainly due to the need of increasing the number of iterations of the algorithm. At the light of the results, it is convenient to study proper initialization methods in order to reduce the computational complexity needed by the algorithm to achieve the lower bound.

6. CONCLUSIONS

In this paper, an efficient way of implementing the Maximum Likelihood Estimator (MLE) of position in GNSS receivers is proposed. This novel approach to position calculation in GNSS receivers was seen to be robust against fading multipath channels [2] and signal blockages [7]. The SAGE algorithm is studied to iteratively approximate the MLE, being a nonlinear and multidimensional function of the unknown parameters. The paper contains a description of the proposed algorithm and a computational cost study. Computer simulation results show that the algorithm is sensitive to proper initialization, thus stressing the need of investigating good initialization strategies. Nonetheless, for proper initial estimates the SAGE algorithm is seen to attain the Cramér-Rao Bound.

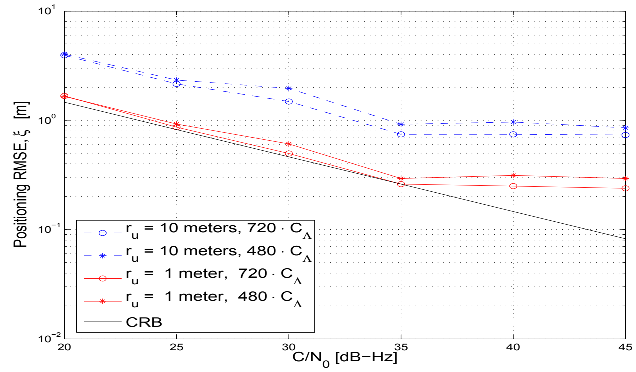

Fig. 2. Position estimation error versus C/N_0 of the satellites in the benchmark scenario considered.

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