TWO-CHANNEL, SEPARABLE, COMPLEX, ORTHOGONAL, PERFECT RECONSTRUCTION FILTER BANKS AND WAVELETS

B. Usevitch, usevitch@ece.utep.edu

University of Texas at El Paso, Department of Electrical Engineering, El Paso, TX 79968-0523

ABSTRACT

This paper demonstrates the design of separable complex orthogonal perfect reconstruction filters (OPRFs), which are defined to be complex OPRFs in which the real or imaginary part is also a valid OPRF. Separable filters greatly simplify the task of designing complex OPRFs by reducing it to a two step process: design a real OPRF (using now well known methods), then finding a "complex complementary" filter that results in an overall complex OPRF. This paper parameterizes, given any real OPRF, filters complex complementary to it. The parameterization is based on simple angle relationships in the lattice representation of each filter. Also shown is a simple method for designing complex wavelets from separable complex OPRFs.

Index Terms— wavelets, filterbanks, perfect reconstruction

1. INTRODUCTION

Original research into perfect reconstruction filterbanks and wavelets concentrated entirely on filters with real coefficients The publication by on complex wavelets by Lawton [1] resulted in additional research into wavelets and filterbanks with complex coefficients. These filterbanks and wavelet filters, hereafter referred to collectively as perfect reconstruction filters (PRFs), have found application in areas such as image compression, signal analysis, and communications. The design of complex PRFs has been aided by the work of Gao, et. al., who have parameterized the entire class of two-channel complex orthogonal PRFs [2]. The focus of this paper is different from prior work in that it develops methods to simplify the potentially "complex" problem of designing complex PRFs. The basic idea for design simplification was inspired by separable filters used in two-dimensional filter design. Although separable filters form only a sub class and are less flexible than general two-dimensional filters, their design and implementation simplicity and computational efficiency have led to their widespread use In this paper we consider only the class of *separable* complex orthogonal PRFs, which are defined to be complex filters where the real (or imaginary) filter is a valid real orthogonal PRF (OPRF). Results are also limited only to the two-channel case.

The idea of using the separable property to simplify the design of complex PRFs was first considered by Hernandez, et. al., in [3]. That work only resulted in a very limited set of separable complex PRFs, none of which were valid wavelet filters. The work in this paper shows that separable OPRFs form a much larger class than that given by the work of Hernandez, and design methods and constraints are given that significantly simplify the task of designing complex OPRFs using the separable property. A design method is also derived for creating complex wavelet filters from complex separable OPRFs.

The remainder of this paper proceeds as follows. Section 2 characterizes the class of separable complex OPRFs in terms of an arbitrary real OPRF and a filter which is "complex complementary" to it. These two filters, as the real and imaginary parts (in either order), form a valid complex OPRF. Section 3 gives a design method for deriving complex wavelets from separable complex OPRFs.

This work was supported by Texas Instruments Foundation Grant Number 26-8112-19

2. SEPARABLE, COMPLEX, ORTHOGONAL, PRFS

Let the OPRF $H(z)$ be defined as

$$H(z) = aA(z) + jbB(z) \qquad (1)$$

where $A(z)$ and $B(z)$ are a real filters and a and b are real constants. The complex filter $H(z)$ is defined to be separable if $A(z)$ or $B(z)$ is a real OPRF. The problem of determining the class of separable, complex OPRFs thus becomes one of determining the constraints on the real filters $A(z)$ and $B(z)$. An alternate way of approaching the problem, which will be important for design simplification and will be the approach emphasized here, is to pick one of the filters $A(z)$ or $B(z)$ to be a real OPRF. Here we assume without loss of generality that this filter is $A(z)$. The question then becomes what are the constraints on $B(z)$ such that $H(z)$ is a complex OPRF? The resulting filter $B(z)$ is hereafter referred to as being complex complementary (CC) to filter $A(z)$. The second method of characterizing complex OPRFs thus leads to a simple design procedure. First, design a real OPRF $A(z)$ using now well known methods. Then determine a CC filter $B(z)$ to give a resulting complex filter $H(z)$. The constraints on the filters $A(z)$ and $B(z)$ are now derived.

Previous results show that for $H(z)$ to be perfect reconstruction and orthogonal, that

$$P(z) + P(-z) = 2 \tag{2}$$

where $P(z) = H(z)\tilde{H}(z)$ and $\tilde{X}(z)$ represents the time reversed filter with all coefficients conjugated: $X_*(z^{-1})$. Substituting (1) into (2), recognizing that $A(z)$ and $B(z)$ are real filters, and simplifying gives

$$\begin{aligned} &P(z) + P(-z) = \\ &\quad a^2[P_A(z) + P_A(-z)] + b^2[P_B(z) + P_B(-z)] \\ &\quad + jab\{A(z^{-1})B(z) + A(-z^{-1})B(-z) \\ &\quad -[A(z)B(z^{-1}) + A(-z)B(-z^{-1})\}. \end{aligned} \tag{3}$$

where we have used

$$P_X(z) = X(z)\tilde{X}(z). \tag{4}$$

Previous research has derived, given a real OPRF $A(z)$, a very limited set of filters CC to it, and none of these filters resulted in complex OPRFs $H(z)$ that led to complex wavelets [3]. The research of this paper derives a much larger class of CC filters and shows that this set can lead to valid complex wavelet filters. The constraints of the filters $A(z)$ and $B(z)$ such that they form complex OPRFs is given by the following theorem.

Theorem 1 *Given a real OPRF $A(z)$ and complex filter $H(z)$ constructed as in Equation (1), for $H(z)$ to be a complex, OPRF it is necessary and sufficient that $B(z)$ be a real OPRF and that the correlation $A(z^{-1})B(z)$ of the filters $A(z)$ and $B(z)$ have even indexed coefficients that are symmetric about zero.*

Proof First assume that $B(z)$ is a real orthogonal PRF and that $A(z^{-1})B(z)$ satisfies the symmetry constraint. For Equation (3) to be satisfied, it is necessary and sufficient that the real and imaginary parts be equal. Since both $A(z)$ and $B(z)$ are OPRFs, the real part of Equation (3) will have only on non-zero coefficient at index z^0 (which can easily be scaled to 2 using a and b). Now define the polynomial

$$Q(z) = A(z^{-1})B(z) + A(-z^{-1})B(-z). \tag{5}$$

$Q(z)$ thus represents the even coefficients of the correlation $A(z^{-1})B(z)$. The imaginary part of (3) can thus be written as

$$ab[Q(z) - Q(z^{-1})].$$

Since the even coefficients of $A(z^{-1})B(z)$ are assumed to be symmetric about zero, the imaginary part of (3) is thus zero.

Now assume that either $B(z)$ is not a real orthogonal PRF, or that $A(z^{-1})B(z)$ has even coefficients that are not symmetric about zero. Thus $B(z)$ is either the all zero filter, or $B(z)B(z^{-1})$ has a nonzero even indexed coefficient that is not zero. In either case $H(z)$ is then not a complex orthogonal PRF since it is either entirely real (if $B(z) = 0$) or the real part of Equation (3) is not satisfied. Also, if $A(z^{-1})B(z)$ violates the symmetry condition, then the imaginary part of (3) is non-zero, and $H(z)$ also violates the necessary condition of Equation (3). **QED**

Since by Theorem 1, $A(z)$ and $B(z)$ must be real OPRFs, one way of deriving valid complex filters $H(z)$ is to represent $A(z)$ and $B(z)$ with their parametric lattice representation, then optimize this representation such that the cross correlation conditions are met. The lattice representations guarantee that $A(z)$ and $B(z)$ satisfy the OPRF condition of Theorem 1. To illustrate this method, consider the following PRF lattice representation of the polyphase matrix of $A(z)$ [4]

$$\mathbf{A}_p(z) = \mathbf{R}_l \mathbf{\Lambda}(z) \mathbf{R}_{l-1} \mathbf{\Lambda}(z) \cdots \mathbf{R}_1 \mathbf{\Lambda}(z) \mathbf{R}_0 \tag{6}$$

where

$$\mathbf{R}_l = \begin{bmatrix} \cos\theta_l & \sin\theta_l \\ -\sin\theta_l & \cos\theta_l \end{bmatrix}, \quad \mathbf{\Lambda}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

The valid angle constraints on the angles ϕ_i representing the filter $B(z)$ such that it is CC to $A(z)$ is given by the following theorem and summarized in Table 1.

Theorem 2 *Given a real, length L OPRF $A(z)$ constructed using angles θ_i, $i \in \{0, \ldots, \frac{L}{2} - 1\}$ of the lattice representation of Equation (6), a length L OPRF $B(z)$ which is complex complementary to $A(z)$ can be constructed by picking an angle ϕ_{j_0}, $0 \leq j_0 \leq \frac{L}{2} - 1$ to be free, then constraining all other angles representing $B(z)$ to satisfy the following*

$$\begin{aligned} \tan\theta_j &= \tan\phi_j \quad & j < j_0 \\ \tan\theta_j &= -\tan\phi_j \quad & j > j_0. \end{aligned}$$

Proof The OPRF $A(z) = A_0(z^2) + z^{-1}A_1(z^2)$ has the following polyphase representation

$$\mathbf{A}_p(z) = \begin{bmatrix} A_0(z) & A_1(z) \\ -A_1(z^{-1})z^{-\frac{L}{2}+1} & A_0(z^{-1})z^{-\frac{L}{2}+1} \end{bmatrix}. \tag{7}$$

Now let $\mathbf{C}_p(z) = \mathbf{A}_p(z)\mathbf{B}_p^T(z^{-1})$ and note that the top left (bottom right) entry of $\mathbf{C}_p(z)$ corresponds to the even coefficients of the correlation $A(z)B(z^{-1})$ ($A(z^{-1})B(z)$). Specifically, the top left corner is $A_0(z)B_0(z^{-1}) + A_1(z)B_1(z^{-1})$. Thus to prove the theorem it must only be shown that this top left corner entry is a symmetric function of z (making the even indexed coefficients of $A(z^{-1})B(z)$ symmetric). The proof is made by induction using the lattice representation of the polyphase filters $\mathbf{A}_p(z)$ and $\mathbf{B}_p^T(z^{-1})$ (see Equation (6)). Starting at the center of this product of matrices, it is seen that

Table 1. Given a length L OPRF $A(z)$ parameterized using angles θ_i, the constraints on the angles ϕ_i of the length L CC filter $B(z)$ are found by equating terms on the top row to corresponding terms on any subsequent row.

$\tan\phi_0$	$\tan\phi_1$	$\dots$	$\tan\phi_{j_0-1}$	$\tan\phi_{j_0}$	$\tan\phi_{j_0+1}$	$\dots$	$\tan\phi_{\frac{L}{2}-1}$
free	$-\tan\theta_1$	$\dots$	$-\tan\theta_{j_0-1}$	$-\tan\theta_{j_0}$	$-\tan\theta_{j_0+1}$	$\dots$	$-\tan\theta_{\frac{L}{2}-1}$
				$\vdots$			
$\tan\theta_0$	$\tan\theta_1$	$\dots$	$\tan\theta_{j_0-1}$	free	$-\tan\theta_{j_0+1}$	$\dots$	$-\tan\theta_{\frac{L}{2}-1}$
				$\vdots$			
$\tan\theta_0$	$\tan\theta_1$	$\dots$	$\tan\theta_{j_0-1}$	$\tan\theta_{j_0}$	$\tan\theta_{j_0+1}$	$\dots$	free

the inner matrices cancel since $\theta_i = \phi_i$ for $i < j_0$. The first set of inner matrices that do not cancel occur at index j_0

$$\mathbf{M}_{j_0}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta_{j_0} & \sin\theta_{j_0} \\ -\sin\theta_{j_0} & \cos\theta_{j_0} \end{bmatrix}$$
$$\cdot \begin{bmatrix} \cos\phi_{j_0} & -\sin\phi_{j_0} \\ \sin\phi_{j_0} & \cos\phi_{j_0} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} \alpha & \beta z \\ -\beta z^{-1} & \alpha \end{bmatrix}$$

where

$$\begin{aligned} \alpha &= \cos\theta_{j_0}\cos\phi j_0 - \sin\theta_{j_0}\sin\phi_{j_0} \\ \beta &= -\cos\theta_{j_0}\sin\phi j_0 + \sin\theta_{j_0}\cos\phi_{j_0}. \end{aligned}$$

Now define $\mathbf{X}_{j-1}(z)$ as

$$\mathbf{X}_{j-1}(z) = \begin{bmatrix} X_{j-1}(z) & Y_{j-1}(z)z \\ -Y_{j-1}(z^{-1})z^{-1} & X_{j-1}(z) \end{bmatrix}$$

where $X_{j-1}(z)$ is a symmetric function in z. Note that $\mathbf{M}(j_0)$ satisfies the form of $\mathbf{X}_{j-1}(z)$ (basis). Now we must only show that (induction step)

$$\mathbf{X}_j(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$
$$\cdot \begin{bmatrix} X_{j-1}(z) & Y_{j-1}(z)z \\ -Y_{j-1}(z^{-1})z^{-1} & X_{j-1}(z) \end{bmatrix} \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$$

also has the same form as $\mathbf{X}_{j-1}(z)$, with $X_j(z)$ being an even function of z. Some algebra shows that

$$X_j(z) = (\cos\theta_j^2 - \sin\theta_j^2)X_{j-1}(z) - \cos\theta_j\sin\theta_j[Y_{j-1}(z)z + Y_{j-1}(z^{-1})z^{-1}]$$

which is indeed an even function. Also, the off diagonal terms of $\mathbf{X}_j(z)$ have the form as that of $\mathbf{X}_{j-1}(z)$. **QED**

The results of Theorem 2, as shown in Table 1, agree with a straightforward degrees of freedom analysis. Specifically, two length L OPRFs have a total of L degrees of freedom ($\frac{L}{2}$ degrees for each filter) as can be seen from considering the lattice representation of Equation 6. The symmetry condition of the cross correlation required to make one filter CC to the other adds another $\frac{L}{2} - 1$ constraints. The result is that a length L separable complex OPRF has a total of $\frac{L}{2} + 1$ degrees of freedom. One method of constructing these complex separable filters is to first design a real length L OPRF, then determine an OPRF that is CC to it. Given that the first filter has $\frac{L}{2}$ degrees of freedom, the design of the CC filter leaves only 1 remaining degree of freedom (the choice of which angle to vary is not considered a degree a freedom here).

The main theme of this paper has been to simplify the process of designing complex OPRFs by considering only the class of separable complex OPRFs. Specifically, complex filters are designed by first specifying a real OPRF, then finding a CC filter that results in a valid complex OPRF. The following theorem gives a very simple way of deriving an OPRF $B(z)$ that is CC to a given filter $A(z)$. This method has the advantage that if $A(z)$ is a wavelet scaling filter, then the CC filter $B(z)$ will also be a wavelet scaling filter.

Theorem 3 *Let $A(z)$ be a real OPRF with polyphase representation $A(z) = A_0(z) + z^{-1}A_1(z)$, and let $B(z)$ be given as $B(z) = A_1(z) + z^{-1}A_0(z)$. The filter $B(z)$ is an OPRF that is CC to $A(z)$.*

Proof Let $A(z)$ be length L with polyphase matrix as given in Equation (7). The determinant of $\mathbf{A}_p(z)$ is

$$\det(\mathbf{A}_p) = [A_0(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})]z^{-\frac{L}{2}+1},$$

which, since $A(z)$ is perfect reconstruction, is equal to z^{k_0} for k_0 an odd integer. The polyphase matrix for $B(z)$ is

$$\mathbf{B}_p(z) = \begin{bmatrix} A_1(z) & A_0(z) \\ -A_0(z^{-1})z^{-\frac{L}{2}+1} & A_1(z^{-1})z^{-\frac{L}{2}+1} \end{bmatrix}$$

which has a determinant that is identical to that of $\mathbf{A}_p(z)$, showing that $B(z)$ is perfect reconstruction. From the structure of $\mathbf{B}_p(z)$ it is also evident that $B(z)$ is orthogonal (the "highpass filter" or more precisely the second row of $\mathbf{B}_p(z)$

is constructed by time reversing and negating every other coefficient of the "lowpass" or upper row of $\mathbf{B}_p(z)$). The only other condition to satisfy for $B(z)$ to be CC to $A(z)$ (from Theorem 1) is symmetry of the cross correlation. It is straightforward to show that for length L filters, the z^{-2l} and z^{2l} coefficients ($0 \leq l \leq \frac{L}{2} - 1$) of the correlation $A(z^{-1})B(z)$ are equal and given by

$$\sum_{k=0}^{\frac{L}{2}-l-1} a_{2k+1}a_{2k+2l} + a_{2k}a_{2k+1+2l}$$

where the a_k's are the coefficients of the filter $A(z)$. **QED**.

Theorem 4 *Given an OPRF $A(z)$ that is also a valid wavelet scaling filter, the filter $B(z)$ CC to $A(z)$ constructed using Theorem 3 is also a valid wavelet scaling filter.*

The proof follows readily since the sum of the even coefficients must equal the sum of the odd coefficients for a filter to be a valid wavelet scaling filter, and swapping the even and odd coefficients maintains this property.

3. COMPLEX WAVELETS FROM OPRFS

This section derives complex orthogonal PRFs that are also result in wavelet filters. In addition to being perfect reconstruction, a wavelet filter must satisfy an additional smoothness constraints which can be stated stated in terms of the lowpass or scaling filter $h(n)$ as [4]

$$\sum_{n=0}^{L-1}(-1)^n n^j h(n) \quad j = \{0, 1, \ldots, p-1\}. \tag{8}$$

In order for a filter to generate a wavelet, it must satisfy Equation 8 for at least $p = 1$. The larger p is, the smoother or better approximating the resulting wavelet. The approximating condition can also be stated equivalently in terms of the number of zeros of $H(z)$ at $z = -1$ (namely p), or in terms of the eigenvalues of a matrix $\mathbf{m}(0)$ [5, 4]. The matrix $\mathbf{m}(0)$ represents a subportion of the infinite decimate by 2 convolution matrix of the scaling filter and its form can be found in [5] or on page 195 of [4]. Approximation of order p corresponds to $\mathbf{m}(0)$ having eigenvalues $1, 2, \ldots, \left(\frac{1}{2}\right)^{p-1}$ with $p = 1$ being the minimum required to result in a wavelet. Given this background, the following theorem shows a method for deriving complex wavelet filters from separable complex OPRFs. The key constraint required is that the original filter $A(z)$ and its CC filter $B(z)$ be valid wavelet scaling filters.

Theorem 5 *Let $A(z)$ and $B(z)$ be real OPRFs that are also wavelet scaling filters, and let $B(z)$ be CC to $A(z)$. Without loss of generality assume that the coefficient sums of these two filters are 1 ($\sum a_i = \sum b_i = 1$). These two filters will always result in a complex wavelet scaling filter given by*

$$H_W(z) = \frac{(1-j)}{\sqrt{2}}(A(z) + jB(z)).$$

Proof Let $H(z) = A(z) + jB(z)$ and let $\mathbf{m}(0)$ be the scaling matrix formed from $H(z)$. It is easy to verify that

$$\begin{aligned} [\ 1 \quad 1 \quad \ldots \quad 1\ ]\mathbf{m}(0) &= [\ 1+j \quad 1+j \quad \ldots \quad 1+j\ ] \\ &= (1+j)[\ 1 \quad 1 \quad \ldots \quad 1\ ] \end{aligned}$$

demonstrating that the matrix $\mathbf{m}(0)$ has an eigenvalue of $1+j$. Since

$$\alpha A x = \alpha \lambda x$$

the scaling matrix $\mathbf{m}(0)$ formed from $H_W(z) = \frac{(1-j)}{\sqrt{2}}H(z)$ will have eigenvalue 1. Thus $H_W(z)$ has an order of approximation of at least one. Is $H_W(z)$ perfect reconstruction? Yes since

$$\begin{aligned} (re^{j\theta}H(z))(\widetilde{re^{j\theta}H}(z)) &= (re^{j\theta}H(z))(re^{-j\theta}H_*(z^{-1})) \\ &= r^2 H(z)\tilde{H}(z) \end{aligned}$$

showing that multiplying by a complex constant maintains the orthogonal perfect reconstruction property. **QED**.

Given a real wavelet scaling function $A(z)$, there are not a lot of wavelet filters $B(z)$ CC to it in order to satisfy Theorem 5. As shown in Section 2, the choice of CC filter allows only one degree of freedom. Requiring the filter to be a wavelet scaling filter uses this degree of freedom to force the angle to sum to be $\frac{\pi}{4}$ (for regularity of $p = 1$). The choice left is which of the $\frac{L}{2} - 1$ "free" angles of Table 1 will be used to give the $\frac{\pi}{4}$ sum. There are only $\frac{L}{2} - 1$ choices since if angle $\phi_{\frac{L}{2}-1}$ is chosen (last row of Table 1) $A(z)$ will equal $B(z)$ and the design method of Theorem 5 will result in a degenerate real wavelet scaling function.

4. REFERENCES

[1] W. Lawton, "Applications of complex valued wavelet transforms to subband decomposition," *IEEE Trans. Signal Process.*, vol. 41, pp. 3566–3568, December 1993.

[2] X. Gao, T. Nguyen, and G. Strang, "A study of two-channel complex-valued filterbanks and wavelets with orthogonality and symmetry properties," *IEEE Trans. Signal Process.*, vol. 50, April 2002.

[3] M. Hernandez and B. Usevitch, "A study of two-channel separable complex-valued wavelets and filter banks," in *Thirty-Eigth Southeastern Symposium on System Theory*, (Cookeville, Tn.), pp. 151–155, March 2006.

[4] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Massachusetts: Wellesley Cambridge Press, 1996.

[5] G. Strang, "Eigenvalues of $(\downarrow \mathbf{H})$ and convergence of the cascade algorithm," *IEEE Trans. Signal Process.*, vol. 44, pp. 233–238, February 1996.