DESIGN OF 2-D DOUBLY COMPLEMENTARY FILTERS BASED ON NONSYMMETRIC HALF-PLANE ALLPASS FILTERS

Yuan-Hau Yang and Ju-Hong Lee

National Taiwan University
Graduate Institute of Communication Engineering
No. 1, Sec. 4, Roosevelt Rd., Taipei, 10617, TAIWAN.
d92942011@ntu.edu.tw, juhong@cc.ee.ntu.edu.tw

ABSTRACT

A parallel-connected structure based on recursive nonsymmetric half-plane (NSHP) digital allpass filters is presented for designing two-dimensional (2-D) recursive doubly complementary (DC) filters. First, the theory of 2-D recursive digital allpass filters (DAFs) with NSHP support region for filter coefficients is developed. Then, the design problem is appropriately formulated to result in a simple optimization problem that minimizes the phase error in the least-squares (L_2) sense with a closed-form solution. The novelty of the presented 2-D NSHP DAFs structure is that it possesses the advantage of better performance in designing DC filters over existing 2-D structures based on quarter plane (QP) DAFs and tremendously saves the computational complexity for designing sampling rate converters. Finally, a design example is provided for conducting illustration and comparison.

Index Terms— Doubly complementary, allpass filter, nonsymmetric half-plane, stability.

1. INTRODUCTION

Recently, the extension of the 1-D allpass structure to the design of 2-D recursive digital filters has been widely considered in the literature [1], [2]. They presented the technique for the design of 2-D recursive circularly symmetric lowpass filters (CS-LPF) based on the parallel combination of allpass subfilters (PCAS) with quarter-plane (QP) support region. Two PCAS structures are cascaded to remove the unwanted passband for designing CS-LPF. However, the PCAS structure composed of QP DAFs is not as general as that composed of NSHP DAFs.

In this paper, 2-D DC filters, composed of parallel connected nonsymmetric half-plane (NSHP) allpass subfilters, are presented. It has been shown in [3] that 2-D recursive NSHP filters outperform 2-D recursive QP filters in terms of approximating more general frequency response specification. We first develop the results of 2-D recursive NSHP digital allpass filters (DAFs). The stability of the designed 2-D recursive NSHP DAFs is guaranteed by focusing the design problem on approximating the desired phase responses that satisfy some stability constraints on the phase of 2-D recursive filters. Several important properties of the developed 2-D recursive NSHP DAFs are investigated. Then, two 2-D recursive NSHP DAFs are parallel connected to develop the 2-D recursive DC filters (DCFs).

With prescribed phase characteristics, the design problem based on the phase error is formulated. After some algebraic manipulation, a linear objective function in the allpass coefficients is presented. A closed-form solution is derived to minimize the phase error in the least-squares sense efficiently and it can approximately satisfy the desired magnitude and approximately linear phase characteristics at the same time.

This work was supported by the National Science Council under Grant NSC95-2221-E002-187.

2. 2-D IIR NSHP ALLPASS FILTERS

2.1. Properties of 2-D Recursive NSHP Digital Allpass Filters

For a 2-D recursive DAF with order $M \times N$, its transfer function is given by

$$A(z_1, z_2) = z_1^{-M} z_2^{-N} \frac{D\left(z_1^{-1}, z_2^{-1}\right)}{D(z_1, z_2)} \tag{1}$$

The above equation implies that the allpass function $A(z_1, z_2)$ is completely determined by the denominator polynomial. Let the phase response of $A(z_1, z_2)$ and that of $D(z_1, z_2)$ be $\theta(\omega_1, \omega_2)$ and $\phi(\omega_1, \omega_2)$, respectively, and we can obtain

$$\phi(\omega_1, \omega_2) = -\left[M\omega_1 + N\omega_2 + \theta(\omega_1, \omega_2)\right]/2 \tag{2}$$

Here, we consider the 2-D IIR allpass filter with nonsymmetric half-plane (NSHP) support region. The denominator polynomial $D(z_1, z_2)$ of $A(z_1, z_2)$ is given by

$$D(z_1, z_2) = \sum_{m=0}^{M} d(m, 0)\, z_1^{-m} + \sum_{m=-M}^{M} \sum_{n=1}^{N} d(m, n)\, z_1^{-m} z_2^{-n} \tag{3}$$

Based on the frequency response of (1), we investigate the frequency characteristics of 2-D recursive DAFs and present several important properties as follows:

Property 1. The phase response $\phi(\omega_1,\omega_2)$ of $D\left(e^{j\omega_1},e^{j\omega_2}\right)$ is equal to zero at the frequency points (termed as the crucial points Ω_{CP} in [2]) in the (ω_1,ω_2) plane.

Property 2. The frequency response of the 2-D recursive NSHP DAF $A\left(e^{j\omega_1},e^{j\omega_2}\right)$ of (1) is restricted to 1 or -1 at the CPs.

It is revealed by *Property 2* that some unwanted passbands or stopbands may be induced. Therefore, the values of M and N for the orders of the 2-D recursive NSHP DAFs must be appropriately specified to avoid the possible unwanted passbands or stopbands.

2.2. Stability of 2-D Recursive Allpass Filters

A necessary and sufficient condition that guarantees the stability of a 1-D IIR DAF [4] is extended to 2-D cases [5], i.e.

$$\int_0^{\pi} -\frac{d}{d\omega_1}\arg\left[A\left(e^{j\omega_1},e^{j\omega_2}\right)\right]d\omega_1 = M\pi, \quad -\pi \le \omega_2 \le \pi \tag{4}$$

and

$$\int_0^{\pi} -\frac{d}{d\omega_2}\arg\left[A\left(e^{j\omega_1},e^{j\omega_2}\right)\right]d\omega_2 = N\pi, \quad -\pi \le \omega_1 \le \pi \tag{5}$$

By enforcing a desired phase response $\theta_d(\omega_1,\omega_2)$ to satisfy these constraints, we can neglect the stability problem and focus on the minimization problem only.

3. DOUBLY COMPLEMENTARY (DC) FILTER PAIR

We consider the 2-D DC filters as follows:

$$G\left(e^{j\omega_1},e^{j\omega_2}\right) = \frac{A_1\left(e^{j\omega_1},e^{j\omega_2}\right)+A_2\left(e^{j\omega_1},e^{j\omega_2}\right)}{2} \tag{6}$$

and

$$H\left(e^{j\omega_1},e^{j\omega_2}\right) = \frac{A_1\left(e^{j\omega_1},e^{j\omega_2}\right)-A_2\left(e^{j\omega_1},e^{j\omega_2}\right)}{2} \tag{7}$$

Based on (6) and (7), the 2-D DC filters are implemented in the form of the sum and difference of two allpass filters as illustrated in Fig. 1. In this paper, the 2-D recursive NSHP DAF proposed in previous section is applied to construct the 2-D DC filter pair.

Let the frequency response of the 2-D recursive NSHP DAF of (1) be expressed by

$$A_i\left(e^{j\omega_1},e^{j\omega_2}\right) = e^{j\theta_i(\omega_1,\omega_2)}, \quad i=1,2 \tag{8}$$

Substituting (8) into (6) yields

$$\begin{aligned} G\left(e^{j\omega_1},e^{j\omega_2}\right) &= \tfrac{1}{2}\left[A_1\left(e^{j\omega_1},e^{j\omega_2}\right)+A_2\left(e^{j\omega_1},e^{j\omega_2}\right)\right] \\ &= \tfrac{1}{2}\left[e^{j\theta_1(\omega_1,\omega_2)}+e^{j\theta_2(\omega_1,\omega_2)}\right] \\ &= \cos\left\{\left[\theta_1(\omega_1,\omega_2)-\theta_2(\omega_1,\omega_2)\right]/2\right\} e^{j[\theta_1(\omega_1,\omega_2)+\theta_2(\omega_1,\omega_2)]/2} \end{aligned} \tag{9}$$

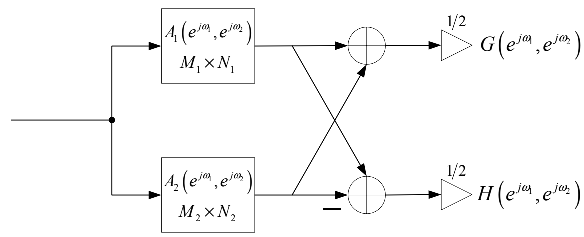

Fig. 1. Implementation of the 2-D doubly complementary filter pair.

Because of the equation given by (2), (9) is further rewritten as

$$\begin{aligned} G\left(e^{j\omega_1},e^{j\omega_2}\right) &= \cos[-\tfrac{M_1-M_2}{2}\omega_1-\tfrac{N_1-N_2}{2}\omega_2 \\ &-\phi_1(\omega_1,\omega_2)+\phi_2(\omega_1,\omega_2)] \times \exp\{j[-\tfrac{M_1+M_2}{2}\omega_1 \\ &-\tfrac{N_1+N_2}{2}\omega_2-\phi_1(\omega_1,\omega_2)-\phi_2(\omega_1,\omega_2)]\} \end{aligned} \tag{10}$$

The frequency response of the complementary filter $H(z_1,z_2)$ can be obtained in the similar manners.

It is noted that $G\left(e^{j\omega_1},e^{j\omega_2}\right)$ and $H\left(e^{j\omega_1},e^{j\omega_2}\right)$ are simultaneously determined by $\theta_m(\omega_1,\omega_2)$ and $\theta_p(\omega_1,\omega_2)$, where

$$\begin{aligned} \theta_m(\omega_1,\omega_2) = &-\frac{M_1-M_2}{2}\omega_1-\frac{N_1-N_2}{2}\omega_2 \\ &-\phi_1(\omega_1,\omega_2)+\phi_2(\omega_1,\omega_2) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \theta_p(\omega_1,\omega_2) = &-\frac{M_1+M_2}{2}\omega_1-\frac{N_1+N_2}{2}\omega_2 \\ &-\phi_1(\omega_1,\omega_2)-\phi_2(\omega_1,\omega_2) \end{aligned} \tag{12}$$

It is revealed in (11) and (12) that the phase responses of denominator polynomials $\phi_1(\omega_1,\omega_2)$ and $\phi_2(\omega_1,\omega_2)$ are also characterized by $\theta_m(\omega_1,\omega_2)$ and $\theta_p(\omega_1,\omega_2)$. Therefore, the design problem of DC filters $G\left(e^{j\omega_1},e^{j\omega_2}\right)$ and $H\left(e^{j\omega_1},e^{j\omega_2}\right)$ is equivalent to approximate the phase responses of denominators while the desired $\theta_{m,d}(\omega_1,\omega_2)$ and $\theta_{p,d}(\omega_1,\omega_2)$ are specified. As a result, the design problems can be formulated as follows:

$$\left\|arg\left\{D_i\left(e^{j\omega_1},e^{j\omega_2}\right)\right\}-\phi_{i,d}(\omega_1,\omega_2)\right\|^p, i=1,2 \tag{13}$$

where $\|x\|^p$ means the p-th norm of x.

4. PROPOSED DESIGN TECHNIQUE

In this section, we present a design technique for solving the resulting minimization problem of (13). With some algebraic manipulation, the objective function given by (13) can be rewritten as shown in the following:

$$Minimize \left\| \begin{aligned} &\sum\sum_{(m,n)\in \mathrm{R}-(0,0)} d_i(m,n)\sin[m\omega_1+n\omega_2 \\ &+\phi_{i,d}(\omega_1,\omega_2)]+\sin[\phi_{i,d}(\omega_1,\omega_2)] \end{aligned} \right\|^p \tag{14}$$

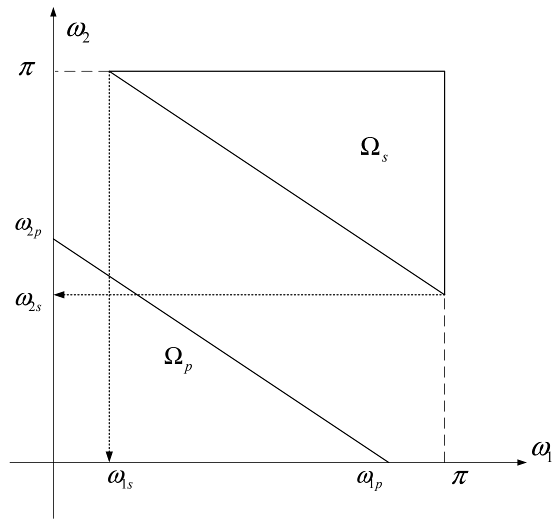

Fig. 2. Passband and stopband for the design of $G\left(e^{j\omega_1}, e^{j\omega_2}\right)$.

(14) can be further formulated in the following matrix form:

$$Minimize \left\|\mathbf{U}_i\mathbf{d}_i - \mathbf{s}_i\right\|^p \tag{15}$$

where $\mathbf{d}_i$ represents an unknown coefficient vector given by

$$\begin{aligned}\mathbf{d}_i = &[d_i(M,N), d_i(M-1,N), \ldots, d_i(-M,N)\\ &, d_i(M,N-1), d_i(M-1,N-1), \ldots, d_i(-M,N-1)\\ &, ..., d_i(M,0), d_i(M-1,0), \ldots, d_i(1,0)]^T\end{aligned} \tag{16}$$

where the superscript T is the matrix-transpose operation. Correspondingly, the rth row of the matrix $\mathbf{U}_i$ in (15) can be expressed as shown in (17) where the frequency pair $(\omega_{1r}, \omega_{2r})$ represents the rth uniformly sampled frequency grid point in the interesting frequency band. The rth element of column vector $\mathbf{s}_i$ is given by

$$\mathbf{s}_{i,r} = -\sin\left(\phi_{i,d}\left(\omega_{1r}, \omega_{2r}\right)\right) \tag{18}$$

If p=2 in (15), i.e., a least-squares solution of the proposed objective function is considered, we can obtain the closed-form solution which minimizes $\left\|\mathbf{U}_i\mathbf{d}_i - \mathbf{s}_i\right\|^2$ as follows

$$\mathbf{d}_i = \left(\mathbf{U}_i^T\mathbf{U}_i\right)^{-1}\mathbf{U}_i^T\mathbf{s}_i \tag{19}$$

5. APPLICATION IN SAMPLING STRUCTURE CONVERSION

It is shown that the diamond-shaped decimation/interpolation filter are good candidates for conversion processing between rectangular and hexagonal sampling structures because they allow a maximum resolution in the horizontal and vertical directions. The ideal diamond-shaped filter, which possesses quadrantal symmetry, is characterized in a quarter plane as illustrated in Fig. 2.

If we set $\omega_{1p}+\omega_{1s} = \omega_{2p}+\omega_{2s} = \pi$, the passband of the shifted version of $G\left(e^{j\omega_1}, e^{j\omega_2}\right)$, i.e., $G\left(e^{j(\omega_1-\pi)}, e^{j(\omega_2-\pi)}\right)$, is located in the passband of $H\left(e^{j\omega_1}, e^{j\omega_2}\right)$. Under this condition, some useful properties that tremendously save the design complexity are presented in the following. From (6), we have

$$\begin{aligned}&G(e^{j(\omega_1-\pi)}, e^{j(\omega_2-\pi)})\\ &=\tfrac{1}{2}\left[A_1(-e^{j\omega_1}, -e^{j\omega_2})+A_2(-e^{j\omega_1}, -e^{j\omega_2})\right]\end{aligned} \tag{20}$$

Combined with the complementary filter $H\left(e^{j\omega_1}, e^{j\omega_2}\right)$ given by (7), (20) can be further rewritten as

$$\begin{aligned}&G(e^{j(\omega_1-\pi)}, e^{j(\omega_2-\pi)})\\ &=\begin{cases} H(e^{j\omega_1}, e^{j\omega_2}), & \text{for } M_1+N_1\text{: even and } M_2+N_2\text{: odd}\\ -H(e^{j\omega_1}, e^{j\omega_2}), & \text{for } M_1+N_1\text{: odd and } M_2+N_2\text{: even}\end{cases}\end{aligned} \tag{21}$$

with

$$d_i(m,n) = 0, \text{for } m+n \text{ is odd.} \tag{22}$$

$G(e^{j\omega_1}, e^{j\omega_2})$ and its shifted version $G(e^{j(\omega_1-\pi)}, e^{j(\omega_2-\pi)})$ also possess the DC properties, i.e., the frequency response of $G\left(z_1, z_2\right)$ possesses the DC symmetry with respect to $(\omega_1, \omega_2) = (\pi/2, \pi/2)$ in the first quarter of the frequency plane. This means that if $G\left(e^{j\omega_1}, e^{j\omega_2}\right) = 0$ in the stopband, and then the frequency response of $G\left(z_1, z_2\right)$ becomes 1 in the passband. Therefore, we only need to approximate the passband or stopband response in the design of $G\left(z_1, z_2\right)$. Moreover, (22) indicates that about half of $A_i\left(z_1, z_2\right)$'s coefficients are zero. Both advantages lead to tremendous savings in computational burden.

6. DESIGN EXAMPLE

In this section, we consider the design example of the diamond-shaped filters illustrated in Fig. 2 with $\omega_{1p} = \pi/2$, $\omega_{2p} = \pi$, and $\omega_{1p} + \omega_{1s} = \omega_{2p} + \omega_{2s} = \pi$. Substituting the following desired responses

$$\theta_{m,d}\left(\omega_1, \omega_2\right) = 0 \tag{23}$$

and

$$\theta_{p,d}\left(\omega_1, \omega_2\right) = -\frac{M_1+M_2}{2}\omega_1 - \frac{N_1+N_2}{2}\omega_2 \tag{24}$$

into (11) and (12) yields the desired denominator phase responses for $(\omega_1, \omega_2) \in \Omega_p$. Then, the proposed design technique discussed in Section 4 is applied to find the coefficients of $A_1\left(z_1, z_2\right)$ and $A_2\left(z_1, z_2\right)$, respectively. The orders of two recursive NSHP DAFs are set to 4×2 and 3×2, respectively. By solving the closed-form solution given by (19),

$$\mathbf{U}_{i,r} = [\sin(M\omega_{1r}+N\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \sin((M-1)\omega_{1r}+N\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \ldots, \sin(-M\omega_{1r}+N\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \sin(M\omega_{1r}+(N-1)\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \sin((M-1)\omega_{1r}+(N-1)\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \ldots, \sin(-M\omega_{1r}+(N-1)\omega_{2r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \ldots, \sin(M\omega_{1r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \sin((M-1)\omega_{1r}+\phi_{i,d}(\omega_{1r},\omega_{2r})), \ldots, \sin(\omega_{1r}+\phi_{i,d}(\omega_{1r},\omega_{2r}))] \quad (17)$$

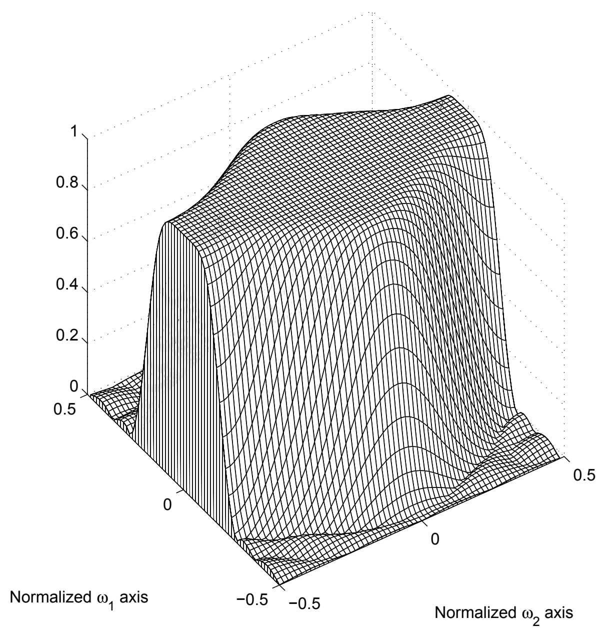

Fig. 3. Magnitude response of $G(z_1, z_2)$.

Fig. 3 shows the resultant magnitude response of the designed diamond-shaped filter. Table 1 lists the significant performance parameters defined as follows: passband magnitude mean-squared errors (PMSE), stopband magnitude mean-squared errors (SMSE), passband phase mean-squared errors (PPMSE_i), and stopband phase mean-squared errors (SPMSE_i). For comparison, the design example is also implemented by using the 2-D PCAS structure with QP support region presented in [1], [2]. The orders of two recursive QP DAFs are set to 7×2 and 6×2, respectively. As shown in Table 1, the design using 2-D DAFs with NSHP support shows more satisfactory performances than that with QP support and confirms the generality of NSHP filters.

7. CONCLUSION

This paper has presented a technique for the design of sampling rate converters based on 2-D DAFs composed of 2-D NSHP DAFs. The design problem is firs formulated as a linear optimization problem of an appropriate objective function for the phase response using the least-squares (L_2) criteria.

Table 1. Significant performance parameters.

	NSHP	QP
PMSE	7.79579×10^{-8}	5.13262×10^{-6}
SMSE	3.12723×10^{-4}	1.83256×10^{-3}
PPMSE_1	3.03131×10^{-5}	1.56331×10^{-4}
PPMSE_2	2.38336×10^{-4}	1.31271×10^{-3}
SPMSE_1	3.01082×10^{-5}	1.55875×10^{-4}
SPMSE_2	2.36940×10^{-4}	1.29814×10^{-3}
Stopband attenuation (dB)	21.47302	11.29495
No. of independent coefficients	19	21

The spectral factorization method [3] is utilized to verify the stability of the designed 2-D NSHP DAFs. Very small stability errors indicate that the stability of our design has been guaranteed.

8. REFERENCES

[1] M. S. Anderson and S. S. Lawson, "Direct design of approximately linear phase (ALP) 2-D digital filters," *Electronics Letters*, vol. 29, no. 9, pp. 804–805, Apr. 1993.

[2] H. Safiri, M. Ahmadi, and V. Ramachandran, "Design of stable, causal 2-D digital filters using real coefficient 2-D allpass building blocks," *IEEE Trans. Circuits and Systems II: Analog and Digital Signal Processing*, vol. 44, no. 5, pp. 409–412, May 1997.

[3] M. P. Ekstrom and J. W. Woods, "Two-dimensional spectral factorization with applications in recursive digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 24, no. 2, pp. 115–128, Apr. 1976.

[4] Z. Zing, "A new method for digital all-pass filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-35, pp. 1557–1564, Nov. 1987.

[5] S. C. Pei and J. J. Shyu, "Eigenfilter design of 1-D and 2-D IIR digital all-pass filters," *IEEE Trans. Signal Processing*, vol. 42, no. 4, pp. 966–968, Apr. 1994.