BOOTSTRAP BASED CONFIDENCE INTERVALS FOR THE CONDITIONAL COHERENCE

Abdelhak M. Zoubir

Signal Processing Group
Technische Universität Darmstadt
Merckstr. 25, D-64283 Darmstadt, Germany
zoubir@ieee.org

ABSTRACT

We propose confidence intervals for the conditional coherence using the bootstrap. The asymptotic distribution of the empirical conditional coherence is inaccurate when signal and noise are non-Gaussian and/or the data size is small. The confidence intervals obtained with the bootstrap are shown to be accurate, maintaining the preset level of confidence.

Index Terms— confidence intervals, conditional coherence, conditional spectra, the bootstrap, non-Gaussian signals.

1. INTRODUCTION

Consider the situation of Figure 1, wherein $S(t)$, $U_1(t)$ and $U_2(t)$, $t = 0, \pm 1, \ldots$ are jointly stationary processes with spectral densities $C_{SS}(\omega), C_{U_1U_1}(\omega), C_{U_2U_2}(\omega)$ and $C_{U_1U_2}(\omega)$. Assume that $U_1(t)$ and $S(t)$ as well as $S(t)$ and $U_2(t)$ are uncorrelated for all $t = 0, \pm 1, \ldots$. Assume further that $h_1(t)$ and $h_2(t), t = 0, \pm 1, \ldots$ are impulse responses of linear time-invariant stable filters with the respective frequency responses $H_1(\omega)$ and $H_2(\omega)$, $-\infty < \omega < \infty$.

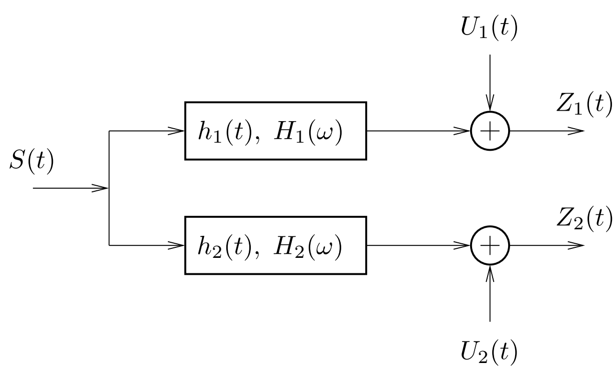

Fig. 1. Linear time-invariant filters driven by a stationary process.

This situation is encountered in many applications, among others, in wireless communications [1], sonar [2], bioscience [3], speech processing [4], optics [5] and seismic exploration [6]. The conditional coherence is useful to determine if the high magnitude squared coherence function of the measurements $Z_1(t)$ and $Z_2(t)$ at a given frequency of interest is caused by a third signal (or signals), $S(t)$ in Figure 1. To appreciate the importance of the concept of the conditional coherence function, we give below its expression for the scenario of Figure 1 and highlight the relationship with the magnitude squared coherence function.

1.1. Preliminaries

Let the spectral density matrix of $(S(t), Z_1(t), Z_2(t))'$,

$$\begin{pmatrix} C_{SS}(\omega) & C_{SZ_1}(\omega) & C_{SZ_2}(\omega) \\ C_{Z_1S}(\omega) & C_{Z_1Z_1}(\omega) & C_{Z_1Z_2}(\omega) \\ C_{Z_2S}(\omega) & C_{Z_2Z_1}(\omega) & C_{Z_2Z_2}(\omega) \end{pmatrix}$$

be non-negative definite for $-\infty < \omega < \infty$. The magnitude squared coherence function (or simply the coherence) of $Z_i(t)$, $i = 1, 2$ and $S(t)$ at frequency ω is given by

$$|R_{Z_iS}(\omega)|^2 = \frac{|C_{Z_iS}(\omega)|^2}{C_{Z_iZ_i}(\omega)C_{SS}(\omega)} = \frac{|H_i(\omega)|^2}{|H_i(\omega)|^2 + \frac{C_{U_iU_i}(\omega)}{C_{SS}(\omega)}} .$$

The coherence function of $Z_i(t)$ and $S(t)$ measures the extent to which $Z_i(t)$, $i = 1, 2$ is determinable from $S(t)$, $t = 0, \pm 1 \ldots$, at frequency ω by linear time-invariant operations and is bounded between 0 and 1. The coherence of $Z_1(t)$ and $Z_2(t)$ is defined through

$$|R_{Z_1Z_2}(\omega)|^2 = \frac{|C_{Z_1Z_2}(\omega)|^2}{C_{Z_1Z_1}(\omega)C_{Z_2Z_2}(\omega)} ,$$

and one can show the relationship given in Eq. (1) where $\mathsf{Re}\,\{z\}$ is the real part of the complex-valued number z and $\overline{z}$ denotes the complex conjugate of z. In the case where $U_1(t)$ and $U_2(t)$ are uncorrelated for $t = 0, \pm 1, \ldots$, then $|R_{Z_1Z_2}(\omega)|^2 \equiv |\tilde{R}_{Z_1Z_2}(\omega)|^2$ and under the condition that $C_{U_1U_1}(\omega) = C_{U_2U_2}(\omega) \ll C_{SS}(\omega)$ at frequency ω, the coherences $|\tilde{R}_{Z_1Z_2}(\omega)|^2$, $|R_{Z_1S}(\omega)|^2$ and $|R_{Z_2S}(\omega)|^2$ are approximately 1.

$$|R_{Z_1Z_2}(\omega)|^2 = |\tilde{R}_{Z_1Z_2}(\omega)|^2 + \frac{|C_{U_1U_2}(\omega)|^2 + 2\cdot\mathrm{Re}\,\{H_1(\omega)\overline{H_2(\omega)C_{U_1U_2}(\omega)}\}C_{SS}(\omega)}{|H_1(\omega)H_2(\omega)|^2C_{SS}(\omega)^2\left\{1+\frac{C_{U_1U_1}(\omega)}{|H_1(\omega)|^2C_{SS}(\omega)}+\frac{C_{U_2U_2}(\omega)}{|H_2(\omega)|^2C_{SS}(\omega)}+\frac{C_{U_1U_1}(\omega)C_{U_2U_2}(\omega)}{|H_1(\omega)H_2(\omega)|^2C_{SS}(\omega)^2}\right\}}, \tag{1}$$

$$|\tilde{R}_{Z_1Z_2}(\omega)|^2 = \frac{1}{1+\frac{C_{U_1U_1}(\omega)}{|H_1(\omega)|^2C_{SS}(\omega)}+\frac{C_{U_2U_2}(\omega)}{|H_2(\omega)|^2C_{SS}(\omega)}+\frac{C_{U_1U_1}(\omega)C_{U_2U_2}(\omega)}{|H_1(\omega)H_2(\omega)|^2C_{SS}(\omega)^2}},$$

The above results show that in the general case there could be a high coherence of the signals $Z_1(t)$ and $Z_2(t)$, but this coherence alone does not explain what caused the high value. To explore this phenomenon, we consider the so-called conditional coherence, also called the partial coherence [7, 8].

1.2. The Conditional Coherence

Consider the conditional spectra

$$\begin{aligned} C_{Z_1Z_1\cdot S}(\omega) &= (1-|R_{Z_1S}(\omega)|^2)C_{Z_1Z_1}(\omega) \\ C_{Z_2Z_2\cdot S}(\omega) &= (1-|R_{Z_2S}(\omega)|^2)C_{Z_2Z_2}(\omega) \\ C_{Z_1Z_2\cdot S}(\omega) &= \left(1-\frac{C_{Z_1S}(\omega)C_{SZ_2}(\omega)}{C_{Z_1Z_2}(\omega)C_{SS}(\omega)}\right)C_{Z_1Z_2}(\omega)\,. \end{aligned}$$

The conditional coherence is given by

$$|R_{Z_1Z_2\cdot S}(\omega)|^2 = \frac{|C_{Z_1Z_2\cdot S}(\omega)|^2}{C_{Z_1Z_1\cdot S}(\omega)C_{Z_2Z_2\cdot S}(\omega)}\,, \tag{2}$$

which can be interpreted as the coherence of $Z_1(t)$ and $Z_2(t)$ after removing the linear effects of $S(t)$, $t = 0, \pm1, \ldots$. The conditional coherence is also bounded between 0 and 1.

In the model given in Figure 1 the conditional coherence $|R_{Z_1Z_2\cdot S}(\omega)|^2$ is identical to $|R_{U_1U_2}(\omega)|^2$, $-\infty < \omega < \infty$, where $|R_{U_1U_2}(\omega)|^2$ is the coherence of $U_1(t)$ and $U_2(t)$ at frequency ω. Consequently $|R_{Z_1Z_2\cdot S}(\omega)|^2 \equiv 0$ in the case where $U_1(t)$ and $U_2(t)$ are uncorrelated for all $t = 0, \pm1, \ldots$. Thus, it is clear that the high coherence of $Z_1(t)$ and $Z_2(t)$ is due to $S(t)$, exciting the two linear time-invariant filters, which can be shown with the conditional coherence only.

The objective of this study is to construct confidence intervals for $|R_{Z_1Z_2\cdot S}(\omega)|^2$. We note that for the sake of simplicity, we consider a scalar $S(t)$, $t = 0, \pm1, \ldots$, however an extension to a vector-valued input is straightforward.

2. ESTIMATION OF THE CONDITIONAL COHERENCE

We consider non-parametric estimation of spectra and cross-spectra. Given independent data records $Z_1(t,l)$, $Z_2(t,l)$ and $S(t,l)$ for $t = 0, \ldots, T-1$ and $l = 1, \ldots, n$, define for for $-\infty < \omega < \infty$ the finite Fourier transforms of the data by $d_{Z_i}(\omega, l) = \sum_{t=0}^{T-1} w(t/T)\ Z_i(t,l)e^{-j\omega t}$, $i = 1, 2$ and $d_S(\omega, l) = \sum_{t=0}^{T-1} w(t/T)\ S(t,l)e^{-j\omega t}$, where $w(\tilde{t})$, $\tilde{t} \in \mathbb{R}$ is a window that is bounded, is of bounded variation and vanishes for $|\tilde{t}| > 1$. Let the estimates of the spectra and cross-spectra $C_{Z_iZ_i}(\omega)$, $C_{Z_iS}(\omega)$ and $C_{Z_1Z_2}(\omega)$ be $\hat{C}_{Z_iZ_i}(\omega)$, $\hat{C}_{Z_iS}(\omega)$ for $i = 1, 2$ and $\hat{C}_{Z_1Z_2}(\omega)$, respectively and obtained by averaging n periodograms. Estimates for the conditional coherence are obtained by replacing true spectra and conditional spectra in Eq. (2) by their estimates.

Under regularity conditions, which include strict stationarity of the vector process $(S(t), Z_1(t), Z_2(t))'$, existence of all moments and absolute summability of all k-th order cumulant functions for all $k = 2, 3, \ldots$, the asymptotic probability density function (pdf) of $|\hat{R}_{Z_1Z_2\cdot S}|^2$ (omitting frequency ω) is given by [8]

$$\begin{aligned} &(n-2)(1-|R_{Z_1Z_2\cdot S}|^2)^{n-1}(1-|\hat{R}_{Z_1Z_2\cdot S}|^2)^{n-3} \\ &{}_2F_1(n-1, n-1; 1; |R_{Z_1Z_2\cdot S}|^2|\hat{R}_{Z_1Z_2\cdot S}|^2)\,, \end{aligned} \tag{3}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. The asymptotic cumulative distribution function (cdf) is given by [8]

$$\begin{aligned} &\left(\frac{1-|R_{Z_1Z_2\cdot S}|^2}{1-|R_{Z_1Z_2\cdot S}|^2|\hat{R}_{Z_1Z_2\cdot S}|^2}\right)^{n-1} |\hat{R}_{Z_1Z_2\cdot S}|^4 \\ &\times \sum_{k=0}^{n-4}\left(\frac{1-|\hat{R}_{Z_1Z_2\cdot S}|^2}{1-|R_{Z_1Z_2\cdot S}|^2|\hat{R}_{Z_1Z_2\cdot S}|^2}\right)^k \\ &(k+1)\cdot {}_2F_1(-k, 3-n; 2; |R_{Z_1Z_2\cdot S}|^2|\hat{R}_{Z_1Z_2\cdot S}|^2)\,. \end{aligned} \tag{4}$$

The pdf and cdf expressions are exact if the vector-valued processes are assumed to be Gaussian distributed. The plot in Figure 2 is an example where one can see the failure of the asymptotic distribution of $|\hat{R}_{Z_1Z_2\cdot S}(\omega)|^2$ in the case where the signal and the noise processes are non-Gaussian. In the figure we compare the density function of the empirical conditional coherence in Eq. (3) and the one obtained using Monte Carlo simulations. Herein, we chose $H_1(\omega) = -1 + 2e^{j\omega} + e^{j2\omega}$ and $H_2(\omega) = 2 + e^{j\omega}$ (see Figure 1). One may use Fisher's z transform proposed by Enochson and Goodman [9], with which one defines the transformation $\breve{R}_{Z_1Z_2\cdot S}(\omega) = \tanh^{-1}|\hat{R}_{Z_1Z_2\cdot S}(\omega)|$ as an approximate normally distributed random variate. Using this approximation an $100(1-\alpha)\%$ confidence interval for $|R_{Z_1Z_2\cdot S}(\omega)|^2$ can be readily obtained. However, the Gaussian approximation also fails to accurately estimate the distribution function of the empirical conditional coherence when the data is non-Gaussian and in particular when n is small.

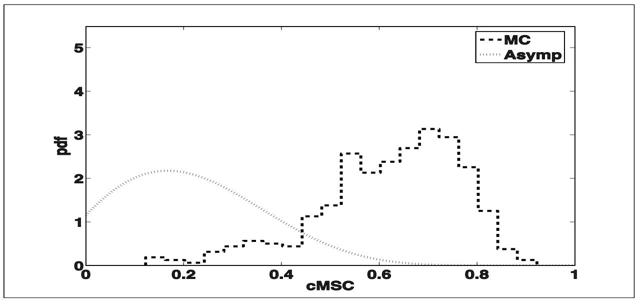

Fig. 2. Asymptotic density function of the conditional coherence (dotted line) vs. the density function obtained via 500 Monte Carlo simulations (dashed line). All processes are jointly independent, Laplace distributed and $T = 64, n = 10$.

In what follows we approximate the distribution function of $|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2$ with the bootstrap to construct confidence intervals. The approach is valid irrespective how large n is and independent of the type of distribution of the processes involved.

3. BOOTSTRAP CONFIDENCE INTERVALS

In a recent paper [10], we devised bootstrap confidence bounds for the magnitude squared coherence function and compared our results with those of a Gaussian approximation and a newly developed iterative method. It was shown that the bootstrap approach is superior as compared to the other two approximations. We follow a similar approach as in [10] to derive bootstrap confidence intervals for the conditional coherence.

Bootstrapping coherences can be performed in two ways. One approach would be to explore the linear regression $d_{Z_i}(\omega) = H_i(\omega) d_S(\omega) + d_{U_i}(\omega),\ i = 1, 2$ (omitting the error term $o_{a.s.}(1)$, which tends to 0 almost surely as $T \to \infty$ [7]), estimate $H_i(\omega)$ through $\hat{H}_i(\omega) = \hat{C}_{Z_i S}(\omega)/\hat{C}_{SS}(\omega)$ and define residuals $d_{\hat{U}_i}(\omega) = d_{Z_i}(\omega) - \hat{H}_i(\omega) d_S(\omega)$, which are used for resampling. This approach would enable us to replicate estimates of $\hat{R}_{Z_1 Z_2 \cdot S}(\omega)$ with the bootstrap and estimate confidence bounds. However, the approach relies on the model of Figure 1. In the absence of any model, we propose the approach of Table 1, which makes use of the input-output data only.

In Table 1, we used a variance stabilising transformation h in order to get more accurate confidence intervals [11]. Its estimation is similar to what we proposed in [10]. It should also be noted that in this particular case we could use the variance stabilising Fisher's z transform $\tanh^{-1}$, which would reduce computations.

4. THE EXPERIMENT

We consider linear time-invariant (LTI) systems as depicted in Figure 1, modeled by finite impulse response (FIR) filters whose frequency transfer functions $H_1(\omega)$ and $H_2(\omega)$ are given above. The LTI systems are driven by identically and independently Laplace distributed noise $S(t)$, with a constant spectral density $C_{SS}(\omega) = \sigma_S^2 = 1$. Independent identically and independently χ_4^2 distributed noise $U_1(t)$ is added to the output of the system with impulse response $h_1(t)$ to generate $Z_1(t)$ (See Figure 1). The process $U_2(t)$ is obtained through linear time-invariant filtering of $U_1(t)$. The filter used has the frequency response $H_U(\omega) = 1 - 0.5e^{j\omega} + 1.5e^{j2\omega}$. Independent noise $V(t)$ of a Gaussian is added to generate $U_2(t)$ and $\mathsf{SNR}_{U_1 V} = 0$ dB.

We construct confidence intervals for the conditional coherence given above as follows. We generate independently $S(t, l)$ and $U_i(t, l)$, $t = 0, \ldots, T-1$ for $l = 1, \ldots, n$ independent data stretches. We filter $S(t, l)$ with the linear time-invariant systems with transfer functions $H_1(\omega)$ and $H_2(\omega)$, add respectively $U_i(t, l)$, to obtain $Z_i(t, l)$, $i = 1, 2$, $t =$

Table 1. The Bootstrap procedure.

Step 0. *Data Collection.* Conduct the experiment and calculate the frequency data $d_S(\omega, 1), \ldots, d_S(\omega, n)$ and $d_{Z_i}(\omega, 1), \ldots, d_{Z_i}(\omega, n)$ for $i = 1, 2$.

Step 1. *Resampling.* Using a pseudo random number generator, draw a random sample, $\mathcal{X}^*(\omega)$ (of the same size), with replacement, from $\mathcal{X}(\omega) = \{(d_S(\omega,1), d_{Z_1}(\omega,1)), d_{Z_2}(\omega,1)) \ldots, (d_S(\omega,n), d_{Z_1}(\omega,n) d_{Z_2}(\omega,n))\}$.

Step 2. *Bootstrap Estimates.* From $\mathcal{X}^*(\omega)$, calculate $|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2$, the bootstrap analogue of $|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2$ estimated by replacing the estimates by their bootstrap counterparts and form $h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2)$ and $h(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)$.

Step 3. *Repetition.* Repeat **Steps 1-2** a large number of times to obtain a total of N bootstrap statistics $|\hat{R}^{*1}_{Z_1 Z_2 \cdot S}(\omega)|^2, \ldots, |\hat{R}^{*N}_{Z_1 Z_2 \cdot S}(\omega)|^2$.

Step 4. *Distribution Function Estimation.* Sort the variance stabilised bootstrap estimates in increasing order to obtain $h^{(1)}(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2) \leq \ldots \leq h^{(N)}(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)$ and approximate the density function of $h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2) - h(|R_{Z_1 Z_2 \cdot S}(\omega)|^2)$ by the density function of $h(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2) - h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2)$.

Step 5. *Confidence Bands Estimation.* For a desired $(1-\alpha)100\%$ bootstrap confidence interval, find critical points of the bootstrap distribution of $h(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2) - h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2)$, $h^{(q_1)}(|\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)$ and $h^{(q_2)}|(\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)$, say, where $q_1 = \lfloor (N+1)\alpha/2 \rfloor$ and $q_2 = N - q_1 + 1$. The confidence interval for $|R_{Z_1 Z_2 \cdot S}(\omega)|^2$ is obtained as $(h^{-1}(h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2) - h^{(q_2)}|(\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)), h^{-1}(h(|\hat{R}_{Z_1 Z_2 \cdot S}(\omega)|^2) - h^{(q_1)}|(\hat{R}^*_{Z_1 Z_2 \cdot S}(\omega)|^2)))$

$0, \dots, T-1$ for $l = 1, \dots, n$. Then, we estimate the conditional coherence. We compute confidence intervals using the procedure described in Table 1. The results below use the following parameters: $\alpha = 5\%$, $N = 399$, $T = 64$, $n = 20$, ($N_1 = 100$, $N_2 = 25$ for the variance stabilising transformation), $\mathsf{SNR} = 0$ dB. All results are based on 500 Monte Carlo replications. Figure 3 shows the 95% confidence bounds obtained with the bootstrap method.

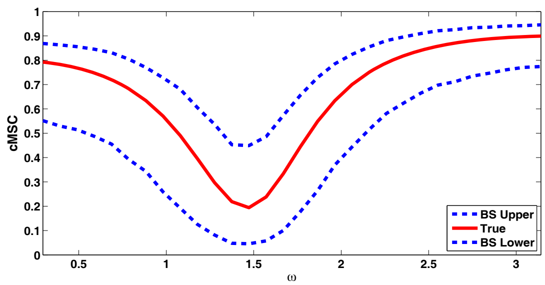

Fig. 3. True conditional coherence (solid) and estimated 95% confidence bounds with the bootstrap (dashed) non-Gaussian signal and noise.

In Figure 4, we show the lower and upper tail coverages as well as the confidence level estimated using the bootstrap. It can clearly be seen that the bootstrap approach maintains the nominal level. Also, The coverage values averaged over all frequencies are shown in Table 2. More experiments with different transfer functions have confirmed this property which makes it more robust than the asymptotic approximation or Fisher's z transform in the absence of knowledge about the probability distribution of signal and noise or when the number of data records is small.

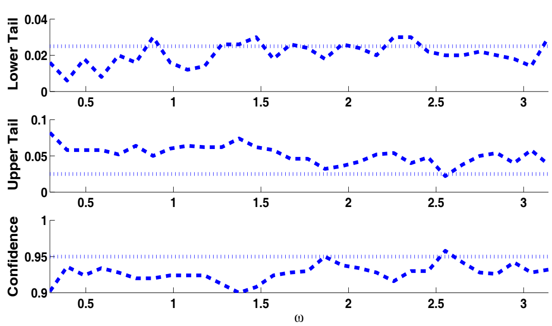

Fig. 4. Estimated lower and upper tail coverages and confidence level with the bootstrap (dashed) non-Gaussian signal and noise. Dotted lines indicate nominal values.

5. CONCLUSIONS

The conditional coherence is an important tool in many applications such as sonar, wireless communications and seismic exploration. We have considered estimation of confidence intervals for the conditional coherence using the bootstrap. The proposed method is more valid than the asymptotic approach as it does neither assume a Gaussian distribution of signal and noise nor does it require large data records. It has been shown in an experiment that the approximation is accurate as the nominal level of confidence is well maintained.

Table 2. Lower and upper tails and confidence level for a nominal 95% confidence interval over all frequencies

	lower tail	confidence	upper tail
Bootstrap	0.0207	0.9273	0.052

6. ACKNOWLEDGMENTS

The author wishes to thank Stefan Leier and Dr Said Aouada for their help with the simulations.

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