ASYMPTOTIC CRAMÉR-RAO BOUND FOR MULTI-DIMENSIONAL HARMONIC MODELS

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1. ABSTRACT

The multi-dimensional harmonic model has attracted considerable attention for a variety of applications in signal processing. Stoica and Nehorai have derived the Asymptotic (*ie.*, for large analysis duration) Cramér-Rao lower Bound (ACRB) which represents the minimal theoretical variance in the estimation of the model parameters for a one-dimensional harmonic model of order M. In this work, we generalize and analyze the ACRB associated to a M-order harmonic model of dimension P with $P > 1$.

Keywords: Parameter estimation, multidimensional signal processing.

2. INTRODUCTION

The one-dimensional harmonic model is very useful in many fields such as signal processing, audio compression, digital communications and others. A generalization of this model to $P > 1$ dimensions can be encountered in several domains such as MIMO channel estimation [1], wireless communications [2], passive localization and radar processing, etc. In addition, we can find in [3, 4] an analysis of the identification problem associated with this model.

In this contribution, we derive Asymptotic (*ie.*, for large analysis duration) expressions of the Cramér-Rao Lower bound (ACRB) for a M-order harmonic model (sum of M waveforms) of dimension P. This work can be viewed as an extension of the seminal work of Stoica and Nehorai [5] for the one-dimensional harmonic model. Although, much work has been done on the determination of the ACRB for small P, *ie.*, for $P = 2$ (two-dimensional harmonic model) [6] or for $P = 3; 4$ in the context of a sensor array [7]. But to our best knowledge, we cannot find a systematic characterization of the ACRB for any dimension P. The philosophy of our approach is similar that of reference [7] since we base our derivation on tensor algebra but our ACRB is dedicated to the multi-dimensional harmonic model of any dimension P and in particular, we examine its asymptotic properties.

We begin by the standard result that a M-order harmonic model of dimension P follows a CanDecomp/Parafac (CP) model [8] of order M. In other words, the rank of the P-order tensor associated with this model is M and can be exactly decomposed into the sum of M rank-1 tensors. Using this formalism, we show that the "order of magnitude" of the ACRB for the the signal parameters (angular-frequency, real amplitude and initial phase) are respectively $O(N^{-P-2})$, $O(N^{-P})$ and $O(N^{-P})$ where N is the analysis duration. We have also shown that the quotient of two consecutive ACRB for the signal parameters, *ie.* $\text{ACRB}_{(P)}(.)/\text{ACRB}_{(P+1)}(.)$, is $O(N)$ and becomes exactly equal to N for larger dimension P. Consequently, we prove the intuitive idea that increasing the dimension of the model decreases the ACRB and thus improves the minimal theoretical variance of the estimated model parameters.

3. DEFINITION OF THE MULTI-DIMENSIONAL HARMONIC MODEL

We define a noisy M-order harmonic model of dimension P according to

$$[\mathcal{Y}]_{n_1 \dots n_P} = [\mathcal{X}]_{n_1 \dots n_P} + \sigma[\mathcal{N}]_{n_1 \dots n_P} \quad (1)$$

where $\mathcal{Y}$, $\mathcal{X}$ and $\mathcal{N}$ are three P-order hypercubic tensors of size N, σ is a positive real scalar and

$$[\mathcal{X}]_{n_1 \dots n_P} = \sum_{m=1}^{M} \alpha_m \prod_{p=1}^{P} e^{i\omega_m n_p}. \quad (2)$$

is the noise-free M-order harmonic model of dimension P with $n_p \in [0 : N-1]$. The m-th complex amplitude is denoted by $\alpha_m = a_m e^{i\phi_m}$ where $a_m > 0$ is the m-th real amplitude and ϕ_m is the m-th initial phase. It is well-known that this model follows a CP model [8, 3, 4] and its associated vectorized expression is

$$y = \text{vec}(\mathcal{Y}) = x + \sigma n \ \in \ \mathbb{C}^{N^P} \quad (3)$$

where $n = \text{vec}(\mathcal{N})$ is the additive white Gaussian noise of parameters $\mathcal{N}(0, I_{N^P})$ and

$$x = \text{vec}(\mathcal{X}) = \sum_{m=1}^{M} \alpha_m \underbrace{\left(d(\omega_m) \otimes \ldots \otimes d(\omega_m)\right)}_{P \text{ times}} \quad (4)$$

in which $\otimes$ denotes the Kronecker product and

$$d(\omega) = \begin{bmatrix} 1 & e^{i\omega} & \ldots & e^{i\omega(N-1)} \end{bmatrix}^T.$$

4. ACRB FOR THE MULTI-DIMENSIONAL HARMONIC MODEL

The noisy observation y in (3) is a function of the real parameter vector θ given by $\theta = \begin{bmatrix}\theta'^T & \sigma^2\end{bmatrix}^T$ where $\theta' = [\omega^T\ a^T\ \phi^T]^T$, $\omega = [\omega_1 \dots \omega_M]^T$, $a = [a_1 \dots a_M]^T$ and $\phi = [\phi_1 \dots \phi_M]^T$. A very famous result [9, 10] is the following. Let $\Gamma = E\left\{(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T\right\}$ be the covariance matrix of an unbiased estimate of θ, denoted by $\hat{\theta}$ then under quite general conditions, $\Gamma - \text{ACRB}_{(P)}(\theta)$, where $\text{ACRB}_{(P)}(.)$ is the Asymptotic Cramér-Rao lower Bound associated with a harmonic model of dimension P, is a positive semidefinite matrix.

The likelihood function of $y \sim \mathcal{N}(x, \sigma^2 I_{N^P})$ is given by

$$L(y) = \frac{1}{(\pi\sigma^2)^{N^P}}\ e^{-\frac{1}{\sigma^2}\|y-x\|^2} = \frac{1}{(\pi\sigma^2)^{N^P}}\ e^{\frac{-1}{\sigma^2}\left\|y - \sum_{m=1}^{M}\alpha_m(d(\omega_m)\otimes\dots\otimes d(\omega_m))\right\|^2}.$$

The log-likelihood function is defined by

$$f(\theta) = \ln(L(y)) = c - \frac{1}{\sigma^2}\left\|y - \sum_{m=1}^{M}\alpha_m\Big(d(\omega_m)\otimes\dots\otimes d(\omega_m)\Big)\right\|^2$$

where c is a given constant. As the signal and nuisance parameters are decoupled, the Fisher Information Matrix (FIM) is given by

$$F_{\theta\theta} = \begin{bmatrix}\frac{2}{\sigma^2}F_{\theta'\theta'} & 0\\ 0 & J_{\sigma^2\sigma^2}\end{bmatrix} \tag{5}$$

where

$$F_{\theta'\theta'} = \begin{bmatrix}J_{\omega\omega} & J_{\omega a} & J_{\omega\phi}\\ J_{\omega a}^H & J_{aa} & J_{a\phi}\\ J_{\omega\phi}^H & J_{a\phi}^H & J_{\phi\phi}\end{bmatrix} \tag{6}$$

in which we have defined

$$J_{pq} = \Re\left\{\left(\frac{\partial x}{\partial p}\right)^H \frac{\partial x}{\partial q}\right\} \tag{7}$$

with $\Re\{.\}$ being the real part of a complex number.

4.1. Asymptotic CRB for a M-order harmonic model of dimension P

In the sequel, we consider large analysis duration ($N \to \infty$) where closed-form expressions of the $\text{ACRB}_{(P)}$ can be obtained.

Theorem 1 *The $ACRB_{(P)}$ for a M-order harmonic model of dimension P defined in (1) with respect to (*wrt.*) the model parameter θ',* ie.*, $ACRB_{(P)}(\theta')$, is given by*

$$ACRB_{(P)}(\omega_\ell) = \frac{6}{PN^{P+2}SNR_\ell}, \tag{8}$$

$$ACRB_{(P)}(a_\ell) = \frac{a_\ell^2}{2N^P SNR_\ell}, \tag{9}$$

$$ACRB_{(P)}(\phi_\ell) = \frac{3P+1}{2N^P SNR_\ell}. \tag{10}$$

where $SNR_\ell = a_\ell^2/\sigma^2$.

Proof: The partial derivatives of the noise-free signal wrt. *the angular-frequency, the real amplitude and the initial phase are given by*

$$\frac{\partial x}{\partial \omega_\ell} = i\alpha_\ell\left(\frac{\partial d(\omega_\ell)}{\partial\omega_\ell}\otimes d(\omega_\ell)\otimes\dots\otimes d(\omega_\ell) + d(\omega_\ell)\otimes\frac{\partial d(\omega_\ell)}{\partial\omega_\ell}\otimes\dots\otimes d(\omega_\ell) + \dots + d(\omega_\ell)\otimes\dots\otimes d(\omega_\ell)\otimes\frac{\partial d(\omega_\ell)}{\partial\omega_\ell}\right)$$

$$\frac{\partial x}{\partial a_\ell} = e^{i\phi_\ell}\left(d(\omega_\ell)\otimes\dots\otimes d(\omega_\ell)\right)$$

$$\frac{\partial x}{\partial \phi_\ell} = i\alpha_\ell\left(d(\omega_\ell)\otimes\dots\otimes d(\omega_\ell)\right)$$

for $\ell \in [1 : M]$ and where we have denoted $\frac{\partial d(\omega_\ell)}{\partial\omega_\ell} = \left.\frac{\partial d(\omega)}{\partial\omega}\right|_{\omega=\omega_\ell}$. Using the asymptotic properties of the harmonic model [5],

$$\frac{1}{N^3}\left(\frac{\partial d(\omega_k)}{\partial\omega_k}\right)^H\frac{\partial d(\omega_\ell)}{\partial\omega_\ell} \overset{N\to\infty}{\longrightarrow} \frac{1}{3}\delta_{k-\ell} \tag{11}$$

$$\frac{1}{N^2}\left(\frac{\partial d(\omega_k)}{\partial\omega_k}\right)^H d(\omega_\ell) \overset{N\to\infty}{\longrightarrow} \frac{1}{2}\delta_{k-\ell} \tag{12}$$

$$\frac{1}{N}d(\omega_k)^H d(\omega_\ell) \overset{N\to\infty}{\longrightarrow} \delta_{k-\ell}, \tag{13}$$

we can derive the (k,ℓ)-th entry of each block of the FIM according to

$$\begin{aligned}[J_{\omega\omega}]_{k\ell} &= \Re\left\{\left(\frac{\partial x}{\partial\omega_k}\right)^H\frac{\partial x}{\partial\omega_\ell}\right\}\\ &\overset{N\to\infty}{\longrightarrow} \Re\left\{\alpha_k^*\alpha_\ell\ P\left(\frac{N^{P+2}}{3} + (P-1)\frac{N^{P+2}}{4}\right)\delta_{k-\ell}\right\}\\ &= \Re\left\{\alpha_k^*\alpha_\ell\ PN^{P+2}\left(\frac{4+3(P-1)}{12}\right)\delta_{k-\ell}\right\}\\ &= \begin{cases} a_k^2\ PN^{P+2}\ \frac{3P+1}{12}, & \textit{for } k=\ell,\\ 0, & \textit{otherwise.}\end{cases}\end{aligned} \tag{14}$$

Similarly, we have

$$[J_{aa}]_{k\ell} = \Re\left\{\left(\frac{\partial x}{\partial a_k}\right)^H\frac{\partial x}{\partial a_\ell}\right\} \overset{N\to\infty}{\longrightarrow} \Re\left\{e^{i(\phi_\ell-\phi_k)}N^P\delta_{k-\ell}\right\} = \begin{cases} N^P, & \textit{for } k=\ell,\\ 0, & \textit{otherwise.}\end{cases}$$

and

$$[J_{\phi\phi}]_{k\ell} = \Re\left\{\left(\frac{\partial x}{\partial\phi_k}\right)^H\frac{\partial x}{\partial\phi_\ell}\right\} \overset{N\to\infty}{\longrightarrow} \Re\left\{\alpha_k^*\alpha_\ell N^P\delta_{k-\ell}\right\} = \begin{cases} a_k^2\ N^P, & \textit{for } k=\ell,\\ 0, & \textit{otherwise.}\end{cases}$$

and

$$
\begin{aligned}
[J_{a\phi}]_{k\ell} &= \Re\left\{\left(\frac{\partial x}{\partial a_k}\right)^H \frac{\partial x}{\partial \phi_\ell}\right\} \\
&\overset{N\to\infty}{\longrightarrow} \Re\left\{ie^{-i\phi_k}\alpha_\ell N^P \delta_{k-\ell}\right\} \\
&= 0, \ \forall k, \ell
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
[J_{\omega a}]_{k\ell} &= \Re\left\{\left(\frac{\partial x}{\partial \omega_k}\right)^H \frac{\partial x}{\partial a_\ell}\right\} \\
&\overset{N\to\infty}{\longrightarrow} -\Re\left\{i\alpha_k^* e^{i\phi_\ell}\frac{P}{2}N^{P+1}\delta_{k-\ell}\right\} \\
&= 0, \ \forall k, \ell.
\end{aligned} \tag{16}
$$

For $k = \ell$, $ie^{-i\phi_k}\alpha_\ell$ in (15) and $i\alpha_k^* e^{i\phi_\ell}$ in (16) are pure imaginary numbers. This explains why $J_{a\phi}$ and $J_{\omega a}$ are null matrices. Finally, we have

$$
\begin{aligned}
[J_{\omega\phi}]_{k\ell} &= \Re\left\{\left(\frac{\partial x}{\partial \omega_k}\right)^H \frac{\partial x}{\partial \phi_\ell}\right\} \\
&\overset{N\to\infty}{\longrightarrow} \Re\left\{\alpha_k^*\alpha_\ell\frac{P}{2}N^{P+1}\delta_{k-\ell}\right\} \\
&= \begin{cases} a_k^2\,\frac{P}{2}\,N^{P+1}, & \textit{for } k=\ell, \\ 0, & \textit{otherwise.}\end{cases}
\end{aligned} \tag{17}
$$

Consequently, the blocks of the FIM are asymptotically diagonal or null and we obtain

$$
\begin{aligned}
J_{\omega\omega} &= \frac{P(3P+1)}{12}N^{P+2}\Delta^2 \\
J_{aa} &= N^P I_M \\
J_{\phi\phi} &= N^P\Delta^2 \\
J_{a\phi} &= J_{\omega a} = 0 \\
J_{\omega\phi} &= \frac{P}{2}N^{P+1}\Delta^2
\end{aligned}
$$

where $\Delta = \text{diag}\{a_1, \ldots, a_M\}$. Finally, the FIM is given by

$$
F_{\theta'\theta'} = \begin{bmatrix} \frac{P(3P+1)}{12}N^{P+2}\Delta^2 & 0 & \frac{P}{2}N^{P+1}\Delta^2 \\ 0 & N^P I_M & 0 \\ \frac{P}{2}N^{P+1}\Delta^2 & 0 & N^P\Delta^2 \end{bmatrix}. \tag{18}
$$

Thanks to the standard inverse of a partitioned matrix [5], analytic expression of $F_{\theta'\theta'}^{-1}$ is possible. Thus,

$$
F_{\theta'\theta'}^{-1} = \begin{bmatrix} \Lambda & 0 & \times \\ 0 & J_{aa}^{-1} & 0 \\ \times & 0 & \Theta\Lambda\Theta + J_{\phi\phi}^{-1} \end{bmatrix} \tag{19}
$$

where

$$
\begin{aligned}
\Lambda &= (J_{\omega\omega} - J_{\omega\phi}J_{\phi\phi}^{-1}J_{\omega\phi})^{-1} \\
&= \left(\frac{P(3P+1)}{12}N^{P+2}\Delta^2 - \frac{P^2}{4}N^{P+2}\Delta^2\right)^{-1} \\
&= \left(\frac{P(3P+1)}{12} - \frac{P^2}{4}\right)^{-1}\frac{1}{N^{P+2}}\Delta^{-2} \\
&= \frac{12}{PN^{P+2}}\Delta^{-2}
\end{aligned}
$$

and $\Theta = J_{\phi\phi}^{-1}J_{\omega\phi} = \left(N^P\Delta^2\right)^{-1}\frac{P}{2}N^{(P+1)}\Delta^2 = \frac{PN}{2}$. So, the $(3,3)$-block of matrix $F_{\theta'\theta'}^{-1}$ is given by

$$
\begin{aligned}
\Theta\Lambda\Theta + J_{\phi\phi}^{-1} &= \frac{PN}{2}\left(\frac{12}{PN^{P+2}}\Delta^{-2}\right)\frac{PN}{2} + \left(N^P\Delta^2\right)^{-1} \\
&= \frac{3P+1}{N^P}\Delta^{-2}.
\end{aligned}
$$

Hence, the inverse of the FIM is

$$
F_{\theta'\theta'}^{-1} = \begin{bmatrix} \frac{12}{PN^{P+2}}\Delta^{-2} & 0 & \times \\ 0 & \frac{1}{N^P}I_M & 0 \\ \times & 0 & \frac{3P+1}{N^P}\Delta^{-2} \end{bmatrix}.
$$

So, the ACRB associated to a M-order harmonic model of dimension P is given by the diagonal terms of the FIM inverse which proves the theorem. ■

4.2. Tabulation of the ACRB for several values of P

The values of the ACRB for several dimension P are given in the following table.

P	$\text{ACRB}_{(P)}(\omega_\ell)$	$\text{ACRB}_{(P)}(a_\ell)$	$\text{ACRB}_{(P)}(\phi_\ell)$
1 [5]	$\frac{6}{N^3\text{SNR}_\ell}$	$\frac{a_\ell^2}{2N\text{SNR}_\ell}$	$\frac{2}{N\text{SNR}_\ell}$
2	$\frac{3}{N^4\text{SNR}_\ell}$	$\frac{a_\ell^2}{2N^2\text{SNR}_\ell}$	$\frac{7}{2N^2\text{SNR}_\ell}$
3	$\frac{2}{N^5\text{SNR}_\ell}$	$\frac{a_\ell^2}{2N^3\text{SNR}_\ell}$	$\frac{5}{N^3\text{SNR}_\ell}$
4	$\frac{3}{2N^6\text{SNR}_\ell}$	$\frac{a_\ell^2}{2N^4\text{SNR}_\ell}$	$\frac{13}{2N^4\text{SNR}_\ell}$
5	$\frac{6}{5N^7\text{SNR}_\ell}$	$\frac{a_\ell^2}{2N^5\text{SNR}_\ell}$	$\frac{8}{N^5\text{SNR}_\ell}$

4.3. Order of magnitude of the ACRB

According to the previous Theorem, we can characterize the $\text{ACRB}_{(P)}(\theta')$ according to the following corollary.

Corollary 1 *Consider a M-order harmonic model of dimension P, then we have*

$$
ACRB_{(P)}(\omega_\ell) \sim O\left(\frac{1}{N^{P+2}}\right) \tag{20}
$$

$$
ACRB_{(P)}(a_\ell) \sim O\left(\frac{1}{N^P}\right) \tag{21}
$$

$$
ACRB_{(P)}(\phi_\ell) \sim O\left(\frac{1}{N^P}\right). \tag{22}
$$

As a consequence of the above result, increasing the dimension of the harmonic model decreases the ACRB.

Corollary 2 *For $i \in [1:3M]$, the quotient of two consecutive ACRB,* ie., *$ACRB_{(P)}(\theta_i')/ACRB_{(P+1)}(\theta_i')$ is of order $O(N)$ for any P and is equal to N for large P.*

Proof: *Using (8), (9) and (10) we have*

$$\frac{ACRB_{(P)}(\omega_\ell)}{ACRB_{(P+1)}(\omega_\ell)} = \left(\frac{P+1}{P}\right) N \tag{23}$$

$$\frac{ACRB_{(P)}(a_\ell)}{ACRB_{(P+1)}(a_\ell)} = N \tag{24}$$

$$\frac{ACRB_{(P)}(\phi_\ell)}{ACRB_{(P+1)}(\phi_\ell)} = \left(\frac{3P+1}{3P+4}\right) N. \tag{25}$$

Hence the above expressions show that $ACRB_{(P)}(\theta_i')/ACRB_{(P+1)}(\theta_i') \sim O(N) \geq 0$ which proves the first part of the corollary. For larger (infinite) P, it is direct to see that the limit of (25) and (27) is N which proves the second part of the corollary. Note that no condition on P is required for the real amplitude parameter. ■

Consequently, for large P, the quotient of two consecutive ACRB follows a geometric progression since this quotient is a constant and we have

$$\text{ACRB}_{(P)}(\theta_i') = \frac{1}{N^{P-1}} \text{ACRB}_{(1)}(\theta_i') \tag{26}$$

where $\text{ACRB}_{(1)}(\theta_i')$ is the bound derived by Stoica and Nehorai [5] for $P = 1$.

5. NUMERICAL SIMULATIONS

In this part, we illustrate the behavior of the ACRB *wrt.* the dimension, $P \in [1:5]$, of a 500-sample first-order harmonic model. We vary the Signal to Noise Ratio (SNR), by taking σ^2 from 1 to 100 and $a_1 = 1$. Fig. 1, Fig. 2-a and b display the ACRB of the angular frequency, the real amplitude and the initial phase, respectively.

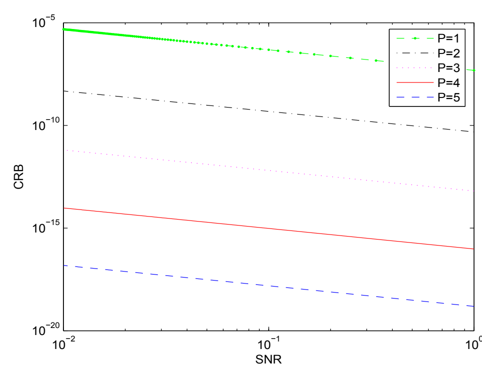
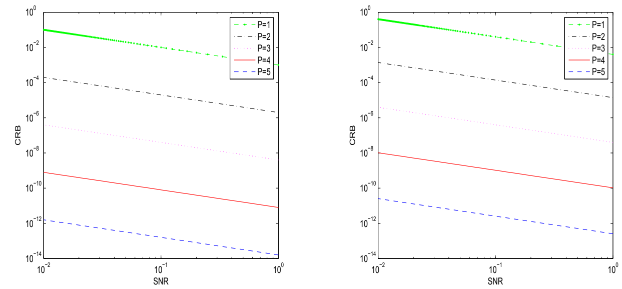


Figure 1: ACRB Vs. SNR (log scale) for the angular frequency of a first order harmonic model of dimension P.

According to these simulations, we can see clearly the gain to consider large dimensions.



(a) (b)

Figure 2: ACRB Vs. SNR (log scale) for a first order harmonic model of dimension P, (a) Real amplitude, (b) Initial phase.

6. CONCLUSIONS

In this work, the asymptotic CRB (ACRB) associated with a M-order harmonic model of dimension P using tensors algebra has been achieved. It has been shown that increasing the dimension of the harmonic model decreases the ACRB and thus improves the minimal theoretical variance of the estimated model parameters. In addition, we have proved that the quotient of two consecutive ACRB for the signal parameters, *ie.* $\text{ACRB}_{(P)}(.)/\text{ACRB}_{(P+1)}(.)$, is $O(N)$ and becomes exactly equal to N for large dimension P. This last result allows us to give a compact expression of the ACRB for any dimension *wrt.* the well-known bound derived by Stoica and Nehorai for $P = 1$.

7. REFERENCES

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