MEAN-SQUARE CONSISTENT ESTIMATION OF THE SPECTRAL CORRELATION DENSITY FOR SPECTRALLY CORRELATED STOCHASTIC PROCESSES

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ABSTRACT

In this paper, the problem of estimating the spectral correlation density of spectrally correlated stochastic processes is addressed. These processes have Loève bifrequency spectrum with spectral masses concentrated on a countable set of support curves in the bifrequency plane. The almost-cyclostationary processes are obtained as a special case when the support curves are lines with unit slope. Spectrally correlated processes find application in wide-band or ultra-wideband mobile communications. It is shown that the cross-periodogram frequency smoothed along a known support curve and properly normalized provides a mean-square consistent estimator of the spectral correlation density of the Loève bifrequency spectrum along that curve.

Index Terms— Spectral analysis, Stochastic processes.

1. INTRODUCTION

Spectral analysis of stochastic processes is of big interest in theory and applications. For the wide-sense stationary processes, the power spectrum constitutes the parameter of interest for spectral analysis [2], [13], [21]. For these processes, the Loève bifrequency spectrum [13] has support on the main diagonal of the bifrequency plane. Thus, no correlation exists between spectral component at distinct frequencies and the density of the Loève bifrequency spectrum on the main diagonal of the bifrequency plane is coincident with the power spectrum. When correlation exists only between spectral components that are separated by quantities belonging to a countable set of values, the process is said almost-cyclostationary (ACS) or almost-periodically correlated [5]. The values of the frequency separations between correlated spectral components are called cycle frequencies and are the frequencies of the Fourier series expansion of the almost-periodically time-variant statistical autocorrelation function [5], [7]. For ACS processes, the Loève bifrequency spectrum has the support contained in lines parallel to the main diagonal of the bifrequency plane and its density on such lines is described by the cyclic spectra [5]. Techniques of spectral analysis have been developed for ACS processes in [4], [5], [7], [8], and [19].

Recently, a new class of nonstationary stochastic processes, the spectrally correlated (SC) processes, has been introduced and characterized in [15]. SC processes exhibit a Loève bifrequency spectrum with spectral masses concentrated on a countable set of curves in the bifrequency plane. Thus, ACS processes are obtained as a special case of SC processes when the support curves are lines with unit slope. In communications, SC processes can arise from some linear time-variant transformations of ACS processes (which are an appropriate model for almost all modulated signals). Examples of such transformations are the "stationary linear time-varying systems" considered in [3] (see also [6]) whose notable particular case is the multipath Doppler channel, that is, the channel that introduces a different complex gain, time-delay, frequency shift, and time-scale factor for each path [15]. It models the mobile radio channel in indoor or outdoor environments when there is a relative motion between transmitter and receiver or when the multipath arises from reflections on moving objects and the involved relative radial speeds (with respect to transmitter and/or receiver) are such that the product signal-bandwidth times data-record length is not much smaller than the ratios between the medium propagation speed and the radial speeds [22]. Therefore, such a model is appropriate in modern mobile communication systems where wide bandwidths are considered and large data-record lengths should be used for blind channel identification or equalization algorithms or for detection techniques in highly noise- and interference-corrupted environments. Further situations where nonunit time-scale factors should be accounted for can be encountered in radar and sonar applications [22, pp. 339-340], communications with wide-band and ultra wide-band (UWB) signals [11], [20], and space communications [17]. In all these cases, SC processes, instead of wide-sense stationary or ACS processes, are more appropriate models for the involved signals [15]. Finally, in [18] it is shown that fractional Brownian motion (fBm) processes have a Loève bifrequency spectrum with spectral masses concentrated on three lines.

In [15], the problem of the spectral correlation density estimation for SC processes is addressed in the case of unknown support curves. It is shown that the (bifrequency) spectral correlation density function of SC processes that are not ACS can be estimated with some degree of reliability by the time-smoothed cross-periodogram only if the departure of the nonstationarity from the almost-cyclostationarity is not too large.

In the present paper, the problem of estimation of the spectral correlation density of SC processes is addressed in the case of known support curves. Specifically, the cross-periodogram frequency smoothed along a given known support curve is proposed as estimator of the spectral correlation density on this support curve. It is shown that the frequency-smoothed cross-periodogram asymptotically (as the data-record length approaches infinity and the spectral resolution approaches zero) approaches the product of the spectral correlation density function and a function of frequency and slope of the support curve (Theorem 4.1). Since the support curve is assumed to be known, such a multiplicative bias term can be compensated. Moreover, it is shown that the asymptotic covariance of the frequency smoothed cross-periodogram approaches zero (Theorem 4.2). Therefore, the properly normalized frequency-smoothed cross-periodogram is a mean-square consistent estimator of the spectral correlation density function. The results of this paper generalize those of [1] and [12] where the case of support lines is considered. Moreover, the well-known result for ACS processes that the frequency-smoothed cyclic periodogram is a mean-square consistent estimator of the cyclic spectrum (see [4], [5], [7], [8], [19]) can be obtained as a special case of the results of this paper.

2. SPECTRALLY CORRELATED PROCESSES

Definition 2.1 Let $x(t)$ be a continuous-time complex-valued second-order harmonizable stochastic process. The *Loève bifrequency spectrum* [13], [21] (see also [8] and [14]) is defined as

$$\mathcal{S}_{xx^*}(f_1, f_2) \triangleq \mathrm{E}\left\{X(f_1)\, X^*(f_2)\right\} \ . \tag{1}$$

In (1), $X(f)$ is the Fourier transform of $x(t)$ and is assumed to exist (at least) in the sense of distributions for almost all sample paths of $x(t)$. Superscript $*$ denotes complex conjugation. □

Definition 2.2 Let $x(t)$ be a complex-valued second-order harmonizable stochastic process. The process is said *spectrally correlated* [15] if its Loève bifrequency spectrum can be expressed as

$$\mathcal{S}_{xx^*}(f_1, f_2) = \sum_{n \in \mathbb{I}} S^{(n)}_{xx^*}(f_1)\, \delta\left(f_2 - \Psi^{(n)}_{xx^*}(f_1)\right) \tag{2}$$

where $\delta(\cdot)$ is Dirac delta, $\mathbb{I}$ is a countable set, the curves $f_2 = \Psi^{(n)}_{xx^*}(f_1)$ describe the support of $\mathcal{S}_{xx^*}(f_1, f_2)$, and the functions $S^{(n)}_{xx^*}(f_1)$, called *spectral correlation density functions*, represent the density of the Loève spectrum on its support curves. □

The class of the SC stochastic processes includes, as a special case, the class of the second-order wide-sense ACS processes which, in turn, includes the class of the second-order wide-sense stationary processes. For the ACS processes, the support of the Loève bifrequency spectrum in the bifrequency plane is constituted by lines with unit slope [5]. Consequently, the separation between correlated spectral components assumes values belonging to a countable set. Such values, say α_n, are called cycle frequencies and are the frequencies of the (generalized) Fourier series expansion of the almost-periodically time-variant statistical autocorrelation function [5], [7]. Thus, for ACS processes, $\Psi^{(n)}_{xx^*}(f_1) = f_1 - \alpha_n$, and the spectral correlation density functions $S^{(n)}_{xx^*}(f)$ are coincident with the cyclic spectra. Note that the class of the ACS processes turns out to be the intersection between the class of the SC processes and that of the generalized almost-cyclostationary (GACS) processes [9], [10]. The GACS processes exhibit an almost-periodically time-variant statistical autocorrelation function whose (generalized) Fourier series expansion has coefficients and frequencies (cycle frequencies) depending on the lag parameter.

Nonstationary (possibly jointly) SC processes that are not (possibly jointly) ACS can arise from linear not-almost-periodically time-variant transformations of ACS processes. For example, an ACS signal transmitted by a moving source and received by two sensors gives rise to two signals that are jointly SC but are not jointly ACS [1]. Moreover, reverberation mechanisms generate coherency relationships ensemblewise between spectral components [14].

In [15], it is shown that an interesting example of generation of a SC stochastic process is given by an ACS process passing through a multipath Doppler channel, that is, a linear time-variant system such that for the input complex-envelope signal $x(t)$ the output complex-envelope $y(t)$ is given by

$$y(t) = \sum_{k=1}^{K} a_k\, x(s_k t - d_k)\, e^{j2\pi\nu_k t} \tag{3}$$

where, for each path of the channel, a_k is the complex gain, d_k the delay, s_k the time-scale factor, and ν_k the frequency shift. Such a model is appropriate to describe the multipath channel when, for each path, the relative radial speeds between transmitter, receiver, and reflecting moving objects can be considered constant in the observation interval [22, pp. 240-242]. In [15], it is shown that when an ACS process passes throughout a multipath Doppler channel, the output process is a SC process whose Loève bifrequency spectrum has support in the bifrequency plane constituted by lines with slopes $s_{k_2}/s_{k_1}, k_1, k_2 \in \{1, \ldots, K\}$.

It can be shown that the time-scale factors s_k can be considered unitary if the condition $BT \ll c/v_k\ \forall k$ is fulfilled, where B is input-signal bandwidth, T is the data-record length, c is the medium propagation speed, and v_k is the relative radial speed for the kth-path [22, pp. 240-242]. In such a case, the channel can be modelled as linear almost-periodically time variant and $y(t)$ is ACS. However, in modern communication systems, wider and wider bandwidths are required to get higher and higher bit rates. Moreover, large data-record lengths are necessary for blind channel identification techniques or detection algorithms in highly noise- and interference-corrupted environments. That is, there are practical situations where $BT \not\ll c/v_k$ and, hence, the time-scale factors cannot be considered unitary so that the output process $y(t)$ must be modelled as SC. For example, in [15] it is shown that in a code-division multiple access (CDMA) system with 512 chip per bit and in the presence of a radial speed of 100 km h^{-1}, if the maximum number of processed bits exceeds few hundreds, then the received signal must be modelled as SC instead of as ACS. Further situations where $BT \not\ll c/v_k$ and, hence, nonunit time-scale factors should be accounted for, can be encountered in radar and sonar applications [22, pp. 339-340], time-delay and Doppler estimation of wide band signals [11], UWB channel modeling [20], and space communications [17]. Spectrally correlated stochastic processes with nonlinear functions $\Psi^{(n)}_{xx^*}$ can be obtained by feeding with ACS processes the "stationary linear time-varying systems" considered in [3]. Furthermore, jointly SC processes with nonlinear functions $\Psi^{(n)}_{yx^*}$ are the input $x(t)$ and output $y(t)$ signals of the "stationary linear time-varying systems" or the linear time-variant systems described in [6]. Finally, fBm processes and their linear time-invariant filtered versions are SC processes with Loève bifrequency spectrum concentrated on the lines $f_2 = f_1$, $f_1 = 0$, and $f_2 = 0$ [18].

3. THE FREQUENCY-SMOOTHED CROSS-PERIODOGRAM

Definition 3.1 Given two stochastic processes $y(t)$ and $x(t)$, their *cross-periodogram* is defined as

$$I_{yx^*}(t; f_1, f_2)_T \triangleq \frac{1}{T}\, Y_T(t, f_1)\, X^*_T(t, f_2) \tag{4}$$

where $Y_T(t, f_1)$ and $X_T(t, f_2)$ are defined according to

$$Z_T(t, f) \triangleq \int_{\mathbb{R}} z(u)\, b_T(u - t)\, e^{-j2\pi f u}\, \mathrm{d}u \tag{5}$$

with $b_T(t)$ denoting a T-duration data-tapering window. □

Assumption 3.1 a) The second-order harmonizable stochastic processes $y(t)$ and $x(t)$ are singularly and jointly (second-order) spectrally correlated, that is, for any choice of z_1 and z_2 in $\{x, x^*, y, y^*\}$ it results

$$\mathcal{S}_{z_1 z_2}(f_1, f_2) = \sum_{n \in \mathbb{I}_{z_1 z_2}} S^{(n)}_{z_1 z_2}(f_1)\, \delta\left(f_2 - \Psi^{(n)}_{z_1 z_2}(f_1)\right) \tag{6}$$

where $\mathbb{I}_{z_1 z_2}$ is a countable set. ***b)*** The fourth-order spectral cumulant can be expressed as

$$\begin{aligned} &\operatorname{cum}\{Y(f_1),\ X^*(f_2),\ Y^*(f_3),\ X(f_4)\} \\ &= \sum_{n\in\mathbb{I}_4} P^{(n)}_{yx^*y^*x}(\boldsymbol{f}')\,\delta\left(f_4 - \Psi^{(n)}_{yx^*y^*x}(\boldsymbol{f}')\right) \end{aligned} \tag{7}$$

where $\mathbb{I}_4$ is a countable set and $\boldsymbol{f}' \triangleq [f_1, f_2, f_3]$. □

Assumption 3.2 ***a)*** For any choice of z_1 and z_2 in $\{x, x^*, y, y^*\}$, the functions $S^{(n)}_{z_1 z_2}(f)$ in (6) are almost everywhere (a.e.) continuous, in $L^\infty(\mathbb{R})$ and such that $\sum_{n\in\mathbb{I}_{z_1 z_2}} \|S^{(n)}_{z_1 z_2}\|_\infty < \infty$, where $\|S\|_\infty$ is the essential supremum of $S(f)$.
b) The functions $P^{(n)}_{yx^*y^*x}(f_1, f_2, f_3)$ in (7) are a.e. continuous, in $L^\infty(\mathbb{R}^3)$ and such that $\sum_{n\in\mathbb{I}_4} \|P^{(n)}_{yx^*y^*x}\|_\infty < \infty$. □

Assumption 3.3 ***a)*** For any choice of z_1 and z_2 in $\{x, x^*, y, y^*\}$, the functions $\Psi^{(n)}_{z_1 z_2}(f)$ in (6) are a.e. derivable with a.e. continuous derivatives. ***b)*** The functions $\Psi^{(n)}_{yx^*y^*x}(f_1, f_2, f_3)$ in (7) posses a.e. all the first-order partial derivatives that are a.e. continuous. □

Assumption 3.4 $b_T(t)$ is a T-duration data-tapering window with Fourier transform $B_{\frac{1}{T}}(f)$, such that $b_T(t) = w_b(t/T)$, $\lim_{T\to\infty} w_b(t/T) = 1\ \forall t \in \mathbb{R}$, $B_{\frac{1}{T}}(f) = T\,W_B(fT)$, where $W_B(f)$, the Fourier transform of $w_b(t)$, is a.e. continuous and regular as $|f| \to \infty$, $W_B(f) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $\int_{\mathbb{R}} W_B(f)\,\mathrm{d}f = w_b(0) = 1$. □

We have the following results from [15]. Let $y(t)$ and $x(t)$ be second-order harmonizable zero-mean singularly and jointly SC stochastic processes with bifrequency spectra and cross-spectra (6). Under Assumptions 3.1, 3.2, 3.3, and 3.4, the expected value of the cross-periodogram (4) is given in [15, Lemma 3.1] (eq. (39) with $\Delta f = 1/T$); the covariance is given in [15, Lemma 3.2] (eqs. (41)–(44)); the asymptotic ($T \to \infty$) expected value is given in [15, Theorem 3.1] (eq. (45)); the asymptotic covariance is given in [15, Theorem 3.2] (eqs. (48)–(50)). These results show that, as for wide-sense stationary and ACS processes, the properly normalized cross-periodogram of SC processes is an asymptotically unbiased but not consistent estimator of the spectral correlation density function.

It is well known that the power spectrum of wide-sense stationary processes can be consistently estimated by the frequency-smoothed periodogram, provided that some mixing assumptions regulating the memory of the process are satisfied [2], [21]. Moreover, for ACS processes, the frequency-smoothed cyclic periodogram at a given cycle frequency is a mean-square consistent estimator of the cyclic spectrum at that cycle frequency [4], [5], [7], [8], [19]. In both cases of wide-sense stationary and ACS processes, the frequency-smoothing procedure consists in considering frequency averages of the cross-periodogram along the support lines (with unity slope) of the Loève bifrequency spectrum. This technique was extended in [1] and [12] to the case of SC processes with support lines with not necessarily unit slope.

By following the above idea, in this paper the cross-periodogram frequency-smoothed along a known given support curve is proposed as estimator of the spectral correlation density of the Loève bifrequency spectrum on this support curve.

Definition 3.2 Given two jointly SC stochastic processes $y(t)$ and $x(t)$, their *frequency-smoothed cross-periodogram* along the support curve $f_2 = \Psi^{(n)}_{yx^*}(f_1)$ is defined as

$$S^{(n)}_{yx^*}(t; f_1)_{T,\Delta f} \triangleq \left[I_{yx^*}(t; f_1, f_2)_T \big|_{f_2 = \Psi^{(n)}_{yx^*}(f_1)} \right] \underset{f_1}{\otimes} A_{\Delta f}(f_1) \tag{8}$$

where $A_{\Delta f}(f_1)$ is a Δf-bandwidth frequency-smoothing window and $\otimes_{f_1}$ denotes convolution with respect to f_1. □

Assumption 3.5 $A_{\Delta f}(f)$ is a Δf-bandwidth frequency-smoothing window such that $A_{\Delta f}(f) = W_A\left(f/\Delta f\right)/\Delta f$ with $W_A(f)$ a.e. continuous, $W_A(f) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\int_{\mathbb{R}} W_A(f)\,\mathrm{d}f = 1$, and $\lim_{\Delta f\to 0} W_A\left(f/\Delta f\right)/\Delta f = \delta(f)$. □

By using the expressions of the expected value and covariance of the cross-periodogram derived in [15, Lemmas 3.1 and 3.2], the following results can be proved [16], where the made assumptions allow to interchange the order of expectation, integral, and sum operations.

Theorem 3.1 Let $y(t)$ and $x(t)$ be second-order harmonizable jointly SC stochastic processes with bifrequency cross-spectrum (6) (with $z_1 = y$, $z_2 = x^*$). Under Assumptions 3.2a, 3.4, and 3.5, the *expected value of the frequency-smoothed cross-periodogram* (8) is given by

$$\begin{aligned} \mathrm{E}\left\{S^{(n)}_{yx^*}(t; f)_{T,\Delta f}\right\} &= \frac{1}{T}\sum_{m\in\mathbb{I}} \int_{\mathbb{R}} S^{(m)}_{yx^*}(\nu) \int_{\mathbb{R}} \mathcal{B}_{\frac{1}{T}}(\lambda - \nu; t) \\ &\quad \mathcal{B}^*_{\frac{1}{T}}\left(\Psi^{(n)}_{yx^*}(\lambda) - \Psi^{(m)}_{yx^*}(\nu); t\right)\, A_{\Delta f}(f - \lambda)\,\mathrm{d}\lambda\,\mathrm{d}\nu \end{aligned} \tag{9}$$

where $\mathcal{B}_{\frac{1}{T}}(f; t) \triangleq B_{\frac{1}{T}}(f)\, e^{-j2\pi f t}$. □

Theorem 3.2 Let $y(t)$ and $x(t)$ be second-order harmonizable zero-mean singularly and jointly SC stochastic processes with bifrequency spectra and cross-spectra (6). Under Assumptions 3.1, 3.2, 3.4, and 3.5, the *covariance of the frequency-smoothed cross-periodogram* (8)

$$\operatorname{cov}\left\{S^{(n_1)}_{yx^*}(t_1; f_1)_{T,\Delta f},\ S^{(n_2)}_{yx^*}(t_2; f_2)_{T,\Delta f}\right\}$$

is the sum of three terms, two of them depending on the functions $S^{(n)}_{z_1 z_2}$ and $\Psi^{(n)}_{z_1 z_2}$, and the third depending on the functions $P^{(n)}_{yx^*y^*x}$ and $\Psi^{(n)}_{yx^*y^*x}$ [16]. □

4. MEAN-SQUARE CONSISTENCY OF THE FREQUENCY-SMOOTHED CROSS-PERIODOGRAM

Staring from the expressions of the expected value (Theorem 3.1) and covariance (Theorem 3.2) of the frequency-smoothed cross-periodogram, the following results can be proved [16], where the made assumptions allow to interchange the order of sum, integral, and limit operations.

Theorem 4.1 Let $y(t)$ and $x(t)$ be second-order harmonizable jointly SC stochastic processes with bifrequency cross-spectrum (6) (with $z_1 = y$, $z_2 = x^*$). Under Assumptions 3.2a, 3.3a, 3.4, and 3.5, the *asymptotic ($T \to \infty$, $\Delta f \to 0$, with $T\Delta f \to \infty$) expected value of the frequency-smoothed cross-periodogram* (8) is given by

$$\lim_{\Delta f\to 0}\lim_{T\to\infty} \mathrm{E}\left\{S^{(n)}_{yx^*}(t; f)_{T,\Delta f}\right\} = S^{(n)}_{yx^*}(f)\,\mathcal{E}^{(n)}(f) \tag{10}$$

where
$$\mathcal{E}^{(n)}(f) \triangleq \int_{\mathbb{R}} W_B(\lambda)\, W_B^*\left(\lambda\, \Psi_{yx^*}^{(n)}{}'(f)\right)\, \mathrm{d}\lambda \qquad (11)$$

with $\Psi_{yx^*}^{(n)}{}'(f)$ denoting the first-order derivative of $\Psi_{yx^*}^{(n)}(f)$. □

Theorem 4.2 Let $y(t)$ and $x(t)$ be second-order harmonizable zero-mean singularly and jointly SC stochastic processes with bifrequency spectra and cross-spectra (6). Under Assumptions 3.1, 3.2, 3.3, 3.4, and 3.5, the *asymptotic covariance of the frequency-smoothed cross-periodogram* (8) is given by

$$\begin{aligned}
&\lim_{\Delta f \to 0} \lim_{T\to\infty} (T\Delta f) \operatorname{cov}\left\{ S_{yx^*}^{(n_1)}(t_1; f_1)_{T,\Delta f},\ S_{yx^*}^{(n_2)}(t_2; f_2)_{T,\Delta f} \right\} \\
&= \sum_{n' \in \mathbb{I}'} \sum_{n'' \in \mathbb{I}''} S_{yy^*}^{(n')}(f_1)\, S_{x^*x}^{(n'')}\left(\Psi_{yx^*}^{(n_1)}(f_1)\right) \\
&\quad \mathcal{J}_1^{(n',n'')}(f_1)\, \bar{\delta}_{\Psi_{yx^*}^{(n_2)}(f_2) - \Psi_{x^*x}^{(n'')}\left(\Psi_{yx^*}^{(n_1)}(f_1)\right)} \\
&\quad \mathscr{J}_1^{(n')}(f_1)\, \delta_{f_2 - \Psi_{yy^*}^{(n')}(f_1)} \\
&+ \sum_{n''' \in \mathbb{I}'''} \sum_{n'^{\nu} \in \mathbb{I}'^{\nu}} S_{yx}^{(n''')}(f_1)\, S_{x^*y^*}^{(n'^{\nu})}\left(\Psi_{yx^*}^{(n_1)}(f_1)\right) \\
&\quad \mathcal{J}_2^{(n''',n'^{\nu})}(f_1)\, \bar{\delta}_{\Psi_{yx^*}^{(n_2)}(f_2) - \Psi_{yx}^{(n''')}(f_1)} \\
&\quad \mathscr{J}_2^{(n'^{\nu})}(f_1)\, \delta_{f_2 - \Psi_{x^*y^*}^{(n'^{\nu})}\left(\Psi_{yx^*}^{(n_1)}(f_1)\right)} \qquad (12)
\end{aligned}$$

where, for notation simplicity, $\mathbb{I}' = \mathbb{I}_{yy^*}$, $\mathbb{I}'' = \mathbb{I}_{x^*x}$, $\mathbb{I}''' = \mathbb{I}_{yx}$, and $\mathbb{I}'^{\nu} = \mathbb{I}_{x^*y^*}$. In (12), $\mathcal{J}_1^{(n',n'')}(f_1)$, $\mathscr{J}_1^{(n')}(f_1)$, $\mathcal{J}_2^{(n''',n'^{\nu})}(f_1)$, and $\mathscr{J}_2^{(n'^{\nu})}(f_1)$ depend on W_A, W_B and the support functions $\Psi_{z_1 z_2}^{(n)}$ [16]. $\delta_f = 1$ if $f = 0$ and $\delta_f = 0$ otherwise; $\bar{\delta}_{g(f)} = 1$ if $g(f) = 0$ in a neighborhood of f and $\bar{\delta}_{g(f)} = 0$ otherwise. □

From Theorem 4.1 it follows that the frequency-smoothed cross-periodogram $S_{yx^*}^{(n)}(t, f)_{T,\Delta f}$, normalized by $\mathcal{E}^{(n)}(f)$, is an asymptotically unbiased estimator of the spectral correlation density function along the same curve. Moreover, from Theorem 4.2 with $n_1 = n_2$, $f_1 = f_2$, and $t_1 = t_2$, it follows that the frequency-smoothed cross-periodogram has asymptotically vanishing variance of the order $\mathcal{O}((T\Delta f)^{-1})$. Therefore, the properly normalized frequency-smoothed cross-periodogram is a mean-square consistent estimator of the spectral correlation density.

5. REFERENCES

[1] J.C. Allen and S.L. Hobbs, "Detecting target motion by frequency-plane smoothing," in *Proc. of Twenty-Sixth Annual Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, October 1992.

[2] D. R. Brillinger, *Time Series. Data Analysis and Theory*. San Francisco, CA: Holden Day, Inc., 1981, and SIAM 2001.

[3] T.A.C.M. Claasen and W.F.G. Mecklenbräuker, "On stationary linear time-varying systems," *IEEE Trans. Circuits Syst.*, vol. CAS-29, pp.169-184, March 1982.

[4] A.V. Dandawaté and G.B. Giannakis, "Nonparametric polyspectral estimators for kth-order almost cyclostationary processes," *IEEE Transactions on Information Theory*, vol. 40, pp. 67-84, January 1994.

[5] D. Dehay and H.L. Hurd, "Representation and estimation for periodically and almost periodically random processes," in *Cyclostationarity in Communications and Signal Processing,* (W.A. Gardner, Ed.). IEEE Press, New York, 1994.

[6] P.A. Franaszek and B. Liu, "On a class of linear time-varying filters," *IEEE Transactions on Information Theory*, vol. IT-13, pp. 477-481, July 1967.

[7] W.A. Gardner, *Statistical Spectral Analysis: A Nonprobabilistic Approach.* Prentice Hall, Englewood Cliffs, NJ, 1988.

[8] N.L. Gerr and J.C. Allen, "The generalized spectrum and spectral coherence of harmonizable time series," *Digital Signal Processing*, vol. 4, pp. 222-238, 1994.

[9] L. Izzo and A. Napolitano, "The higher-order theory of generalized almost-cyclostationary time-series," *IEEE Trans. Signal Processing*, vol. 46, pp. 2975-2989, November 1998.

[10] L. Izzo and A. Napolitano, "Linear time-variant transformations of generalized almost-cyclostationary signals, Part I: Theory and method", *IEEE Transactions on Signal Processing*, vol. 50, pp. 2947-2961, December 2002.

[11] Q. Jin, K.M. Wong, and Z.Q. Luo, "The estimation of time delay and Doppler stretch of wideband signals," *IEEE Transactions on Signal Processing*, vol. 43, pp. 904-916, April 1995.

[12] K.-S. Lii and M. Rosenblatt, "Spectral analysis for harmonizable processes," *The Annals of Statistics*, vol. 30, N. 1, pp. 258-297, 2002.

[13] M. Loève, *Probability Theory*. Van Nostrand, Princeton, NJ, 1963.

[14] D. Middleton, "A statistical theory of reverberation and similar first-order scattered fields, Part II: Moments spectra, and special distributions," *IEEE Transactions on Information Theory*, vol. IT-13, July 1967.

[15] A. Napolitano, "Uncertainty in measurements on spectrally correlated stochastic processes", *IEEE Transactions on Information Theory*, vol. 49, pp. 2172-2191, September 2003.

[16] A. Napolitano, "Spectral analysis for spectrally correlated stochastic processes," in preparation.

[17] J. Oberg, "Titan calling," *IEEE Spectrum*, vol. 41, pp. 28-33, October 2004.

[18] T. A. Øigård, L. L. Sharf, and A. Hanssen, "Time-frequency and dual-frequency representation of fractional Brownian motion," Proc. of *IEEE Statistical Signal Processing Workshop (SSP'05)*, Bordeaux, France, July 17-20, 2005.

[19] B.M. Sadler and A.V. Dandawaté, "Nonparametric estimation of the cyclic cross spectrum," *IEEE Transactions on Information Theory,* vol. 44, pp. 351-358, January 1998.

[20] R. A. Schlotz, "Problems in modeling UWB channels," in Proc. of *Thirty-Sixth Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, 3-6 November, 2002.

[21] D.J. Thomson, "Spectrum estimation and harmonic analysis," *Proc. of the IEEE*, vol. 70, pp. 1055-1096, September 1982.

[22] H.L. Van Trees, *Detection, Estimation, and Modulation Theory, Part III*. John Wiley & Sons, Inc., New York, 1971.