Influence of Random Carrier Phase on True Cramer-Rao Lower Bound for Time Delay Estimation

Wei He-Wen, Ye Shangfu

Southwest Electronics and Telecommunication Technology Research Institute,
Chengdu 610041, China
Email: hwei05@gmail.com

Wan Qun

University of Electronic Science and Technology of China,
Chengdu 610054, China
Email: wanqun@uestc.edu.cn

Abstract

The true Cramer-Rao lower bound (CRB) for the time delay estimation has been obtained for narrowband signal with the carrier phase as a deterministic parameter already. However, the carrier phase is usually a random parameter in non-coherent receiver in the applications such as radar, sonar and communication systems. And accuracy of the time delay estimation will be affected by this nuisance carrier phase. In this paper, the true Cramer-Rao lower bound for the time delay estimation is derived and analyzed in the presence of a random carrier phase. The new bound is tighter than the one obtained under the condition that the carrier phase is not random. We show that this relation indicates the influence of the random carrier phase on the true CRB, and the penalty resulting from this random carrier phase increases severely with decreasing signal-to-noise ratio. The explanation about this influence is also given from the point of information theory. Simulations are provided to support the theoretical results.

Index Terms— Cramer-Rao lower bound (CRB), time delay estimation

1. Introduction

The problem of the time delay estimation (TDE) is one of fundamental importance in radar, sonar and communication systems. It's necessary to know the ultimate accuracy of TDE theoretically. The Cramer-Rao lower bound is one of the most fundamental limits on the minimum achievable variances of any unbiased estimates of deterministic parameters [1, 2]. Unfortunately, the received signal usually depends on other unwanted or nuisance parameters which are of no interest, and serve only to perturb the estimation of the desired parameters.

When the observation depends on nuisance parameters, the computation of the true CRB requires the determination of marginal probability density function (pdf), an operation that is generally very hard to perform analytically. For this reason, other bounds have been proposed, namely, the modified CRB (MCRB) [3, 4]. Although easier to obtained, the MCRB is generally looser than the true CRB. Another way to obviate the analytical difficulties associated with the computation of the true CRB is evaluation of high SNR limits of this bound [2, 5].

As far as the time delay estimation is concerned in the presence of the carrier phase, there are three scenarios about the nuisance carrier phase:

1) the carrier phase is known as prior information,
2) the carrier phase is a unknown parameter, and
3) the carrier phase is a random valuable.

The CRBs of TDE for 1) and 2) have been studied for several decades [1-2, 5]. The CRB for TDE under the high signal-to-noise ratio (SNR) was presented in [5] for the third scenario. Although Noels and Tavres *et al*. have derived the CRB for TDE in the presence of the random carrier phase [6, 7], their results were just suitable for the linearly modulated waveforms. Inspired by the work in [7], we extended their work for all narrowband signals in radar, sonar and communication systems in this paper, obtained a relatively simply analytical expression of the true CRB of TDE in the presence of a random phase. We also compared this new bound with other scenarios. Further explanations and conclusions are also given.

2. Signal Model and Problem Formulation

Consider a source signal over an additive white Gaussian noise channel with unknown time delay in the presence of a random carrier phase. Let assume that $s(t)$ is the source signal, which is narrowband. The complex analytical representation of the received signal may be modeled as

$$r(t) = as(t-\tau)e^{j\varphi} + w(t) \tag{1}$$

where a is a real attenuation constant, φ is a random carrier phase. τ denotes the time delay. $w(t)$ is the ergodic, zero-mean , complex white Gaussian noise with variance σ_w^2, and with independent real and imaginary part, each with variance $\sigma^2 = \sigma_w^2/2$, and $SNR = E_s/\sigma_w^2$, where E_s is the energy of the received signal (without noise)

$$E_s = a^2 \int_{-\infty}^{\infty} |s(t)|^2 \, dt \tag{2}$$

Suppose one is able to produce from the received signal $\mathbf{r}$ an unbiased estimate $\hat{\tau}$ of τ, then the estimation error variance satisfies $\mathrm{var}\{\hat{\tau}\} \geq CRB(\tau)$ with [1]:

$$CRB(\tau) = E_{\mathbf{r}} \left\{ \left[\left. \frac{\partial \ln p(\mathbf{r};\hat{\tau})}{\partial \hat{\tau}} \right|_{\hat{\tau}=\tau} \right]^2 \right\}^{-1} \tag{3}$$

where $\ln p(\mathbf{r};\hat{\tau})$ is the log-likelihood function. The expectation $E_{\mathbf{r}}\{\}$ in (3) is with respect to $p(\mathbf{r};\hat{\tau})$. Here the observation $\mathbf{r}$ depends not only on the τ to be estimated but also on a nuisance parameter φ, the $p(\mathbf{r};\hat{\tau})$ is obtained by averaging $p(\mathbf{r}\,|\,\varphi;\hat{\tau})$ of the vector (φ,τ) over the a priori distribution $p(\varphi)$ of the nuisance parameter:

$$p(\mathbf{r};\hat{\tau}) = \int p(\mathbf{r}\,|\,\varphi;\hat{\tau}) \cdot p(\varphi) d\varphi \tag{4}$$

With this signal model, the joint likelihood function $p(\mathbf{r}\,|\,\varphi;\hat{\tau})$ based on $\mathbf{r}$, without a factor not dependent on $\hat{\tau}$, is given by

$$p(\mathbf{r}\,|\,\varphi;\hat{\tau}) = \exp\left\{ \frac{2}{\sigma_w^2} \Re\left(e^{-j\varphi} u(\hat{\tau}) \right) \right\} \tag{5}$$

where $\Re\{\}$ and $\Im\{\}$ stand for real and imaginary part of a complex-valued function, respectively, and $u(\hat{\tau})$ is

$$u(\hat{\tau}) \triangleq \int_{-\infty}^{\infty} a r(t) s^*(t-\hat{\tau}) dt \tag{6}$$

where "*" denotes the complex conjugation. The carrier phase without prior information is often assumed to have a uniform pdf in $(-\pi,\pi)$. Then the marginal likelihood for the estimation of τ is obtained as from (5)

$$p(\mathbf{r};\hat{\tau}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left\{ \frac{2}{\sigma_w^2} \Re\left(e^{-j\varphi} u(\hat{\tau}) \right) \right\} d\varphi = I_0\left(\frac{2}{\sigma_w^2} |u(\hat{\tau})| \right) \tag{7}$$

where $I_n(z)$ is the modified Bessel function of the first kind and order n. In next section, the bound (3) is computed from the log-likelihood function $\ln p(\mathbf{r};\hat{\tau})$.

3. True CRB for TDE with Random Carrier Phase

3.1. Derivation

Before deriving CRB of the time delay, we first define some symbols which will be used later:

$$\overline{\omega} = \frac{a^2}{E_s} \Im \int_{-\infty}^{\infty} s^*(t) \dot{s}(t) dt \tag{8}$$

$$\overline{\omega^2} = \frac{a^2}{E_s} \int_{-\infty}^{\infty} |\dot{s}(t)|^2 \, dt \tag{9}$$

where $\dot{s}(t) = ds(t)/dt$, and $\beta^2 = \overline{\omega^2} - \overline{\omega}^2$. These symbols have been defined already [2], but we do not expatiate here.

Taking partial derivations and using the identity $\partial I_0(z)/\partial z = I_1(z)$, yield:

$$\left. \frac{\partial \ln p(\mathbf{r};\hat{\tau})}{\partial \hat{\tau}} \right|_{\hat{\tau}=\tau} = \frac{1}{\sigma^2} \frac{I}{|u(\tau)|} \times \Re\left\{ u(\tau) \left. \frac{\partial u^*(\tau)}{\partial \hat{\tau}} \right|_{\hat{\tau}=\tau} \right\} \tag{10}$$

where

$$I \triangleq I_1\left(\frac{1}{\sigma^2} |u(\tau)| \right) \Big/ I_0\left(\frac{1}{\sigma^2} |u(\tau)| \right) \tag{11}$$

From (1) and (6), we can get

$$\begin{aligned} u(\tau) &= e^{j\varphi} [\int_{-\infty}^{\infty} a^2 s(t-\tau) s^*(t-\tau) dt \\ &\quad + \int_{-\infty}^{\infty} a w(t) s^*(t-\tau) e^{-j\varphi} dt] \\ &\triangleq e^{j\varphi}(E_s + v) \end{aligned} \tag{12}$$

$$\begin{aligned} \left. \frac{\partial u(\hat{\tau})}{\partial \hat{\tau}} \right|_{\hat{\tau}=\tau} &= e^{j\varphi} [\int_{-\infty}^{\infty} -a^2 s(t-\tau) \dot{s}^*(t-\tau) dt \\ &\quad + \int_{-\infty}^{\infty} -a w(t) \dot{s}^*(t-\tau) e^{-j\varphi} dt] \\ &\triangleq e^{j\varphi}(A + v_1) \end{aligned} \tag{13}$$

where

$$v \triangleq \int_{-\infty}^{\infty} a w(t) s^*(t-\tau) e^{-j\varphi} dt \tag{14}$$

$$A \triangleq \int_{-\infty}^{\infty} -a^2 s(t-\tau) \dot{s}^*(t-\tau) dt \tag{15}$$

$$v_1 \triangleq \int_{-\infty}^{\infty} -a w(t) \dot{s}^*(t-\tau) e^{-j\varphi} dt \tag{16}$$

Note that, the multiplication by the complex exponential $e^{-j\varphi}$ does not modified the statistical properties of the noise processes, so we may consider the processes in the previous definitions of v and v_1 as the original noise process $w(t)$.

Using (10-16), (3) becomes

$$CRB(\tau) = \left\{ E_v \left[\frac{1}{\sigma^4} \frac{I^2}{|E_s + v|^2} E_{v_1} \{ (\Re\{(E_s+v)(A+v_1)^*\})^2 \,|\, v \} \right] \right\}^{-1} \tag{17}$$

The result of the derivation of the conditional expectation in (17) is given by in [7, eq. (I.13)]

$$CRB(\tau) = \left\{ \frac{1}{\sigma^2} (B - \frac{\Im^2\{A\}}{E_s}) \cdot F(\frac{E_s}{2\sigma^2}) \right\}^{-1} \tag{18}$$

where

$$B = \int_{-\infty}^{\infty} a^2 \dot{s}(t-\tau) \dot{s}^*(t-\tau) dt \tag{19}$$

and the function is defined as

$$\begin{aligned} F(x) &\triangleq E_\rho\{ I_1^2(2x\rho) / I_0^2(2x\rho) \} \\ &= 2x e^{-x} \int_0^{\infty} \rho e^{-x\rho^2} \frac{I_1^2(2x\rho)}{I_0(2x\rho)} d\rho \end{aligned} \tag{20}$$

From (8-9), (15) and (19), the relations are shown

$$B = \overline{\omega^2} E_s, \; \Im\{A\} = -\overline{\omega} E_s \tag{21}$$

Plugging $SNR = E_s / \sigma_w^2$, and (21) into (18) gives the desired CRB for the time delay estimation as

$$CRB(\tau) = \frac{1}{2SNR}\frac{1}{F(SNR)}\frac{1}{\beta^2} \qquad (22)$$

It is worth noting that this new bound extends previous work by Tavres [7] that considers the true CRB for linearly modulated signal only. This new CRB pertains not only to linearly modulated waveforms, but also to all other narrowband signals. For a linearly modulated waveform,

$$s(t) = \sum_{k=0}^{K-1} a_k h(t-kT) \qquad (23)$$

where $\{a_k\}_{k=0}^{K-1} = \mathbf{a}$ is a sequence of K symbols from an arbitrary M-ary constellation, T is the symbol duration, and $h(t)$ is a real-valued square-root Nyquist pulse with the energy E_s, the CRB in (22) becomes

$$CRB(\tau) = \left[2\frac{E_s}{\sigma_w^2} F\left(K\frac{E_s}{\sigma_w^2} \right) \left(\mathbf{a}^H \ddot{\mathbf{G}} \mathbf{a} - \frac{\Im^2\{\mathbf{a}^H \dot{\mathbf{G}} \mathbf{a}\}}{K} \right) \right]^{-1} \qquad (24)$$

where $\dot{\mathbf{G}}$ and $\ddot{\mathbf{G}}$ are $K \times K$ matrices with entries $[\dot{\mathbf{G}}]_{ij} = \dot{g}[(i-j)T]$, $[\ddot{\mathbf{G}}]_{ij} = -\ddot{g}[(i-j)T] : i, j = 0, \cdots, K-1$, and $g(t) = p(t) \otimes p(-t)$. And $\dot{g}(t)$ and $\ddot{g}(t)$ are the first and second derivates of this pulse. This bound (24) is consist with the true CRB in [7, eq. (8)].

3.2. Asymptotic Behavior

Here, the nature of the bound (22) is investigated. Since $x = SNR$, it suffices to find asymptotic approximations for $F(x)$ as $x > 0$. Without going the details, we state a number of useful properties of $F(x)$ which may be proved using standard analysis techniques: $0 < F(x) < 1$ and for all $x > 0$, $F(x) \approx x/(1+x)$.

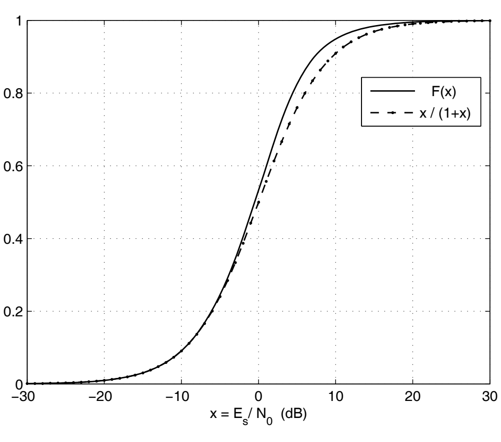

Fig.1 Function $F(x)$ and the corresponding approximation

The function $F(x)$ is plotted in Fig.1, together with the corresponding approximation for all $x > 0$, which is more meaningful and simple than the approximations at low and high SNR in [7], respectively. From the approximation of $F(x)$, we may write for (22) the asymptotic expression shown as

$$CRB(\tau) \approx \frac{1+SNR}{2SNR^2}\frac{1}{\beta^2}, \quad \text{for all } SNR \qquad (25)$$

This asymptotic expression in (25) is more practical and advisable, because the effect of the SNR can be derived easily from (25).

3.3. Relations between CRBs

Let us now compare the new bound in (22) to the results of the other scenarios, as described in the Introduction. The CRBs of time delay estimation for the first and second scenarios was discussed in [1-2, 6]:

$$CRB(\tau)_1 = \frac{1}{2SNR}\frac{1}{\overline{\omega^2}} \qquad (26)$$

$$CRB(\tau)_2 = \frac{1}{2SNR}\frac{1}{\beta^2} \qquad (27)$$

From above results, we can find that

$$\frac{1}{\beta^2 \cdot F(SNR)} > \frac{1}{\beta^2} = \frac{1}{\overline{\omega^2} - \overline{\omega}^2} \geq \frac{1}{\overline{\omega^2}} \qquad (28)$$

Hence

$$CRB(\tau)_3 > CRB(\tau)_2 \geq CRB(\tau)_1 \qquad (29)$$

where $CRB(\tau)_1$, $CRB(\tau)_2$ and $CRB(\tau)_3$ denote the CRBs for the three scenarios in the Introduction, respectively. Obviously, these relations indicate the influence of the random carrier phase on the true Cramer-Rao lower bound, and the penalty resulting from the random carrier phase, increases severely with decreasing SNR.

In fact, the above relations are not occasional. In statistical sense, knowing the information of the other things can reduce the uncertainty of itself. This is well-known theorem in information theory: conditioning reduces entropy [8]. So in our problem, with the decreasing of the obtained information about the unwanted carrier phase, the estimate variance should increase at the same time.

4. Numerical Results and Discussions

All numerical results reported in this section were obtained considering the narrowband signal given by:

$$s(t) = \sqrt[4]{\pi}\exp\{-\tfrac{1}{2}t^2 + j\omega_c t + j\tfrac{1}{2}\mu t^2\} \qquad (30)$$

where $\omega_c = 1$, $\mu = 1$ .we can get $\overline{\omega^2} = 1/2 + \omega_c^2 + \mu^2/2$ and $\overline{\omega} = \omega_c$. For simplicity, we set $\tau = 0$. The measurement time is $-5 \leq t \leq 5$ and the sampling rate is 0.1.

The true CRBs of the time delay estimation from the three scenarios about the carrier phase are plotted in Fig. 2 as a function of SNR. It is seen that, the new bound in (22) has the largest variance of the time delay estimation, because the third scenario with a random carrier phase has the least information about the carrier phase among these three scenarios. Furthermore, the penalty resulting from the random carrier phase increases severely with deceasing SNR. And this phenomenon is considerable in practice. Because the generalized use of channel coding and weak signal detection pushes system operation toward low SNR.

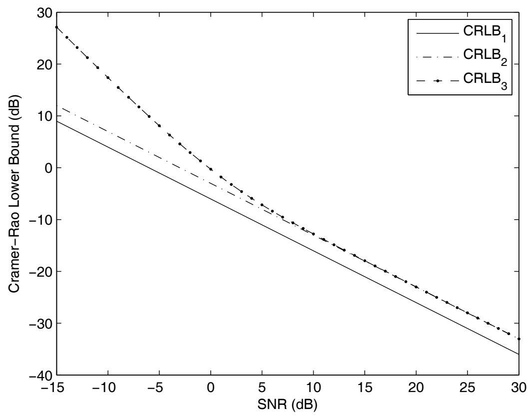

Fig. 2 True CRBs of TDE for the three scenarios: (1) denotes known phase; (2) denotes unknown phase; and (3) denotes random phase

A comparison of actual variance of time delay estimator with the new bound will be presented. Because the maximum likelihood (ML) estimator can attain the CRB asymptotically, we use them to validate the new CRB in our simulation. From (7), the ML estimates are obtained as

$$\hat{\tau}_{ML} = \arg\max_{\tau} I_0\left(\frac{2}{\sigma_w^2}\left|u(\tau)\right|\right) \tag{31}$$

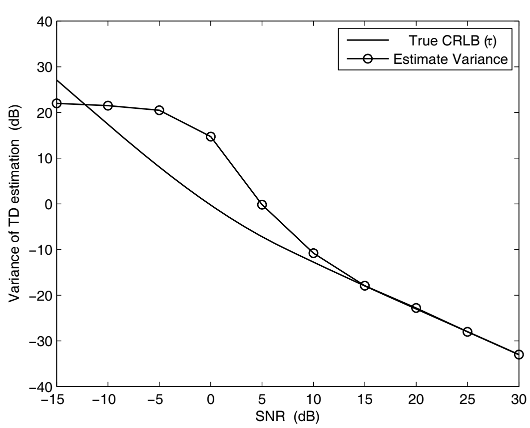

Fig.3 Variance of estimate and corresponding CRB with the random carrier phase

The simulated variance of estimates in (31) is shown in Fig. 3. As the SNR increases, the observation data lead to estimate with variance that approaches the corresponding lower bound. With decreasing SNR, the variance of estimate approaches the threshold, which is the variance of the priori information of time delay determined. We conclude that the performance of ML estimate is very close to the corresponding theoretical limit.

5. Conclusion

In this paper, we have derived the true CRB for time delay estimation of a narrowband signal with a random carrier phase. The new bound is tighter than the ones obtained under the assumption that the carrier phase is a known or unknown parameter. The penalty resulting from the random carrier phase increases severely with decreasing SNR. We have given an explanation about the influence from the viewpoint of information theory. Comparison between the performance of the ML estimator and the new bound shows that this bound is attainable by a practical estimator.

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