ESTIMATION OF THE MAGNITUDE SQUARED COHERENCE SPECTRUM BASED ON REDUCED-RANK CANONICAL COORDINATES

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ABSTRACT

In this paper, a new technique for the estimation of the magnitude squared coherence (MSC) spectrum is proposed. The method is based on the relationship between the MSC and the canonical correlation analysis (CCA) of stationary time series. Particularly, the canonical correlations coincide asymptotically with the squared roots of the MSC, which is exploited in the paper to obtain an estimate of the MSC based on a reduced-rank version of the estimated coherence matrix. The proposed technique provides a higher spectral resolution than the well-known Welch's method, and it also avoids the signal mismatch problem associated to the minimum variance distortionless response (MVDR) based approach. Finally, the performance of the proposed method is evaluated by means of some numerical examples.

***Index Terms*—** Magnitude squared coherence (MSC) spectrum, cross-spectrum, canonical correlation analysis (CCA), coherence matrix, reduced-rank estimation.

1. INTRODUCTION

The magnitude squared coherence (MSC) spectrum and the cross-spectrum are very useful in a large variety of applications. For instance, the MSC provides a measure of the mutual information between two signals [1]. However, only a reduced number of methods have been proposed to estimate them [2, 3]. Specifically, the technique proposed in [2] is based on the well-known averaged periodogram method [4], which can be interpreted as a bank of data and frequency independent filters. On the other hand, the method proposed in [3] is based on the direct application of the minimum variance distortionless response (MVDR) approach [5], which provides a set of data and frequency dependent analysis filters.

The main advantage of the MVDR based technique over the method proposed in [2] is its increased spectral resolution. However, the MVDR analysis filters are independent of the cross-correlation between the signals, and in some situations high spikes in non-sampled frequencies could remain undetected. This drawback is exactly the same as the well-known problem of direction of arrival (DOA) mismatch [6], which arises in beamforming applications. Several solutions to the DOA mismatch problem have been proposed in the literature [7, 8]; however, the estimation of the MSC usually requires the evaluation of a large number of frequencies, which precludes the direct application of these methods.

In this paper, the close relationship between canonical correlation analysis (CCA) and the MSC spectrum is exploited to propose a new MSC and cross-spectrum estimator. Specifically, the proposed technique is based on a reduced-rank version of the estimated coherence matrix, which resembles the reduced-rank estimation techniques proposed in [1, 9–11]. Alternatively, this approach can also be viewed as a subspace method where the signal and noise subspaces correspond to the space of correlated and non-correlated signals, respectively. The benefits of the proposed approach are twofold: first, it reduces the noise in the estimates at the non-correlated frequencies; and second, it provides a better spectral resolution than the Welch's method without introducing the signal cancelling problems associated with the MVDR estimator. In the paper we also show that the proposed MSC estimate can be written in terms of the discrete Fourier transforms (DFT) of the main canonical vectors. Interestingly, this can be interpreted as a set of data and frequency dependent analysis filters which, unlike the MVDR based approach and its variants [7], not only depend on the autocorrelation functions, but also on the cross-correlation between the two signals. Finally, the performance of the proposed technique is illustrated by means of some simulation examples, which show that it outperforms the methods in [2, 3].

2. CROSS-SPECTRUM AND MSC SPECTRUM

Let us consider two stationary complex time series $x_1[n]$ and $x_2[n]$ with Fourier representations

$$x_1[n] \leftrightarrow X_1(\omega), \qquad x_2[n] \leftrightarrow X_2(\omega), \qquad 0 \le \omega < 2\pi.$$

Defining the stationary random vectors $\mathbf{x}_1 = [x_1[0], \ldots, x_1[n]]^T$ and $\mathbf{x}_2 = [x_2[0], \ldots, x_2[n]]^T$, whose dimensions increase without bound as $n \to \infty$, the associated infinite Toeplitz correlation matrices have Fourier representations

$$\mathbf{R}_{x_i x_i} = E[\mathbf{x}_i \mathbf{x}_i^H] \leftrightarrow S_{x_i x_i}(\omega) = |X_i(\omega)|^2, \qquad i = 1, 2,$$

and the cross-spectrum $S_{x_1 x_2}(\omega)$ between $x_1[n]$ and $x_2[n]$ at frequency ω is defined as

$$\mathbf{R}_{x_1 x_2} = E[\mathbf{x}_1 \mathbf{x}_2^H] \leftrightarrow S_{x_1 x_2}(\omega),$$

which satisfies $|S_{x_1 x_2}(\omega)| \le |X_1(\omega)|\,|X_2(\omega)|$, for all $0 \le \omega < 2\pi$. Finally, the magnitude squared coherence (MSC) spectrum is defined as the ratio

$$\gamma^2_{x_1 x_2}(\omega) = \frac{|S_{x_1 x_2}(\omega)|^2}{|S_{x_1 x_1}(\omega)|\,|S_{x_2 x_2}(\omega)|} \le 1,$$

and it provides a measure of the rate at which one signal brings information about the other [1].

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2.1. Welch's and MVDR Estimates

In practical situations, the number of available samples of $x_1[n]$ and $x_2[n]$ is limited and then, the cross-spectrum and the MSC spectrum must be estimated. A first approach consists on the direct application of the Welch's averaged periodogram method [2, 4], which can be used to obtain the estimates of the cross-correlation matrices, $\hat{\mathbf{R}}_{x_i x_j}$, from the vectors $\mathbf{x}_i[n] = [x_i[n], \ldots, x_i[n-L+1]]^T$, where L is the window length parameter. Using these estimates, and defining the K Fourier vectors

$$\mathbf{f}_k = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & e^{j\omega_k} & \cdots & e^{j\omega_k(L-1)} \end{bmatrix}^T, \quad k = 0, \ldots, K-1,$$

with $\omega_k = 2\pi k/K$, we can estimate the cross-spectrum $\hat{S}_{x_1 x_2}(\omega_k)$ and the spectra functions $\hat{S}_{x_1 x_1}(\omega_k)$, $\hat{S}_{x_2 x_2}(\omega_k)$ as

$$\hat{S}_{x_i x_j}(\omega_k) = \mathbf{f}_k^H \hat{\mathbf{R}}_{x_i x_j} \mathbf{f}_k,$$

and, finally, the MSC spectrum is estimated as

$$\hat{\gamma}^2_{x_1 x_2}(\omega_k) = \frac{\left| \hat{S}_{x_1 x_2}(\omega_k) \right|^2}{\left| \hat{S}_{x_1 x_1}(\omega_k) \right| \left| \hat{S}_{x_2 x_2}(\omega_k) \right|}. \tag{1}$$

The main drawback of the averaged periodogram based method is that the Fourier analysis filters $\mathbf{f}_k$ are both data and frequency independent [12], which can cause large interferences among different frequencies due to spectral leakage. In order to avoid this problem, in [3] the authors have proposed an estimator based on a minimum variance distortionless response (MVDR) approach [5]. Basically, the MVDR data dependent analysis filters $\mathbf{g}_{ik}$, $i = 1, 2$, are designed to maintain a unit response at frequency ω_k (i.e. $\mathbf{f}_k^H \mathbf{g}_{ik} = 1$), while minimizing the estimated output energy $\mathbf{g}_{ik}^H \hat{\mathbf{R}}_{x_i x_i} \mathbf{g}_{ik}$, where the estimated correlation matrices are now obtained by simple averaging

$$\hat{\mathbf{R}}_{x_i x_j} = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{x}_i[n] \mathbf{x}_j^H[n], \qquad i, j = 1, 2, \tag{2}$$

and N is the number of available data samples. Finally, the estimates of the cross-spectrum and spectra functions are obtained as

$$\hat{S}_{x_i x_j}(\omega_k) = \mathbf{g}_{ik}^H \hat{\mathbf{R}}_{x_i x_j} \mathbf{g}_{jk} = \frac{\mathbf{f}_k^H \hat{\mathbf{R}}_{x_i x_i}^{-1} \hat{\mathbf{R}}_{x_i x_j} \hat{\mathbf{R}}_{x_j x_j}^{-1} \mathbf{f}_k}{\left[\mathbf{f}_k^H \hat{\mathbf{R}}_{x_i x_i}^{-1} \mathbf{f}_k \right] \left[\mathbf{f}_k^H \hat{\mathbf{R}}_{x_j x_j}^{-1} \mathbf{f}_k \right]},$$

and the MSC is estimated by means of (1).

Although the MVDR approach improves the low spectral resolution of the averaged periodogram based method, it suffers from a signal mismatch problem. This means that a sinusoidal component present in both signals (and therefore perfectly correlated) can be cancelled if its frequency do not coincide with the frequency grid given by $\omega_k = 2\pi k/K$. This drawback is the analogous of the well-known problem of direction of arrival (DOA) mismatch, which is encountered in beamforming applications [6, 7].

3. CANONICAL CORRELATION ANALYSIS

Canonical correlation analysis (CCA) is a well-known technique in multivariate statistical analysis which has been widely used in communications and statistical signal processing [1, 10, 11] problems. Given two random vectors $\mathbf{x}_1 \in \mathbb{C}^{L\times 1}$, $\mathbf{x}_2 \in \mathbb{C}^{L\times 1}$, CCA can be defined as the problem of finding two canonical vectors $\mathbf{h}_1 \in \mathbb{C}^{L\times 1}$, $\mathbf{h}_2 \in \mathbb{C}^{L\times 1}$ maximizing the canonical correlation

$$\lambda = \mathbf{h}_1^H \mathbf{R}_{x_1 x_2} \mathbf{h}_2,$$

and subject to the constraints $\mathbf{h}_1^H \mathbf{R}_{x_1 x_1} \mathbf{h}_1 = \mathbf{h}_2^H \mathbf{R}_{x_2 x_2} \mathbf{h}_2 = 1$. This is the so-called main or first CCA solution. The procedure can be generalized to obtain L CCA solutions $\mathbf{h}_1^{(q)}$, $\mathbf{h}_2^{(q)}$, $\lambda^{(q)}$, ($q = 1, \ldots, L$) by imposing, for $r = 1, \ldots, q-1$, the additional orthogonality constraints

$$\mathbf{h}_1^{(q)H} \mathbf{R}_{x_1 x_1} \mathbf{h}_1^{(r)} = \mathbf{h}_2^{(q)H} \mathbf{R}_{x_2 x_2} \mathbf{h}_2^{(r)} = \mathbf{h}_1^{(q)H} \mathbf{R}_{x_1 x_2} \mathbf{h}_2^{(r)} = 0.$$

Considering the cross-correlation matrix associated to the whitened version of $\mathbf{x}_1$ and $\mathbf{x}_2$, which is known as the coherence matrix

$$\mathbf{C}_{x_1 x_2} = \mathbf{R}_{x_1 x_1}^{-1/2} \mathbf{R}_{x_1 x_2} \mathbf{R}_{x_2 x_2}^{-1/2},$$

and taking its singular value decomposition (SVD) $\mathbf{C}_{x_1 x_2} = \mathbf{F}_1 \mathbf{\Lambda} \mathbf{F}_2^H$, the CCA solutions are $\mathbf{\Lambda} = \text{diag}(\lambda^{(1)}, \ldots, \lambda^{(L)})$, and

$$\left[\mathbf{h}_1^{(1)} \cdots \mathbf{h}_1^{(L)} \right] = \mathbf{R}_{x_1 x_1}^{-1/2} \mathbf{F}_1, \qquad \left[\mathbf{h}_2^{(1)} \cdots \mathbf{h}_2^{(L)} \right] = \mathbf{R}_{x_2 x_2}^{-1/2} \mathbf{F}_2,$$

i.e., the canonical correlations are given by the singular values of the coherence matrix, and the singular vectors can be interpreted as the whitened canonical vectors.

Interestingly, when the random vectors $\mathbf{x}_1 = [x_1[0], \ldots, x_1[n]]^T$, $\mathbf{x}_2 = [x_2[0], \ldots, x_2[n]]^T$ are associated to stationary time series, and in the cases of $n \to \infty$ [1], or circulant channels [9], the associated whitened canonical vectors are the Fourier vectors $\mathbf{f}_k$, and the MSC is given by the square of the canonical correlations. Nevertheless, in practical situations where the channel is not circulant and only a finite number of samples is available, the canonical vectors do not coincide with the Fourier vectors.

4. MSC ESTIMATE BASED ON REDUCED-RANK CCA

In this section we propose a new MSC estimator based on a low-rank approximation of the coherence matrix. Moreover, we provide an interpretation of the proposed estimator as a weighted sum of the cross product between the Fourier transform of the canonical vectors. Let us start by writing

$$\gamma_{x_1 x_2}(\omega_k) = \mathbf{f}_k^H \mathbf{R}_{x_1 x_1}^{-1/2} \mathbf{R}_{x_1 x_2} \mathbf{R}_{x_2 x_2}^{-1/2} \mathbf{f}_k = \mathbf{f}_k^H \mathbf{C}_{x_1 x_2} \mathbf{f}_k,$$

and consider the SVD decomposition of the estimated coherence matrix

$$\hat{\mathbf{C}}_{x_1 x_2} = \hat{\mathbf{R}}_{x_1 x_1}^{-1/2} \hat{\mathbf{R}}_{x_1 x_2} \hat{\mathbf{R}}_{x_2 x_2}^{-1/2} = \hat{\mathbf{F}}_1 \hat{\mathbf{\Lambda}} \hat{\mathbf{F}}_2^H.$$

The basic idea of the estimator is to suppress the subspace associated to the lowest canonical correlations, which spans those components with low correlation and is more affected by estimation errors due to finite sample size effects. The interesting point to stress is that for MSC and cross-spectrum estimation the right coordinate system for this truncation or subspace separation is the system of canonical coordinates. In this way, the method resembles other reduced-rank approaches based on canonical coordinates and used for coding, filtering or estimation problems in [1, 9–11].

Specifically, we use a reduced-rank version of the coherence matrix

$$\tilde{\mathbf{C}}_{x_1 x_2} = \tilde{\mathbf{F}}_1 \tilde{\mathbf{\Lambda}} \tilde{\mathbf{F}}_2^H,$$

where $\tilde{\mathbf{F}}_i$ ($i = 1, 2$) contains the main p singular vectors in $\hat{\mathbf{F}}_i$, and $\tilde{\mathbf{\Lambda}}$ is a diagonal matrix containing the p largest singular values in $\hat{\mathbf{\Lambda}}$.

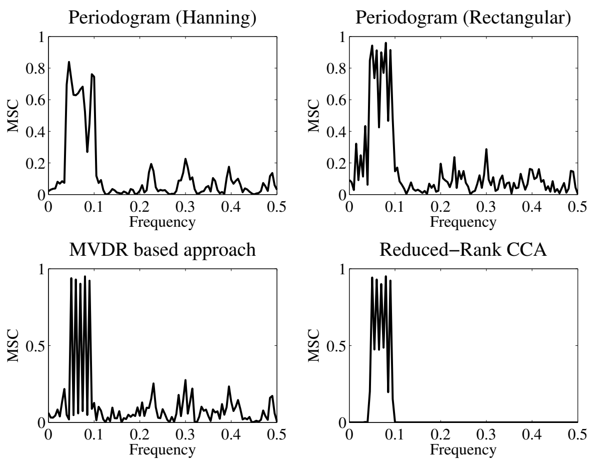

Fig. 1. MSC estimates for the first example. $N_f = 5$ common tones located at sampled frequencies $\omega_k = 2\pi k/K$.

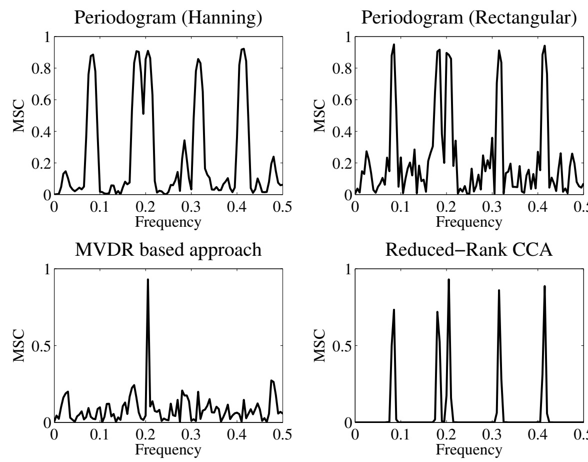

Fig. 2. MSC estimates for the second example. $N_f = 5$ common tones at non-sampled frequencies.

Thus, the estimated MSC is obtained from

$$\hat{\gamma}_{x_1x_2}(\omega_k) = \mathbf{f}_k^H \tilde{\mathbf{C}}_{x_1x_2} \mathbf{f}_k. \tag{3}$$

Finally, the estimate of the cross-spectrum based on the reduced-rank coherence matrix is

$$\hat{S}_{x_1x_2}(\omega_k) = \mathbf{f}_k^H \hat{\mathbf{R}}_{x_1x_1}^{1/2} \tilde{\mathbf{C}}_{x_1x_2} \hat{\mathbf{R}}_{x_2x_2}^{1/2} \mathbf{f}_k. \tag{4}$$

For sinusoidal processes, the rank p is related to the number of correlated complex exponentials. In a general case this value must be selected from the canonical correlations analogously to other order estimation methods used in parametric spectral analysis. We will illustrate this point in the next section.

4.1. Interpretation of the Reduced-Rank Estimates

Let us rewrite (3) as

$$\hat{\gamma}_{x_1x_2}(\omega_k) = \sum_{q=1}^{p} \hat{\lambda}^{(q)} \hat{f}_1^{(q)}(\omega_k) \hat{f}_2^{(q)^*}(\omega_k),$$

where $(\cdot)^*$ denotes the complex conjugate,

$$\hat{\mathbf{f}}_i^{(q)} = \left[\hat{f}_i^{(q)}(\omega_0), \ldots, \hat{f}_i^{(q)}(\omega_{K-1})\right]^T, \qquad i = 1, 2,$$

is the discrete Fourier transform (DFT) of the q-th whitened canonical vector in $\hat{\mathbf{F}}_i$, and $\hat{\lambda}^{(q)}$ is the q-th estimated canonical correlation. Thus, the proposed method for the estimation of the MSC can be seen as a weighted sum of the products between the DFTs of the estimated canonical vectors, where the weights are given by the estimated canonical correlations. Furthermore, taking (4) into account and rewriting $\hat{S}_{x_1x_2}(\omega_k) = \mathbf{g}_{1k}^H \hat{\mathbf{R}}_{x_1x_2} \mathbf{g}_{2k}$, it is straightforward to show that the new analysis filters for the estimation of the cross-spectrum are given by

$$\mathbf{g}_{ik} = \hat{\mathbf{R}}_{x_ix_i}^{-1/2} \tilde{\mathbf{F}}_i \tilde{\mathbf{F}}_i^H \hat{\mathbf{R}}_{x_ix_i}^{1/2} \mathbf{f}_k, \qquad i = 1, 2,$$

where we can see that, unlike the MVDR based approaches, the analysis filters not only depend on the two data signals, but also on their cross-correlation.

5. SIMULATION RESULTS

In this section we evaluate the performance of the proposed estimator, the averaged periodogram method [2, 4], and the MVDR approach proposed in [3]. In all the simulation examples, we have considered $N = 1024$ data samples, the window length has been selected as $L = 100$, and the MSC has been evaluated at $K = 200$ equispaced frequencies. The periodogram based method has been tested with Hanning and rectangular windows with 50% overlap, and the order of the CCA rank-reduction technique is $p = 10$.

In the two first examples, the signals are generated as

$$x_1[n] = w_1[n] + \sum_{i=1}^{N_f} \cos(2\pi\nu_i n),$$

$$x_2[n] = w_2[n] + \sum_{i=1}^{N_f} \cos(2\pi\nu_i n + \phi_i),$$

where $w_1[n]$, $w_2[n]$ are two independent zero-mean and real Gaussian random processes with unit variance, and the phases ϕ_i are random. We have considered $N_f = 5$ common frequencies. The theoretical MSC should be equal to 1 at frequencies $\nu_1, \ldots, \nu_{N_f}$ and zero at the others.

In the first example, the spectrum is exactly sampled at the common frequencies ($\nu_1 = 0.05$, $\nu_2 = 0.06$, $\nu_3 = 0.07$, $\nu_4 = 0.08$ and $\nu_5 = 0.09$). The MSC estimates are shown in Fig. 1, where we can see that the best results are obtained by the proposed technique, which eliminates the spurious correlations at the rest of frequencies; and by the MVDR based approach, which provides the highest spectral resolution.

In the second example, we consider a more realistic scenario where the correlated frequencies do not coincide with the Fourier grid. In particular, the common frequencies are located at $\nu_1 = 0.083$, $\nu_2 = 0.182$, $\nu_3 = 0.205$, $\nu_4 = 0.316$ and $\nu_5 = 0.414$. The simulation results are shown in Fig. 2. As can be seen, the proposed technique provides the best results whereas the performance of the MVDR based approach is severely degraded, which is due to the

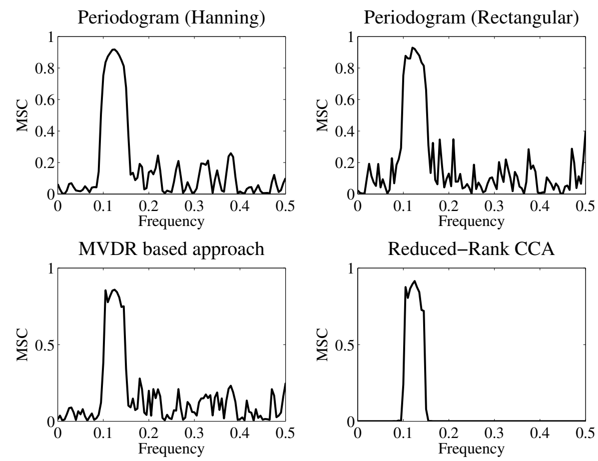

Fig. 3. MSC estimates for the third example. Narrowband signal.

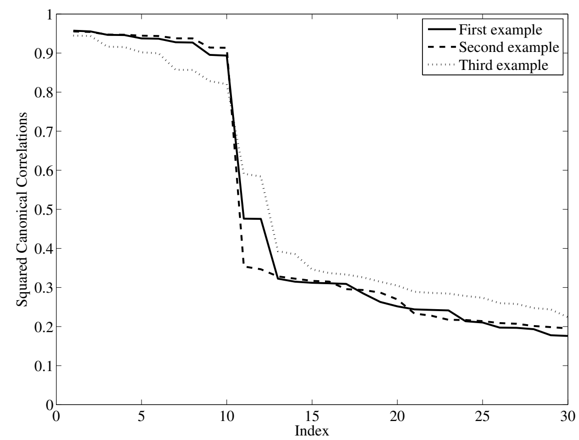

Fig. 4. Main canonical correlations for the three examples presented.

cancellation of most of the common frequencies. As pointed out before, this problem is analogous to the DOA mismatch problem encountered in beamforming [6, 7].

In the final example, the two signals are generated as

$$x_1[n] = s[n] + w_1[n], \qquad x_2[n] = s[n] + w_2[n],$$

where $w_1[n]$, $w_2[n]$ are two independent zero-mean and real Gaussian random processes with unit variance, and the common signal $s[n]$ is a narrowband zero-mean real Gaussian process with unit variance and passband between 0.1 and 0.15. It is easy to prove that in this case the theoretical MSC is $1.1^{-2} = 0.8264$ in the common band and zero at the remaining frequencies. Fig. 3 shows the simulation results in this case, where we can see that the reduced-rank CCA approach provides the most accurate estimate.

Finally, Fig. 4 shows the 30 main squared canonical correlations obtained in the previous examples. In all the examples the number of dominant canonical correlations is 10, which justifies our election of the order for the reduced-rank CCA technique.

6. CONCLUSIONS

In this paper we have proposed a new technique for the estimation of the cross-spectrum and the magnitude squared coherence spectrum. The proposed technique is based on a reduced-rank approximation of the estimated coherence matrix. The method provides a higher spectral resolution than the well-known averaged periodogram technique, and it avoids the problem of signal cancelling associated to the minimum variance distortionless response approach. The obtained results suggest that, like in other coding and filtering problems, the right coordinate system to suppress the noise and the non-correlated signals is the system of canonical coordinates. Further research lines include the application of a weighted version of the proposed technique, which can provide better results in situations where the canonical correlations decay smoothly.

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