ZERO-FORCING BASED SEQUENTIAL MUSIC ALGORITHM

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ABSTRACT

Many works has been made in the context of the recursive or sequential MUSIC algorithm for bearing estimation. Indeed, in some difficult scenarios as for closely spaced bearings, correlated sources and at low SNRs, the accuracy of the MUSIC algorithm can be improved by sequentially cancel the previous estimated bearings. In this paper, we propose a new algorithm, called zero-forcing based sequential MUSIC, which efficiently tackles this problem.

Keywords: Iterative methods, direction of arrival estimation.

1. INTRODUCTION

Bearings estimation using a sensor array is an important topic in various applications as for instance mobile/source location or as for the analysis of the human brain from measurements of scalp potentials or electroencephalogram (EEG) and external magnetic fields or magnetoencephalogram (MEG).

This work focuses on the MUltiple SIgnal Classification (MUSIC) algorithm introduced in 1979 by Schmits [5]. This algorithm estimates the bearings (source location) from the noisy array response by minimizing the orthogonal condition between the array manifold vector and the noise subspace. To improve the performance of this algorithm in difficult scenarios as for closely spaced bearings, correlated sources and at low SNRs, several authors have proposed a sequential version of the MUSIC algorithm. These methods are based on sequential optimization criterion in which each bearing is found as the global optimum of a different cost function.

More precisely, the S-MUSIC (Sequential-MUSIC) of Oh *et al.* [4] and IES-MUSIC (ImprovEd Sequential-MUSIC) of Stoica *et al.* [6] are based on the Alternative Projection technique. By estimating the bearings sequentially rather than simultaneously, these approaches removes effectively the spatial interferences among sources and has a better resolution capabilities at the price of higher computational cost. The Recursive-MUSIC (R-MUSIC) [2] and its improved version, the Recursively Applied and Projected-MUSIC (RAP-MUSIC) [3] algorithm work by applying a MUSIC search to a modified problem in which we first project both the estimated signal subspace and the array manifold vector away from the subspace spanned by the sources that have already been found. Finally, note that two important points are (1) all of these methods avoid the delicate search of several optimum in the MUSIC pseudo-spectrum and (2) they are based on the signal subspace deflation principle.

In this paper, we present a new sequential MUSIC algorithm, called Zero-Forcing MUSIC (ZF-MUSIC) algorithm. Our approach is different from the others since we do not perform a deflation of the signal subspace and we directly scale the MUSIC criterion. By means of Monte-Carlo simulations, we show that our approach has a lower variance than the other sequential MUSIC algorithms for closely spaced bearings.

2. MUSIC ALGORITHM FOR PARAMETRIC MODEL

2.1. The Multi-Input Multi-Output (MIMO) model

Assume there are M narrowband plane waves (sources) simultaneously incident on an L sensor Uniform Linear Array (ULA). The array response for the t-th snapshot is given by

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_L(t) \end{bmatrix} = A \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_M(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ \vdots \\ n_L(t) \end{bmatrix}$$

where

$$A = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & \dots & 1 \\ e^{i\phi_1} & \dots & e^{i\phi_M} \\ \vdots & & \vdots \\ e^{i\phi_1(L-1)} & \dots & e^{i\phi_M(L-1)} \end{bmatrix} \quad (1)$$

is the steering manifold. The L-sensor steering vector

$$p_L(\phi) = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & e^{i\phi} & \dots & e^{i\phi(L-1)} \end{bmatrix}^T \quad (2)$$

is parameterized by $\phi = -2\pi(\Delta/c)\sin(\theta)$ where θ is the bearing, Δ is the distance between two consecutive sensors and c is the wavelength. The noisy observations on each sensor, $x_\ell(t)$, is collected in a vector given by $x(t) = [x_1(t) \dots x_L(t)]^T$. In a similar way, we define the noise vector $n(t) = [n_1(t) \dots n_L(t)]^T$ in which each $n_\ell(t)$ is the contribution of the noise on the ℓ-th sensor which is assumed to be a zero-mean white Gaussian process of variance σ^2. The sources, $\alpha_m(t)$, are collected in vector $\alpha(t) = [\alpha_1(t) \dots \alpha_M(t)]^T$ and the number of sources, M, is assumed to be known or previously estimated [7]. Finally, the MIMO model for T snapshots is

$$X = \begin{bmatrix} x(1) & \dots & x(T) \end{bmatrix} = A\Lambda + N \quad (3)$$

where $\Lambda = [\alpha(1) \dots \alpha(T)]$ and $N = [n(1) \dots n(T)]$.

2.2. The MUSIC algorithm

From (3), the sample spatial covariance of the noisy observation is

$$\hat{R}_X = \frac{1}{T} X X^H \quad (4)$$
$$= A R_\Lambda A^H + \sigma^2 I \quad (5)$$

where $R_\Lambda = \frac{1}{T}\Lambda\Lambda^H$ is the sample source covariance. By eigen-decomposing the rank-L matrix $\hat{R}_X$, we have

$$\hat{R}_X = \sum_{\ell=1}^{L} \lambda_\ell u_\ell u_\ell^H \quad (6)$$

where $\lambda_1 \geq \ldots \geq \lambda_M > \lambda_{M+1} \geq \ldots \geq \lambda_L$ are the ordered eigen-values and u_ℓ is the eigen-vector of the sample spatial covariance. Let $G = [u_{M+1} \ldots u_L]$ be the matrix constituted from the eigen-vector associated to the $L - M$ smallest eigen-values. Then, the noise projector is defined by

$$\Pi^\perp = GG^H = \sum_{\ell=M+1}^{L} u_\ell u_\ell^H. \quad (7)$$

Consequently, the well-known optimization criterion of the spectral-MUSIC criterion [5, 7] is

$$\arg\max_\phi \frac{1}{p_L(\phi)^H \Pi^\perp p_L(\phi)}. \quad (8)$$

since $\|p_L(\phi)\|^2 = 1$.

3. ZERO-FORCING SEQUENTIAL MUSIC ALGORITHM

In this part, we modify the classical MUSIC criterion, given in expression (8), according to the following definition.

Definition 1 *The spectral form of the zero-forcing sequential MUSIC (ZF-MUSIC) algorithm is given by*

$$\phi_m = \arg\max_\phi \frac{\mathcal{F}_m^{(L_z)}(\phi)}{p_L(\phi)^H \Pi^\perp p_L(\phi)} \quad (9)$$

for $m \in [1 : M]$ and with L_z being a positive integer and $\mathcal{F}_m^{(L_z)}(\phi)$ a quadratic function defined according to

$$\mathcal{F}_m^{(L_z)}(\phi) = p_{L_z}(\phi)^H P_m^\perp p_{L_z}(\phi) \quad (10)$$

where $P_m^\perp$ is the noise projector associated to the space of the $m-1$ previously estimated bearings, denoted by $\mathcal{R}(A_m)$. This projector is defined according to

$$P_m^\perp = I - P_m \quad (11)$$

in which

$$P_m = A_m (A_m^H A_m)^{-1} A_m^H \quad (12)$$

and

$$A_m = \begin{bmatrix} p_{L_z}(\phi_1) & \ldots & p_{L_z}(\phi_{m-1}) \end{bmatrix}. \quad (13)$$

Then the ZF-MUSIC algorithm can be described according to

`Init.` Apply the spectral MUSIC algorithm and estimate ϕ_1 with $A_1 = I$, *ie.*, $\mathcal{F}_1^{(L_z)}(\phi) = 1$. Next, compute projector $P_2^\perp = I - p_{L_z}(\phi_1) p_{L_z}(\phi_1)^H$ with $L_z \gg L$.

`Loop` For $m \in [2 : M]$, compute the zero-forcing function $\mathcal{F}_m^{(L_z)}(\phi)$ based on expressions (10)-(13) and solve criterion (9).

4. ANALYSIS OF THE ZERO-FORCING FUNCTION

4.1. Asymptotic behavior of the zero-forcing function

Property 1 *The zero-forcing function has the following properties:*

$$\mathcal{F}_m^{(L_z)}(\phi) = \begin{cases} 0 & \text{for } p_{L_z}(\phi) \in \mathcal{R}(A_m) \\ 1 & \text{otherwise and for large } L_z. \end{cases} \quad (14)$$

*In words, the zero-forcing function is equal to zero for all the previously estimated bearing and is asymptotically (*ie.*, for large L_z) one everywhere else. So, the MUSIC pseudo-spectrum, given in (8), is forced to be zero for previously estimated bearings and is unchanged for the other ones.*

Proof:

1. *Let $p_{L_z}(\phi) \in \mathcal{R}(A_m)$ (meaning that $p_{L_z}(\phi)$ has been already estimated), then it is straightforward to see that*

$$\mathcal{F}_m^{(L_z)}(\phi) = 1 - p_{L_z}(\phi)^H P_m p_{L_z}(\phi) = 0 \quad (15)$$

since $P_m p_{L_z}(\phi) = p_{L_z}(\phi)$.

2. *Let $\alpha_{L_z}(\phi_i, \phi_j) \overset{\text{def}}{=} \langle p_{L_z}(\phi_i), p_{L_z}(\phi_j)\rangle = p_{L_z}(\phi_i)^H p_{L_z}(\phi_j)$ where $\langle ., . \rangle$ defined the Hermitian inner product. Observe [9] that*

$$\alpha_{L_z}(\phi_i, \phi_j) = \frac{1}{L_z} \sum_{\ell=0}^{L_z-1} e^{i(\phi_j - \phi_i)\ell} \overset{L_z\to\infty}{\longrightarrow} \delta_{\phi_i - \phi_j} \quad (16)$$

where $\delta_{\phi_i-\phi_j} = 1$ if $\phi_i = \phi_j$ and 0 otherwise. So, for $p_{L_z}(\phi) \notin \mathcal{R}(A_m)$, we have

$$(A_m^H A_m)^{-1} = \begin{bmatrix} 1 & \ldots & \alpha_{L_z}(\phi_1, \phi_{m-1}) \\ \vdots & \ddots & \vdots \\ \alpha_{L_z}^*(\phi_1, \phi_{m-1}) & \ldots & 1 \end{bmatrix}^{-1} \overset{L_z\to\infty}{\longrightarrow} I_{m-1}$$

where I_{m-1} is the $(m-1)$-rank identity matrix and

$$A_m^H p_{L_z}(\phi) = \begin{bmatrix} \alpha_{L_z}(\phi_1, \phi) \\ \vdots \\ \alpha_{L_z}(\phi_{m-1}, \phi) \end{bmatrix} \overset{L_z\to\infty}{\longrightarrow} 0_{m-1} \quad (17)$$

where 0_{m-1} is the $(m-1) \times 1$ null vector. Consequently, thanks to definition (12), we have $p_{L_z}(\phi)^H P_m p_{L_z}(\phi) \overset{L_z\to\infty}{\longrightarrow} 0$ and thus

$$\mathcal{F}_m^{(L_z)}(\phi) = 1 - p_{L_z}(\phi)^H P_m p_{L_z}(\phi) \quad (18)$$
$$\overset{L_z\to\infty}{\longrightarrow} 1. \quad (19)$$

■

4.2. Width and attenuation of the zero-forcing function

Now, without loss of generality assume that L_z is large, then the zero-forcing function can be rewritten according to

$$\mathcal{F}_m^{(L_z)}(\phi) = 1 - \frac{1}{L_z^2}\mathcal{G}_{L_z}(\phi) \tag{20}$$

where

$$\mathcal{G}_{L_z}(\phi) = \mathcal{Q}_{L_z}(\phi) * \left(\sum_{\ell=1}^{m-1} \delta_{\phi-\phi_\ell}\right) \tag{21}$$

with $*$ denoting the convolution product and

$$\mathcal{Q}_{L_z}(\phi) = \frac{1-\cos(L_z\phi)}{1-\cos(\phi)}. \tag{22}$$

So, function $\mathcal{F}_m^{(L_z)}(\phi)$ is the sum of translated periodic functions centered around the desired ϕ's. The width of the first lobe is obtained by considering the distance between two consecutive zeros of function $\mathcal{Q}_{L_z}(\phi)$ centered around $\phi = 0$, *ie.*, we look for δ which is the solution of

$$\mathcal{Q}_{L_z}(\pm\delta) = 0 \iff \left\{ \begin{array}{rcl} 1-\cos(\pm\delta) & \neq & 0, \\ 1-\cos(\pm\delta L_z) & = & 0. \end{array} \right. \tag{23}$$

Consequently, $\delta = \pm\frac{2\pi}{L_z}$ and thus the width of the first lobe is given by

$$\Delta = 2\delta = \frac{4\pi}{L_z}. \tag{24}$$

This quantity is inversely proportional to parameter L_z. As this parameter is assumed to be large, the first lobe is tight. Thus, the ZF-MUSIC allows an accurate cancelling of the previously estimated bearing.

In addition, we can determine the attenuation of the second lobe which is approximatively centred around $3\pi/L_z$. So, the attenuation in percent is given by

$$Q = \frac{100}{L_z^2}\mathcal{G}_{L_z}\left(\phi_m \pm \frac{3\pi}{L_z}\right) = \frac{100}{L_z^2 \sin^2\frac{3\pi}{2L_z}}. \tag{25}$$

or equivalently as L_z is large, $Q = \frac{400}{9\pi^2} \approx 4,5\%$ which is a quite small quantity.

5. NUMERICAL SIMULATIONS

The context of these simulations is an Uniform and Linear Array (ULA), with a half wavelength, $L = 10$ sensors and $T = 100$ snapshots with two uncorrelated sources, *ie.*, R_Λ is diagonal in (6). We denote by θ_1 the bearing of the first source and by θ_2 the bearing of the second one. We consider two situations. The bearing of the first source, θ_1, can be either exactly known (*cf.* Fig. 1) or estimated, (*cf.* Fig. 2). In the largely (*resp.* closely) spaced situation, we consider $\theta_1 = 0.1$ and $\theta_2 = 0.9$ rad (*resp.* $\theta_1 = 0.595$ and $\theta_2 = 0.6$ rad). In this last case, the "distance" between the two bearings is small and under the Rayleigh resolution [8], *ie.*, $|\phi_1 - \phi_2| = |\pi(\sin(\theta_2) - \sin(\theta_1))| \approx 0.013 \ll 0.49 \approx 2\sqrt{\frac{6}{L^2-1}}$. So, in our simulation context, the MUSIC algorithm cannot resolve the two bearings. The performance criterion is the Mean Square Error (MSE) which is computed by averaging over 500 Monte-Carlo trials for each Signal to Noise Ratio (SNR).

5.1. Improve estimation when θ_1 is known

We first begin by considering that θ_1 is known (estimated without error). This situation is sometimes realistic [1] and by doing this, we focus only on the errors introduced on the desired bearing, θ_2. In Fig. 1-(a), we consider largely spaced bearings. In that case, all algorithms are equivalent. However, according to Fig. 1-(b),(c) and (d) where we have considered different L_z and the bearings are extremely close, the ZF-MUSIC becomes more accurate in comparison to the S-MUSIC and the RAP-MUSIC algorithms, especially for low SNRs. Note that to improve the rejection of θ_1, the ZF-MUSIC algorithm needs to have a tight zero-forcing function. Toward this end, we can increase parameter L_z since we have seen in section 4.2 that the width of the zero-forcing function is inversely proportional to L_z (*cf.* Fig. 3-(a)). Obviously, increasing this parameter leads to a higher complexity cost. This is a price to pay to have a better accuracy.

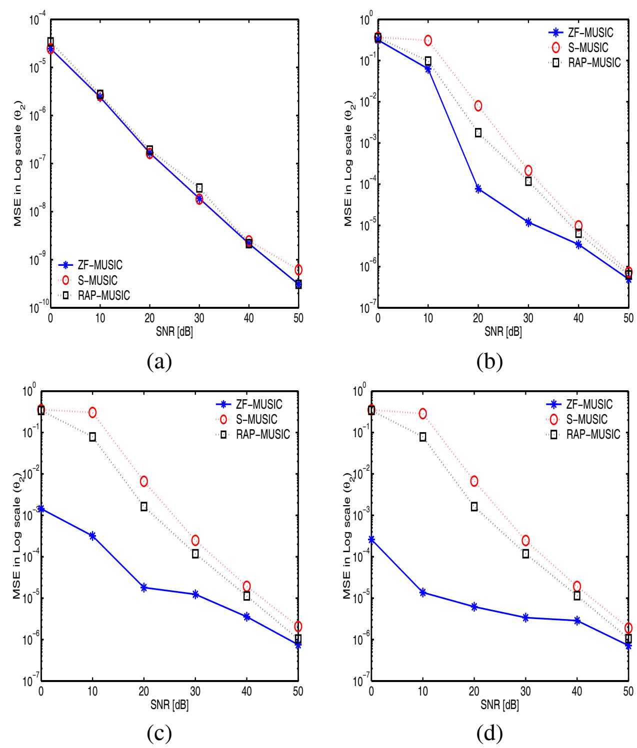
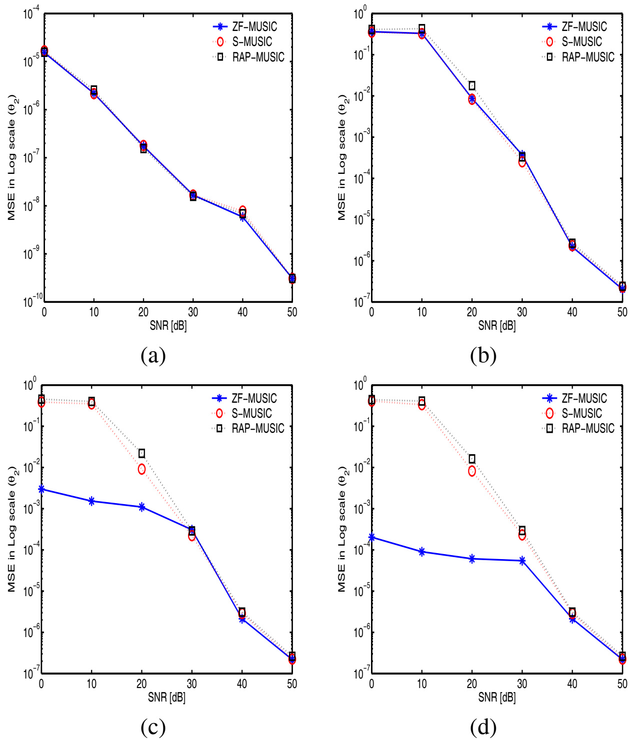


Fig. 1. MSE Vs. SNR, (a) Largely spaced bearings with $L_z = L$, (b) Closely spaced bearings with $L_z = L$, (c) Closely spaced bearings with $L_z = 2L$, (d) Closely spaced bearings with $L_z = 10L$.

5.2. Global estimation in the sequential case

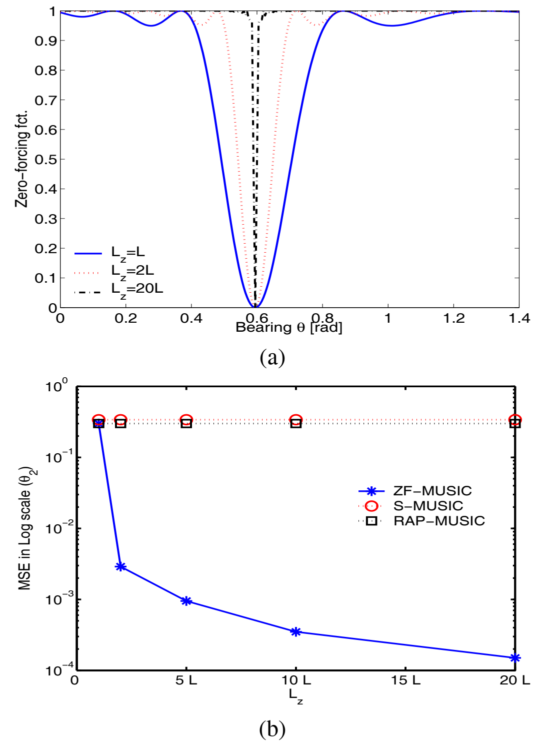
In this part, θ_1 and θ_2 are sequentially estimated. According to Fig. 2-(a) and (b), the accuracy of the ZF-MUSIC algorithm is comparable to the two others approaches for $L_z = L$ and for largely or closely spaced bearings. Here again, by increasing parameter L_z, the estimation of θ_2 is improved as we can see on Fig. 2-(c) and (d) where we have considered $L_z = 2L$ and $L_z = 20L$, respectively. In addition, to further illustrate the effect of parameter L_z, we have reported on Fig. 3-(b), the MSE of θ_2 *wrt.* L_z and for 0 dB of SNR.

We see clearly the gain. Thus, the ZF-MUSIC algorithm can control the rejection level of the previously estimated bearings thought parameter L_z.



(a) (b) (c) (d)

Fig. 2. MSE Vs. SNR, (a) Largely spaced bearings with $L_z = L$, (b) Closely spaced bearings with $L_z = L$, (c) Closely spaced bearings with $L_z = 2L$, (d) Closely spaced bearings with $L_z = 20L$.



(a) (b)

Fig. 3. (a) Zero-forcing fct. for different value of L_z, (b) MSE Vs. L_z for 0 dB of SNR and for closely spaced bearings.

6. CONCLUSION

Traditionally, the key principle of the sequential MUSIC algorithms is to deflate the signal subspace. In other terms, the dimension of this subspace is reduced by projecting the observed signal (or a matrix-based representation) onto the noise subspace associated to the previously estimated bearings. In this work, we do not follow this line and we leave unchanged the signal subspace but we scale the MUSIC criterion by an appropriate function. The latter is zero for all previously estimated bearings and asymptotically one elsewhere. We show in this work that this approach, called Zero-Forcing Sequential MUSIC algorithm, outperforms all the existing methods for closely spaced bearings.

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