A NEW PARAMETRIC METHOD FOR TIME DELAY ESTIMATION

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ABSTRACT

In this paper, a new parametric method for time delay estimation is proposed. The method, classified under the generalized cross-correlation (GCC) approach, uses a couple of identical FIR filters to process the data from each sensor. A set of filters is designed to maximize the output cross-correlation at each lag. The lag associated with the maximum of all filtered cross-correlation is the estimate of the time delay,and its associated FIR filter is the optimum processor. The proposed method is implemented in the time domain and does not need spectral information as for the classical GCC methods. Its implementation uses eigen-decomposition algorithms and requires one input parameter which is the order of the FIR filter. It is equivalent to applying a data-driven bandpass filter to the cross-correlogram to emphasis the source signal. The proposed method is compared with standard methods in a simulation study. Simulation results show very good performance for the case of short data records and at low to moderate SNR levels.

Index Terms— Time delay estimation, Generalized cross correlation.

1. INTRODUCTION

The problem of measuring the time delay between noisy signals received at two or more remote sensors has diverse applications in areas such as sonar, acoustics, geophysics, and biomedical engineering. Specifically, let the two received signals be:

$$\begin{aligned} x_1(n) &= s(n) + \epsilon_1(n) \\ x_2(n) &= \alpha s(n-D) + \epsilon_2(n). \end{aligned} \tag{1}$$

The source signal $s(n)$ and the corrupting noise $\epsilon_1(n)$ and $\epsilon_2(n)$ are assumed jointly uncorrelated. D is the time delay between the received signals and α is an unknown attenuation factor. The problem here is to estimate D based on the measurements $\{x_1(n), x_2(n)\}$, $n = 0, 1, \dots, N-1$, and without any *a priori* knowledge of the source signal.

Many of the methods developed for time delay estimation (TDE) are related through the *generalized cross correlation* (GCC) method. In this approach, the estimated cross correlation sequence of the received signals is convolved with a weighting filter. The time lag for which the filtered cross-correlogram is maximized is the time delay estimate. Optimal weighting filters include the maximum likelihood (ML) estimator originally proposed by Hanann and Thomson [1]. Knap and Carter [2], in a comparative analysis with other estimators, derived the same filter and showed that it is equivalent to applying two prefilters on each of the signals in (1) followed by cross-correlation.

It is difficult to design the true optimal weighting filter since exact knowledge of source and noise spectra are needed. In practice, this is usually done using the non-parametric spectral estimation method of Welch's modified periodogram [3]. Comparative simulation studies have shown that the degradation in TDE performance due to filter estimation error can be sufficiently large to preclude the use of GCC technique and favour the standard (unfiltered) cross-correlogram [4], [5].

To avoid spectral estimation, an alternative approach called *time delay parameter estimation* (TDPE) is proposed in [6]. This approach models the delay between the two signals by a finite impulse response (FIR) filter in one of the signals as input, say for example $x_1(n)$. The weights of the filter are adaptively adjusted to minimize the mean-squares difference between $x_2(n)$ and the output of the filter. The filter tap number corresponding to the maximum absolute weight is the TDPE delay estimate.

TDPE is very attractive in estimating non-stationary time delay and is implemented in the time domain. However, care should be paid in the choice of the filter length p, which should be such that $p > |D|$. In the case of large D, or when exact knowledge on the bound of D is not available, p is to be chosen relatively large. This will increase the variance of the estimated FIR filter's parameters and a difficult convergence monitoring of the LMS algorithm is needed. This makes estimation of the time delay less accurate and difficult.

In this paper, we propose a new method for time delay estimation that can be classified under the GCC approach. A parametric model is used instead of a non-parametric one to determine the appropriate prefilters. The two prefilters, supposed identical without loss of generality, are modeled as FIR filters. A set of FIR filters is designed to maximize the output of the generalized cross-correlator at each lag. The lag corresponding to the maximum of all maximized generalized cross-correlation outputs is then chosen as the estimate of the time delay and its associated FIR filter is the optimum prefilter. We present theoretical arguments supported by empirical results that the new method, named *parametric generalized cross-correlation* (PGCC), is *approximately* equivalent to a bandpass weighting filter whose center frequency is around the peak of the source spectrum and whose bandwidth is determined by the filter length.

2. GENERALIZED CROSS-CORRELATION

In the GCC approach, the sensor outputs in (1) with cross-spectra $G_{x_1x_2}$, are respectively prefiltered using two real causal filters $f_1(n)$ and $f_2(n)$ such as:

$$y_i(n) = \sum_{k=0}^{\infty} f_i(k) x_i(n-k), \quad i = 1, 2. \tag{2}$$

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The cross-spectrum of $y_1(n)$ and $y_2(n)$ is computed as:

$$G_{y_1y_2}(\omega) = F_1(\omega)F_2^*(\omega)G_{x_1x_2}(\omega), \tag{3}$$

where * denote the complex conjugate. The GCC function is then obtained by taking the inverse Fourier transform (IFT) of (3), i.e,

$$r_{y_1y_2}(\tau; W) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)G_{x_1x_2}(\omega)e^{j\omega\tau}d\omega, \tag{4}$$

where $W(\omega) = F_1(\omega)F_2^*(\omega)$. The GCC approach can be alternatively viewed as applying a window function to the cross-power spectrum before computing the IFT. The argument τ that maximizes $r_{y_1y_2}(\tau; W)$ is then the desired estimate of the time delay D.
Time delay estimation under the GCC approach aims to design an appropriate weighting filter $W(\omega)$. In practice, W is real valued, therefore F_1 and F_2 have the same phase. When $W(\omega) = 1$, GCC reduces to the standard cross correlation (SCC) method.

3. PROPOSED METHOD

Without loss of generality, we will assume that $F_1{=}F_2{=}H_p$. Where H_p is an FIR causal filter of order p such that

$$H_p(\omega) = \sum_{n=0}^{p} h_p(n)e^{-j\omega n}. \tag{5}$$

Define $\mathbf{h}_p = \big(h_p(0), h_p(1), \ldots, h_p(p)\big)^t$, where $(.)^t$ denotes transpose. The cross-correlation between the filter's outputs $y_1(n)$ and $y_2(n)$ at lag τ is given by:

$$\begin{aligned} r_{y_1y_2}(\tau; \mathbf{h}_p) &= \mathbb{E}\Big\{y_1(n)y_2(n+\tau)\Big\} \\ &= \sum_{k=0}^{p}\sum_{m=0}^{p} h_p(k)h_p(m)\mathbb{E}\Big\{x_1(n-k)x_2(n+\tau-m)\Big\} \\ &= \sum_{k=0}^{p}\sum_{m=0}^{p} h_p(k)h_p(m)r_{x_1x_2}(\tau+k-m) \\ &= \mathbf{h}_p^t\mathbf{R}_p(\tau)\mathbf{h}_p. \end{aligned} \tag{6}$$

$\mathbf{R}_p(\tau)$ is a square matrix of dimension $(p+1)$ whose (k, m) entry is $r_{x_1x_2}(\tau + k - m)$. The objective is to find the *real filter* $\mathbf{h}_p(\tau)$, that maximizes the cross-correlation of $y_1(n)$ and $y_2(n)$ at lag τ. Because $r_{y_1y_2}(\tau, \mathbf{h}_p)$ can be increased arbitrary by simply multiplying $\mathbf{h}_p$ by a constant, we restrict $\mathbf{h}_p$ to be of unit norm, that is $\mathbf{h}_p^t\mathbf{h}_p = 1$. Hence the maximization problem becomes

$$\text{maximize } |\mathbf{h}_p^t\mathbf{R}_p(\tau)\mathbf{h}_p| \text{ such that } ||\mathbf{h}_p||^2 = 1 \text{ and } \mathbf{h}_p \in \mathbb{R}^r \tag{7}$$

Proposition 1: Define the matrix $\widetilde{\mathbf{R}}_p(\tau) = \big(\mathbf{R}_p(\tau) + \mathbf{R}_p^t(\tau)\big)/2$. The solution to (7) is:

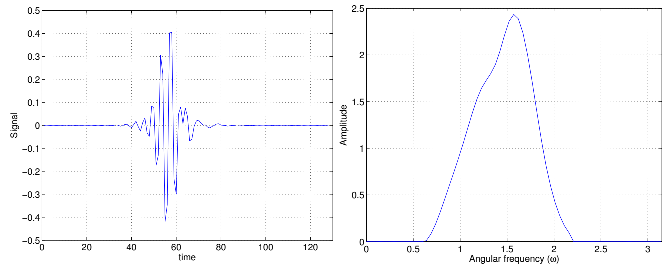
$$\mathbf{h}_p(\tau) = \mathbf{v}_p^{(1)}(\lambda) \quad \text{and} \quad r_{y_1y_2}(\tau; \mathbf{h}_p(\tau)) = \lambda_p^{(1)}(\tau),$$

where $\mathbf{v}_p^{(1)}(\tau)$ is the eigenvector of $\widetilde{\mathbf{R}}_p(\tau)$ associated with the eigenvalue $\lambda_p^{(1)}(\tau)$. The eigenvalues of $\widetilde{\mathbf{R}}_p(\tau)$ are sorted such as

$$|\lambda_p^{(1)}(\tau)| \geq |\lambda_p^{(2)}(\tau)| \geq \ldots \geq |\lambda_p^{(p+1)}(\tau)|.$$

Proof: The solution to the problem

$$\text{maximize } |\mathbf{x}^t\mathbf{R}\mathbf{x}| \text{ such as } ||\mathbf{x}||^2 = 1 \text{ and } \mathbf{x} \in \mathbb{R}^r,$$



(a) (b)

Fig. 1: Source signal (a) and amplitude spectrum (b).

where $\mathbf{R}$ is an $r \times r$ matrix, requires an eigenvalue decomposition of the matrix $\mathbf{R}$. The solution $\mathbf{x}$ corresponds to the eigenvector of $\mathbf{R}$ associated with the eigenvalue having the maximum absolute value [7]. The matrix $\mathbf{R}$ is not necessarily symmetric, and since $\mathbf{x} \in \mathbb{R}^r$, it is required to involve a symmetric matrix in the eigen-decomposition to obtain a feasible solutions. Recall that

$$\mathbf{x}^t(\mathbf{R}\mathbf{x}) = (\mathbf{R}\mathbf{x})^t\mathbf{x} = \mathbf{x}^t\mathbf{R}^t\mathbf{x}.$$

The cost function can be rewritten as

$$\mathbf{x}^t\widetilde{\mathbf{R}}\mathbf{x}, \quad \text{where} \quad \widetilde{\mathbf{R}} = \big(\mathbf{R} + \mathbf{R}^t\big)/2.$$

$\widetilde{\mathbf{R}}$ is a symmetric matrix and has real eigenvectors. The solution $\mathbf{x}$ corresponds to the eigenvector of $\widetilde{\mathbf{R}}$ associated with the eigenvalue which has the maximum absolute value $\square$.
The PGCC method determines the time delay estimate such that

$$\hat{D} = \arg\max_{\tau} |\lambda_p^{(1)}(\tau)|. \tag{8}$$

The optimal FIR filter has an impulse response determined by the vector $\mathbf{h}_p(\hat{D})$. The only input parameter for the PGCC method is the filter order p. We note that for $p = 0$, PGCC reduces to the SCC.

4. SIMULATION

We consider the two-sensors modelof (1). We investigate a typical situation, encountered in many applications, in which only *short data records* are available. So, we use a data sample size $N = 128$ and we set the time delay parameter to 10 samples. The time delay is assumed to be an integer multiple of sampling period. A parabolic fit to the peak of the generalized cross-correlogram may be performed for fine estimation of TDE, but we will not consider this option in our simulations.
The source signal $s(n)$ [Fig.1(a)] has a narrowband spectrum [Fig.1(b)] contrary to active time delay estimation, like in radar for example, where broad band source are common. Two independent white zero-mean Gaussian random sequences, with identical variance σ_n^2, are generated to construct the noise signals $\epsilon_1(n)$ and $\epsilon_2(n)$. The variance of the noise generator is adjusted to have the desired SNR level of $10\log_{10}(\sigma_s^2/\sigma_n^2)$, where $\sigma_s^2 = \sum_{n=0}^{N-1} s^2(n)/N$.
We compare the performance of the proposed method with the standard cross-correlation (SCC) method and the maximum likelihood estimator (ML) [2]. For the ML method, the cross-spectra and the coherence function [8] are estimated by partitioning the data into four segments of 64 point each (50% overlap). Each segment is multiplied by a Hanning window to reduce frequency leakage. For the PGCC method, a filter order of $p = 10$ is used. In all the simulations we search for the time delay D in the interval [-20,20].

A performance measure of each TDE method is the empirical distribution (histogram) of the selected time delay $\hat{D}$. We run a Monte Carlo simulation that consists of 1000 trials using SNR = -3 dB. Results are shown in Fig(2). PGCC achieves the highest percentage of correct selection [Fig.2(a)], followed by SCC [Fig.2(b)] and then ML [Fig.2(c)]. The PGCC distribution of $\hat{D}$ shows shorter tails, as compared with the other methods, and has the lowest variance. This may be an interesting feature for robust detection. On the other hand, ML achieves the lowest performance, mainly because of the poor spectral estimation at the SNR level due to the short data records.
To investigate the effect of the noise level, we repeat the same Monte Carlo simulation for different values of SNR. The probability of correct selection achieved by the three methods is shown in Fig. 3. For very low SNR, all the methods have identical performances. The same thing happens at high SNR, where PGCC and SCC have identical results starting from 5 dB. ML reaches the two other methods at higher SNR level. PGCC outperforms the other methods for low and moderate SNR. For an SNR level ranging in [-5,0] dB, PGCC achieves a performance equivalent to an average increase of 1.25 dB in the SNR with respect to SCC and 2.5 dB with respect to ML. Similar results are obtained if we examine the bias and variance of the estimator $\hat{D}$ for each method (figures not shown).

5. DISCUSSION

5.1. Interpretation of PGCC

In this section, we present both theoretical and empirical arguments to explain the filtering effect of PGCC. From equation (4), and using the symmetry of the problem, it is straightforward to show that for real signals $x_1(n)$ and $x_2(n)$, we have

$$r_{y_1y_2}(\tau, W) = \frac{1}{\pi}\int_0^{\pi} W(\omega)|G_{x_1x_2}(\omega)|\cos\left[\phi(\omega)+\omega\tau\right]d\omega, \qquad (9)$$

where $\phi(\omega)$ is the phase of the cross-spectrum $G_{x_1x_2}(\omega)$. For ease of notation, let $B_\tau(\omega) = |G_{x_1x_2}(\omega)|\cos\left[\phi(\omega)+\omega\tau\right]$. The main idea of this work is to design $W_\tau(\omega)$ over [0, π] that maximizes $|r_{y_1y_2}(\tau, W)|$.

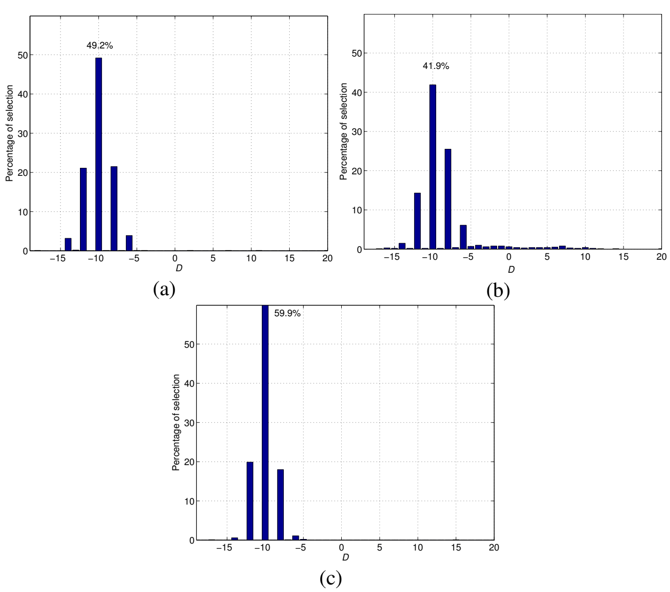


Fig. 2: Histogram of the estimated time delay. (a) SCC, (b) ML, (c) PGCC

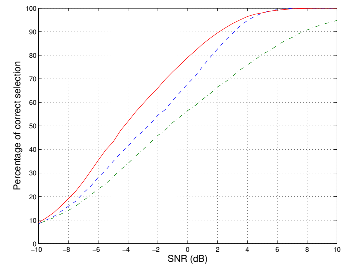


Fig. 3: Percentage of correct selection. (−) PGCC, (−−) SCC, (−.−) ML

Proposition 2: Let $\omega_\tau \in [0, \pi]$ such that $K_\tau = |B(\omega_\tau)| \geq |B(\omega)|$. The weighting function satisfying :(i) $W(\omega) \geq 0$, (ii) $1/\pi \int_0^\pi W(\omega)d\omega = 1$ which maximizes $|r_{y_1y_2}(\tau, W)|$ is given by[1]

$$W_\tau(\omega) = \pi\delta(\omega - \omega_\tau), \qquad (10)$$

where $\delta(.)$ is the Kronecker function.
proof: A direct substitution of (10) in (9) gives

$$|r_{y_1y_2}(\tau, W_\tau)| = \pi|B(\omega_\tau)| = \pi K_\tau.$$

Using the triangule inequality, one can show that

$$\begin{aligned} |r_{y_1y_2}(\tau, W)| &\leq \frac{1}{\pi}\int_0^\pi W(\omega)d\omega \int_0^\pi |B(\omega)|d\omega \\ &\leq \pi K_\tau = |r_{y_1y_2}(\tau, W_\tau)|. \end{aligned}$$

The global maximizer of (9) over a range of τ is $W_{\hat{D}}(\omega)$, where $\hat{D}$ is the time delay estimate using the class of generalized functions as weighting filters.□
This filter corresponds to a single cosine oscillating at frequency $\omega_{\hat{D}}$. It is non-causal and has an infinite impulse response. However in this paper, we proposed to choose the weighting filter $W(\omega)$ from the class of FIR filters obtained by convolving a causal FIR filter of order p with its time reversal. Let us denote this causal FIR filter of order p by $q_p(n)$ and its Fourier transform (FT) by $Q_p(\omega)$. Let also $U_p(\omega)$ be the FT of a rectangular window $u_p(n)$ over $0 \leq n \leq p$. $U_p(\omega)$ is a lowpass filter with a passband of $2\pi/p$. If we approximate $q_p(n)$ by truncation, the filter whose square Q_p^2 best approximates $W_{\hat{D}}$ defined in equation (10), in the mean-square sense is [9]

$$Q_p(\omega) = \sqrt{\pi}\delta(\omega - \omega_{\hat{D}}) * U_p(\omega) = \sqrt{\pi}U_p(\omega - \omega_{\hat{D}}).$$

Hence $Q_p(\omega)$ is a bandpass filter with peak frequency at $\omega_{\hat{D}}$ and bandwidth of $4\pi/p$. Combining the results of proposition 1 and 2, we can so far argue that Q_p approximate the filter H_p of the PGCC method.[2] To investigate this point, we use the same data simulated in section 4 to plot the normalized $|H_p(\omega)|$, defined in Proposition 1, for different values of p. The normalized source spectrum $|S(\omega)|$ is also plotted in superposition [Fig. 4]. We notice that the peak frequency of $H_p(\omega)$ is very close to the peak frequency of the source's spectrum. The larger the value of p, the narrower $|H_p(\omega)|$ is.
Broadly speaking, based on the above arguments we can say the that PGCC method bandpass filters the cross-correlogram with an FIR-based estimate of the source spectrum.

[1] In here, ω_τ is assumed unique over the interval [0,π] to ensure the uniqueness of the solution in (10).
[2] The approximation is better for the limiting case of large p.

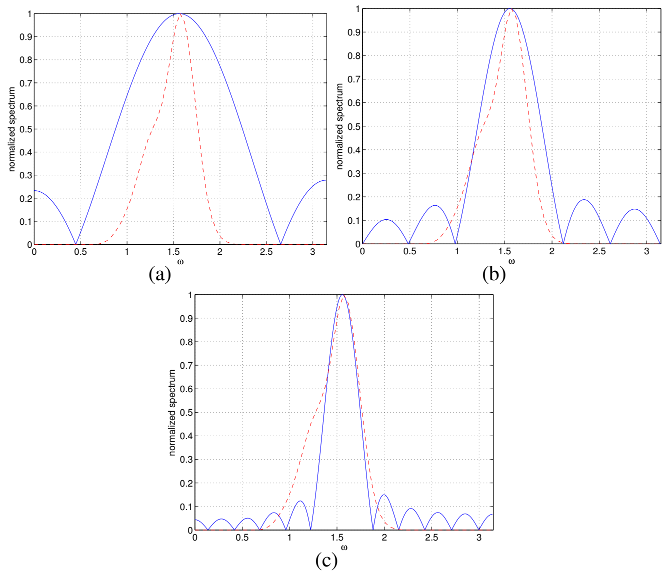


Fig. 4: Normalized spectra, $(-)$ $|H_p(\omega)|$, $(--)$ $|S(\omega)|$. (a) $p=4$, (b) $p=10$, (c) $p=20$.

5.2. Selection of Filter Order p

In the PGCC method, the only parameter left to the user's choice is the filter order p. It has been argued in the preceding subsection that this choice depends mainly on the bandwidth of the source signal. The narrower the bandwidth, the larger the order that should be used. However, in practice we may not have reliable knowledge about the source bandwidth. Therefore, the user needs to investigate several values of p. This may be found helpful to give an idea about the robustness of $\hat{D}_p$ vis-a-vis p and a rough estimate of the source bandwidth.

Let's assume a uniform *a priori* probability for D over $[D_{\min}, D_{\max}]$. The *a posteriori* probability distribution of D is then proportional to the likelihood, which was approximated in the derivation of the ML estimator [2]. Adopting this result, we can claim an approximation to the *a posteriori* probability that the time delay is equal to τ, when using a weighting function $\mathbf{h}_p(\hat{D}_p)$, that is given by

$$\mathrm{P}(\tau,p)=\frac{1}{C_{(p,\tau)}}\exp\Big\{-\big|r_{y_1y_2}\big[\tau;\mathbf{h}_p(\hat{D}_p)\big]\big|\Big\}, \qquad (11)$$

where $C_{(p,\tau)}$ is the normalizing factor given by

$$C_{(p,\tau)}=\sum_{\tau=D_{\min}}^{D_{\max}}\exp\Big\{-\big|r_{y_1y_2}\big[\tau;\mathbf{h}_p(\hat{D}_p)\big]\big|\Big\}$$

An image plot of (11), using for each $p\in[0,20]$, a single realization from a simulation experience identical to the one performed in section 4, is shown in Fig. 5. It seems that the value of p does not affect the selection of $\hat{D}$, being equal to -10, for a wide range of p ($p\geq 2$). This provides an excellent robustness property with respect to a reliable choice of p, though care should be taken not to use very large values of p, which may result in an oscillating filtered cross-correlogram and causes peak ambiguity and therefore a false detection. As a rule of thumb, the bandwidth of a filter of order p is $4\pi/p$. The source signal used in the simulation [Fig. 1(b)] has a bandwidth of about 0.5π. Hence, the best filter to use would be of order $p=4/0.5=8$. This finding is confirmed in Fig.5, where the maximum *a posteriori* probability points toward a filter order of $p=8$.

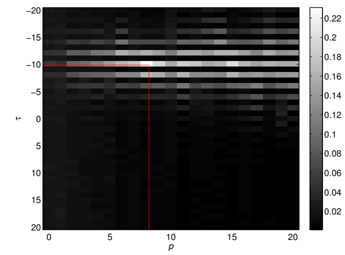


Fig. 5: Image of the a posteriori probability as function of τ and p

6. CONCLUSION

Time delay estimation for short data records or narrow-band signals is a challenging problem. In this paper, we propose a new parametric method tailored to solve such problem. The method is classified under the generalized cross-correlation approach, but does not require knowledge of source and noise spectra. An adaptive FIR filter maximizes the output cross-correlation at each lag. Effectively, it applies a band-pass filter to the cross-correlogram to emphasis the source signal. The filter order is the only control parameter, however results are robust to its choice. Generally speaking, the narrower the source spectrum, the larger the filter order should be. As compared with standard techniques, the new method performs well for low to moderate SNR ratios. Online extension of this method is straightforward by block data processing.

7. REFERENCES

[1] E.J. Hannan and P.J. Thomson, "Estimating group delay," *Biometrika*, vol. 60, pp. 241–253, 1973.

[2] C.H. Knap and G.C. Carter, "The generalized crorrelation method for estimation of time delay," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 24, pp. 320–327, August 1976.

[3] P. D. Welch, "The use of FFT for estimation of power spectra: A method based on time averaging over short, modified periodogram," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 15, pp. 70–73, Augest 1967.

[4] K. Scarbrough, N. Ahmed, and G.C. Carter, "On the simulation of a class of time delay estimation algorithms," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 29, pp. 534–540, June 1981.

[5] J.C. Hassab and R.E. Boucher, "A quantitative study of optimum and sub-optimum filters in the generaized correlator," in *International Conference on Acoustics, Speech, Signal Processing*. ICASSP, 1979, vol. I, pp. 124–127.

[6] Y.T. Chan, J.M. Riley, and J.B. Plant, "A parameter estimation approach to time-delay estimation and signal detection," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 28, pp. 8–16, February 1980.

[7] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 1996.

[8] G. C. Carter, C. H. Knap, and A. H. Nuttall, "Estimation of the magnitude-squared coherence function via overlapped fast fourier transform processing," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 21, pp. 337–344, Augest 1973.

[9] I. M. Gelfand and G. E. Shilov, *Generalized functions Vol. 1: Properties and operators*, Academic Press, 1964.