# COLORED RANDOM PROJECTIONS FOR COMPRESSED SENSING

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## ABSTRACT

The emerging theory of compressed sensing (CS) has led to the remarkable result that signals having a sparse representation in some known basis can be represented (with high probability) by a small sample set, taken from random projections of the signal. Notably, this sample set can be smaller than that required by the ubiquitous Nyquist sampling theorem. Much like the generalized Nyquist sampling theorem dictates that the sampling rate can be further reduced for the representation of bandlimited signals, this paper points to similar results for the sampling density in CS. In particular, it is shown that if additional spectral information of the underlying sparse signals is known, colored random projections can be used in CS in order to further reduce the number of measurements needed. Such *a priori* information is often available in signal processing applications and communications. Algorithms to design colored random projection vectors are developed. Further, an adaptive CS sampling method is developed for applications where non-uniform spectral characteristics of the signal are expected but are not known *a priori*.

***Index Terms*—** Compressed sensing, sparse signals, colored noise, random projections, signal reconstruction.

## 1. INTRODUCTION

Compressed sensing provides a new way to acquire and represent sparse signals that requires less sampling resources than traditional approaches [1, 2, 3]. Given a $T$ sparse signal $x \in \mathcal{R}^N$ on some basis $\Psi = [\psi_1, \psi_2, \ldots, \psi_N]$, such that $x$ can be approximated by a linear combination of $T$ vectors from $\Psi$ with $T \ll N$, the theory of compressed sensing shows that $x$ can be recovered from $M$ random projections with high probability when $M = CT \log N \ll N$, where $C \geq 1$ is the oversampling factor. The projections are given by $y = \Phi x$, where $\Phi$ is a $M \times N$ random measurement matrix with its rows incoherent with the columns of $\Psi$. Commonly used random measurement matrices for CS are random Gaussian matrices ($\Phi_{ij} \in \{\mathcal{N}(0, 1/N)\}$), Rademacher matrices ($\Phi_{ij} \in \{\pm 1/\sqrt{N}\}$) and partial Fourier matrices. In [2], it is shown that a matrix satisfying the incoherent condition is so ubiquitous that "nearly all matrices are CS matrices".

Signal reconstruction is achieved by solving a $l_1$ norm minimization problem: $\min \|\theta\|_1$ subject to $\Phi\Psi\theta = y$. $V = \Phi\Psi$ is called the holographic basis. Minimizing the $l_1$ norm yields solutions that are zero except at a small number of isolated values and can be solved by efficient optimization algorithms which include Basis Pursuit, Matching Pursuit (MP), and Orthogonal Matching Pursuit (OMP) [3, 4]. MP and OMP are greedy-based algorithms that iteratively reconstruct the sparse signal.

CS relies on the fundamental assumption that the underlying signals are sparse on some known basis $\Psi$ and that the measurements are i.i.d. random projections [2]. The random projections and the subsequent signal reconstruction do not utilize any characteristics of the signal other than the sparsity. In many applications, however, additional *a priori* information on the underlying signal characteristics is available, in addition to their sparsity. Most images have higher energy in low frequency bands; signals in narrowband communication have well defined spectral shapes, etc.

This paper shows that if the spectral characteristics of the underlying signals are not expected to be uniform, we can recover the signal with much less measurements than expected by conventional CS, much like the generalized Nyquist sampling theorem dictates that the sampling rate can be further reduced for the representation of a bandpass signal [5]. Such result is in accordance with similar conclusions drawn from a different context [6, 7]. To this end, this paper extends the projection mechanisms in CS. Rather than using i.i.d. spectrally white noise projections, colored noise with well defined spectral shapes are used in the projections. Exact knowledge of the signal spectral structure is not required. Furthermore, an adaptive CS sampling method is developed for applications where non-uniform spectral characteristics of the underlying signal are expected but are not known *a priori*.

## 2. CS WITH COLORED RANDOM PROJECTIONS

Consider a discrete time signal $x = [x_0, x_1, \cdots, x_{N-1}]^T \in \mathcal{R}^N$, $x$ can be expressed in the Fourier domain $\mathcal{F}$ as:

$$x(n) = \sum_{i=0}^{N-1} \hat{x}(i) e^{j2\pi ni/N}, \tag{1}$$

where $\hat{x}(i) \in \mathcal{C}^N$. Let the measurement matrix be $\Phi = [\Phi_0^T; \Phi_1^T; \cdots; \Phi_{M-1}^T]$ with $\Phi_i = [\Phi_{i0}, \Phi_{i1}, \cdots, \Phi_{iN-1}]^T$ for $i = 0, 1, \cdots, M-1$. Without loss of generality, assume $\Phi_{ij} \in \Phi$ to be i.i.d. normal dis-

[†]Prepared through collaborative participation in the Communications and Networks Consortium sponsored by the U. S. Army Research Laboratory under the Collaborative Technology Alliance Program, Cooperative Agreement DAAD19-1-2-0011. The U.S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon. [‡]This work was also supported in part by the Fondo Nacional de Ciencia, Tecnología e Investigación - FONACIT under grant Nro. 2005000342.

tributied samples. From [8], if $x$ is sparse in a known basis $\Psi$, then $x$ can be reconstructed from the set of random projections $y = \Phi x$ when the number of projections $M \geq CT \log N$ with constant $C \geq 1$.

Let $W$ be the Fourier transform matrix with $W_{ij} = e^{-j2\pi ij/N}$. Given the Fourier transform of $\Phi$, $\hat{\Phi} = \Phi W^T$, $y$ can be expressed as:

$$y = \Phi x = \hat{\Phi}^* (W^{-1})^H W^{-1} \hat{x} = \frac{1}{N} \hat{\Phi}^* \hat{x}. \tag{2}$$

Now suppose there exists a discrete time signal $x_B \in \mathcal{R}^N$ that is both $T$-sparse in $\Psi$ and bandpass in $\mathcal{F}$, that is:

$$x_B(n) = \sum_{i=1}^{2B} \hat{x}_{k_i} e^{j2\pi n k_i / N}, \tag{3}$$

where $\hat{x}_{k_i} \in \mathbf{C}^N$, $k_i \in \Omega = [k_1, k_1 + B - 1] \cup [N - k_1, N - k_1 + B - 1]$ with $|\Omega| = 2B < N$ and $0 < k_1 < N/2$. The following theorem applies.

**Theorem 2.1.** *Given a signal $x_B$ that is sparse in $\Psi$ and band-pass in $\mathcal{F}$, as defined above, then $x_B$ can be reconstructed with high probability by CS using colored random projections such that the projection vectors and $x_B$ have the same pass-band and the number of measurements $M$ satisfies the condition*

$$M \geq CT \log 2B. \tag{4}$$

*Proof.* Let the measurement matrix $\Delta = [\Delta_0^T; \Delta_1^T; \cdots; \Delta_{M-1}^T]$ with $\Delta_i = [\Delta_{i0}, \Delta_{i1}, \cdots, \Delta_{i(N-1)}]^T$ for $i = 0, 1, \cdots, M-1$ be the ensemble of samples from a band-pass normal distribution. The Fourier transform of $\Delta_i$ is given by

$$\begin{aligned} \hat{\Delta}_i &= W^{-1} \Delta_i \\ &= [0, \ldots, 0, \hat{\Delta}_{i(k_1)}, \ldots, \hat{\Delta}_{i(k_1+B-1)}, 0, \cdots, 0, \\ &\quad \hat{\Delta}_{i(N-k_1)}, \cdots, \hat{\Delta}_{i(N-k_1+B-1)}, 0, \ldots, 0]^T. \end{aligned}$$

By expressing each row of $\Delta$ and $x_B$ in the Fourier domain and removing all the zero entries corresponding to the stop-band, $y$ can be represented as:

$$y = \Delta x_B = \frac{1}{N} \Gamma \hat{x}_B, \tag{5}$$

where $\Gamma = [\Gamma_1^T; \Gamma_2^T; \cdots; \Gamma_{M-1}^T]$ with $\Gamma_i = [\hat{\Delta}_{ik_1}^*, \cdots, \hat{\Delta}_{i(k_1+B-1)}^*, \hat{\Delta}_{i(N-k_1)}^*, \cdots, \hat{\Delta}_{i(N-k_1+B-1)}^*]^T$ and $\hat{x}_B = [\hat{x}_{k_1}, \cdots, \hat{x}_{k_1+B-1}, \hat{x}_{N-k_1}, \cdots, \hat{x}_{N-k_1+B-1}]^T$. $\Gamma_{ij} \in \Gamma$ has the same distribution as $\hat{\Phi}_{ij}^* \in \hat{\Phi}^*$. From (2) and (5), it is easy to see that only $M \geq CT \log 2B$ measurements are enough to reconstruct $\hat{x}_B$ and then $x_B$ accordingly. □

Theorem 2.1 shows that when the signal has a non-uniform spectrum, the number of necessary measurements can be reduced by using colored random projections. The measurement matrix $\Delta$ contains bandlimited colored random vectors. As an example, Fig. 1(a) shows a realization of a 1024-point colored random vector, (b) its colored spectrum and (c) the spectrum of an i.i.d random projection vector for comparison. The colored random projection vector has its energy concentrated in the frequency band $0.15 < \omega < 0.35$.

Colored random projections can be thought of as bandpass type signal filtering, thus they extract the important frequency contents of the signal and fewer measurements would suffice to capture the salient information of the signal of interest. Given $\Phi$, any CS reconstruction algorithm can be used to reconstruct $x_B$ from the set of colored random projections.

Note that the set of colored random projections leads to the holographic dictionary $V$ equal to $\hat{\Delta} = (\Delta W')^*$, where the nonzero entries only exist on columns $q \in \Omega$. This effectively reduces the search space needed to find the sparse representation of the signal, which in turn reduces the number of necessary measurements. Compared with conventional CS, the structure of $V$ from colored random projections leads to faster convergence rate when using iterative reconstruction algorithms.

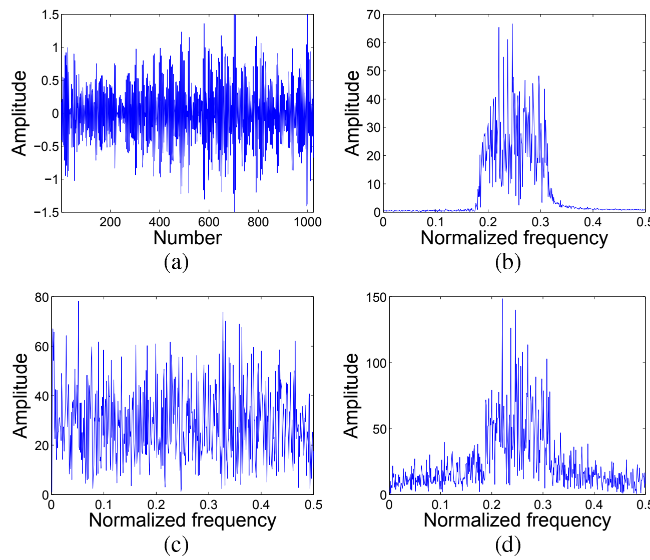


**Fig. 1**. (a) Colored random projection vector; (b) Spectrum of the colored vector; (c) Spectrum of an i.i.d random vector; (d) Spectrum of the colored random vector after binarization.

## 3. GENERATION OF COLORED RANDOM VECTORS

If the spectral profile of the signal to be reconstructed is known, colored random projection vectors can be generated via the structure shown in Fig. 2, where the bandpass filter is tailored to match the spectral profile of the signal. The filter $h(n : f_c, B_w)$ is a function of the signal bandwidth $B_w$ and central frequency $f_c$. The projection vectors can be taken from non-overlapping segments of the filter output $z(n)$. Note that to construct the filter $h(n : f_c, B_w)$, only the upper bound and lower bound of the signal frequency band are needed. However, as it will be shown later, the generation of a desired colored projection vector does not require exact knowledge of the signal's spectrum.

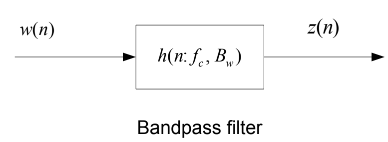


**Fig. 2**. Generation of colored random projection vectors.

Real-world signals are not strictly bandlimited. It is thus not necessary to design colored random projection vectors that have strict bandlimited characteristics, nor is it desirable. It is more favorable to design them having energy throughout all frequencies but having higher energy concentration on a limited band. Projection vectors designed in this fashion will improve the robustness of CS reconstruction as the random vectors contain all frequency components.

A simple and effective method to generate colored random projection vectors having such desired spectrum structure is to quantize the output $z(n)$ to a limited number of levels. Quantization will preserve the spectral structure of the vector. Figure 1(d) shows the spectrum of the vector in Fig. 1(a), after it is quantized to 2 levels.

Quantization is also desirable for simplicity in the hardware implementation. The random projection vectors with limited levels can simplify the computation of the projections. The projection vectors can be further normalized. In the following discussion, we will use normalized binary colored random projection vectors.

From the above discussion, it is clear that exact knowledge of the location of the signal pass-band is not necessary. The filter $h(n : f_c, B_w)$ can be designed in such a way that it matches the signal spectrum approximately. With less accurate information of the signal spectrum, colored random projection still reduces the necessary number of measurements, as is demonstrated in Sec. 5.

## 4. ADAPTIVE COLORED RANDOM PROJECTION

If the spectral characteristics of the signal are not known *a priori*, an adaptive procedure to select the colored random projection vectors can be developed. Let $s \in \mathcal{R}^N$ be a discrete time signal with unknown non-uniform spectrum. Let $\mathcal{M}$ be a binary $N \times N$ colored random measurement matrix such that each row $\mathcal{M}_i \in \mathcal{M}$ is not only random in the spatial domain, but also has different spectral energy concentration. $\mathcal{M}_i$ is a normalized vector having its energy concentrated on a frequency band with central frequency $f_i$ and bandwidth $B_w$. As the row index $i$ increases, the central frequency $f_i$ increases linearly from 0 to 0.5 (normalized frequency). $\mathcal{M}_i$ is then used as colored random projection vector. The objective in the design of $\mathcal{M}$ is to provide a set of projection vectors that is universal for signals with non-uniform spectra. To this end, an empirical value for the filter bandwidth $B_w$ is chosen to be $B_w = 0.1$ which achieves measurement efficiency without loss of generality in most situations. The value of $B_w$ can be tuned according to the application at hand. As an illustrative example, Fig. 3(a) shows a sample of $\mathcal{M}$ with dimension $256 \times 256$. Fig. 3(b) shows the ensemble average of the spectra of four colored random vectors taken from the $3th$, $70th$, $140th$, and $200th$ rows of the matrix for 100 realizations. These row vectors exhibit their energy concentration at increasing frequency locations.

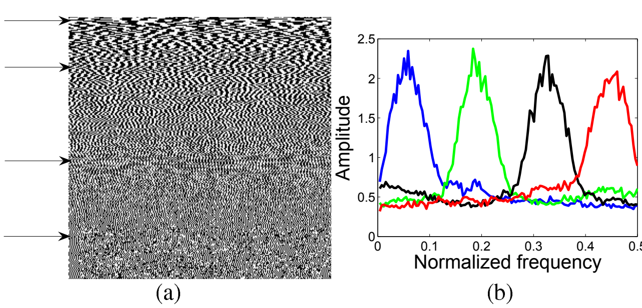


**Fig. 3**. (a) Colored random matrix with rows having increasing frequency energy concentration; (b) Spectrum ensemble average of four colored random vectors pointed in (a).

To measure a signal $s$, a set of random vectors should be chosen from $\mathcal{M}$ such that these vectors have a spectral profile that is similar to that of the signal of interest. An adaptive procedure is developed here to select these projection vectors iteratively when the spectral characteristics of the signal of interest is unknown.

Let $\mathcal{G} = \{G_1, G_2, \cdots, G_L\}$ be a set of $L$ row vectors extracted from $\mathcal{M}$ with the row index equally spaced and monotonically increasing. Given $B_w$, $L$ should satisfy the condition that $B_w L \approx 1$ in order to guarantee that the frequency components of the vectors $G_l \in \mathcal{G}$ span the entire frequency space. The projection of the signal $s$ on the set of vectors in $\mathcal{G}$ is then evaluated. The larger the absolute value of the projection, the stronger the correlation between $s$ and $G_l$ $(1 \leq l \leq L)$. Let the index of the vector in $\mathcal{G}$ which is most correlated with $s$ be $l_{max} = \arg_l \max |\langle G_l, s\rangle|$. The signal $s$ is expected to have a similar spectral profile as that of the row vector $G_{l_{max}}$ in $\mathcal{G}$.

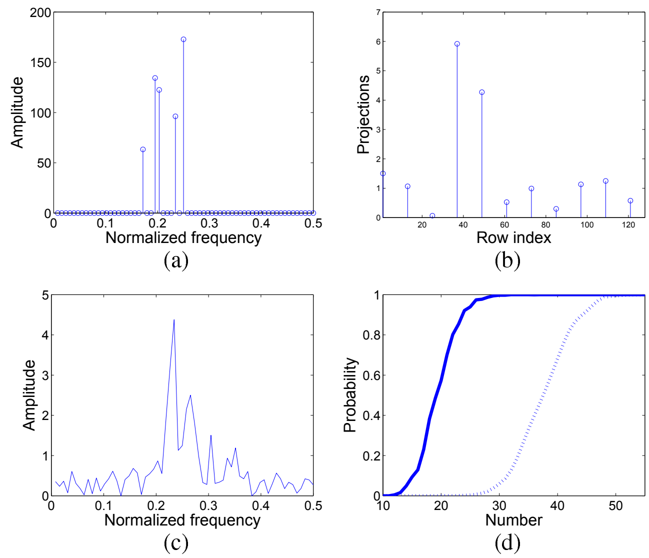


**Fig. 4**. (a) Signal spectrum; (b) magnitude of projections $\langle G_l, s\rangle$; (c) spectrum of $G_{l_{max}}$; (d) probability of successful reconstruction using: (-) colored random CS; $(\cdots)$ i.i.d. random CS.

As an example, Fig. 4(a) shows the spectrum of a 128-point sparse signal. Eleven row vectors are extracted from $\mathcal{M}$. Figure 4(b) shows the absolute values of the projections. The 37th row vector in $\mathcal{M}$, with its spectrum shown in Fig. 4(c), is identified as the most correlated to the signal in this iteration.

Next, $G_{l_{max}}$ is excluded from $\mathcal{G}$ and two more random vectors from $\mathcal{M}$ are added to $\mathcal{G}$ between $G_{l_{max}-1}$ and $G_{l_{max}+1}$. Let $ind(G_l)$ be the row index of vector $G_l$ in $\mathcal{M}$. The new vectors have their row index in $\mathcal{M}$ as $i = \lfloor (ind(G_{l_{max}-1}) + ind(G_{l_{max}}))/2 \rfloor$ and $i = \lfloor (ind(G_{l_{max}}) + ind(G_{l_{max}+1}))/2 \rfloor$, respectively.

With the updated $\mathcal{G}$, the search for the vector that is most correlated to $s$ is performed again. When no new vectors can be selected around $G_{l_{max}}$, new vectors are chosen such that the second largest absolute value of the projection is found, and so on. This procedure is repeated until $M$ colored random projection vectors are obtained. Note that in each iteration, only two new projections need to be performed and the selection of the colored random vectors adapts to the signal under measurement. The algorithm to select the colored random vectors is computationally efficient and robust to most signals with any non-uniform spectral shape. For band-pass signals, the number of measurements $M$ satisfies the condition $CT \log 2B \leq M \leq CT \log N \ll N$ and only a small number of row vectors from $\mathcal{M}$ need to be chosen when the signal of interest is sparse.

## 5. SIMULATION RESULTS

Simulation results are presented illustrating the effectiveness of colored random projections in CS. All simulations use the Basis Pursuit algorithm for signal reconstruction. Simulation 1 verifies Theorem 2.1. The signal of interest is a 128-point, bandpass sparse signal $s \in \mathcal{R}^N$ with a normalized pass-band $0.1 < \omega < 0.2$ and $N = 128$. The signal is sparse in the frequency domain with sparsity $T = 10$. Five sets of colored Gaussian random vectors are generated from bandpass normal distributions. In the frequency domain, each set has a normalized bandwidth of $\beta = 0.1, 0.2, 0.3, 0.4, 0.5$, respectively ($\beta = 0.5$ corresponds to i.i.d. random noise). Furthermore, the pass-band of the random projection vectors includes the pass-band of the signal. Figure 5(a) shows the relationship between the number of measurements and the probability of successful signal reconstruction. From left to right, each curve corresponds to an increasing $\beta$. At each data point for simulation, 1000 trials are performed and the reconstruction probability is the fraction of the 1000 trials that results in success. A trial is successful when the error $\varepsilon = s - \bar{s}$ between the original signal $s$ and the reconstructed signal $\bar{s}$ has its norm $\|\varepsilon\|_2 \leq 0.01\|s\|_2$.

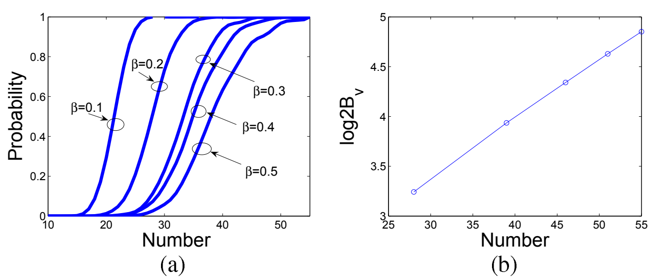


**Fig. 5**. (a) Reconstruction by colored random CS. (b) Number of measurements versus $\log 2B_v$.

Note that when $\beta$ increases, the necessary number of measurements for high probability of signal reconstruction also increases. Let $B_v$ be the bandwidth of the colored random projection vectors with $B_v = \beta N$. In Fig. 5(b), the $x$ axis shows the number of measurements for 99% of successful reconstruction; the $y$ axis shows $\log 2B_v$. The linear relationship between the number of measurements $M$ and $\log 2B_v$ can be easily observed. Furthermore, the constant $C$ in (4) is estimated to be 1.03 in this case.

Simulation 2 focuses on reconstructing the 128-point bandpass signal within a frequency band $0.15 < \omega < 0.25$ (see Fig. 4(a)). The signal is sparse in the frequency domain with $T = 10$. A $128 \times 128$ colored random matrix is generated and adaptive colored random projections are applied to measure the signal. Figure 4(d) shows the relationship between the number of measurements and the probability of successful signal reconstruction. For comparative purpose, the probability of successful reconstruction as a function of the number of i.i.d. Rademacher random projections is also shown in Fig. 4(d). As can be seen from Fig. 4(d), colored random projections greatly reduce the number of measurements needed for signal reconstruction.

In the third simulation, the objective is to show that colored random CS is effective when the signal is not sparse in the frequency domain, as long as the signal has energy concentration on some frequency band. The input is a 128-point piecewise constant signal 'Blocks', as shown in Fig. 6(a). It has a sparse representation in the Haar wavelet domain ($T \approx 20$) and also has its energy concentrated in the low frequency band. Adaptive colored random projections are used to measure the signal. The simulation result is shown in Fig. 6(b). On average, the number of measurements is reduced by more than 10% by using colored random projections for the same probability of successful signal reconstruction compared with i.i.d. random projections.

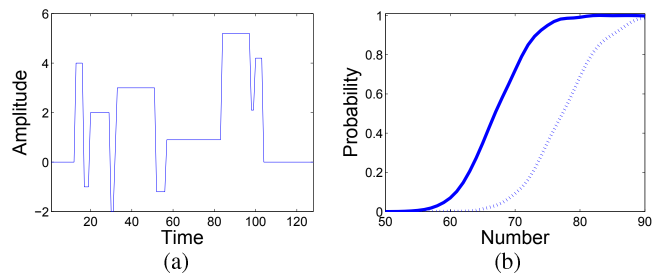


**Fig. 6**. (a) Signal 'Blocks'; (b) Probability of successful reconstruction by: (-) adaptive colored random CS; $(\cdots)$ i.i.d. random CS.

## 6. CONCLUSION

This paper shows that when a signal to be measured has a non-uniform spectral profile, colored random projections can reduce the number of CS measurements for successful signal reconstruction compared with conventional CS. The algorithm to generate colored random projection vectors is described. A computationally efficient adaptive CS sampling method is developed for signals with unknown non-uniform spectral profiles. These concepts can be easily extended to 2-D signal reconstruction problems.

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