COMPUTATION OF THE DUAL FRAME: FORWARD AND BACKWARD GREVILLE FORMULAS

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ABSTRACT

We study the computation of the dual frame for oversampled filter banks (OFBs) by exploiting *Greville*'s formula, which was derived in 1960 to compute the pseudo inverse of a matrix when a new row is appended. In this paper, we first develop the backward Greville formula to handle the case of row deletion. Based on Greville's formula, we then study the dual frame computation of the *Laplacian pyramid*. Through the backward Greville formula, we investigate OFBs for robust transmission over erasure channels. The necessary and sufficient conditions for OFBs robust to one erasure channel are derived. A post-filtering structure is also presented to implement the dual frame when the transform coefficients in one subband are completely lost.

Index Terms— Error resilience, Frames, Greville formula, Laplacian pyramid, Oversampled filter banks.

1. INTRODUCTION

Over the past few years, there have been increased interests in the study of frame expansions due to their extra design freedom and improved noise immunity (e.g., [1–4] and the references therein). From a signal-processing point of view, frames in $l^2(\mathbb{Z})$ correspond to perfect reconstruction (PR) oversampled filter banks (OFBs). Consider an N-channel OFB with the sampling factor of M, where $M < N$. Let the $N \times M$ polynomial matrix $\mathbf{E}(z)$ and the $M \times N$ polynomial matrix $\mathbf{R}(z)$ denote the analysis and synthesis polyphase matrices, respectively. For PR-OFBs, $\mathbf{E}(z)$ and $\mathbf{R}(z)$ should satisfy

$$\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}_M. \tag{1}$$

In other words, $\mathbf{R}(z)$ is the left inverse of $\mathbf{R}(z)$. Unlike conventional filter banks, given $\mathbf{E}(z)$, there are multiple $\mathbf{R}(z)$ which meet (1). Among them, the most significant one is its *para-pseudo inverse*, or the *dual frame*, which is given by [2]

$$\mathbf{E}^{\dagger}(z) = \left(\mathbf{E}^H(z)\mathbf{E}(z)\right)^{-1}\mathbf{E}^H(z), \tag{2}$$

where the superscripts H and † represent respecitvely, the Hermitian transpose and the dual frame. The dual frame correspond to synthesis filters with the minimum norm. Besides, in the ideal case when the quantization noise is uncorrelated and white with equal variance, the dual frame is the optimal solution to minimize the reconstruction errors [2, 4]. However, its computation in (2) requires the inversion of the polynomial matrix $\mathbf{E}^H(z)\mathbf{E}(z)$, which could be a costly task. Although some state-of-the-art software can provide numerical approximations, they cannot generate closed-form solutions.

Note that in the special case when $\mathbf{E}(z)$ is a zero-order matrix, i.e., when $\mathbf{E}(z) = \mathbf{E}$, the dual frame reduces to its pseudo inverse. An efficient way to calculate the pseudo inverse is through the recursive Greville formula [5], which updates its pseudo inverse when a matrix is augmented by a row (or column) vector. Based on the Greville formula, we derive the *closed-form* solution of the dual frame for the Laplacian pyramid [6]. We further develop the backward Greville formula to compute the pseudo inverse when a row (or column) of a matrix is deleted. This is mainly motivated by the fact that frames can be used as error-resilient tools to combat erasures. In the presence of coefficient loss, the signal can still be reconstructed using the dual frame of the remaining analysis bank [7]. The backward Greville formula facilitates the analysis of OFBs in the presence of erasures.

The rest of this paper is organized as follows. In Section 2, we present the Greville formula and derive the backward Greville formula. Applications of Greville formulas are demonstrated in Section 3 and Section 4, respectively. Specifically, we study the Laplacian pyramid [6] in Section 3, and examine OFBs for erasure channels in Section 4. Finally, we draw conclusions in Section 5.

2. FORWARD AND BACKWARD GREVILLE FORMULAS

2.1. The Greville Formula

Suppose that $\mathbf{E}_N(z)$ is an $N \times M$ analysis polyphase matrix of a PR OFB. Let us partition it into

$$\mathbf{E}_N(z) = \begin{bmatrix} \mathbf{E}_{N-1}(z) \\ \mathbf{e}_N(z) \end{bmatrix} \tag{3}$$

where $\mathbf{E}_{N-1}(z)$ is the $(N-1) \times M$ submatrix and $\mathbf{e}_N(z)$ is its last row. Assume further that we know $\mathbf{E}^{\dagger}_{N-1}(z)$. The Greville formula [5] computes $\mathbf{E}^{\dagger}_N(z)$ from $\mathbf{E}^{\dagger}_{N-1}(z)$ and $\mathbf{e}_N(z)$ as follows:

Set

$$\mathbf{d}(z) = \mathbf{e}_N(z)\mathbf{E}^{\dagger}_{N-1}(z)$$

and

$$\mathbf{c}(z) = \mathbf{e}_N(z) - \mathbf{d}(z)\mathbf{E}_{N-1}(z);$$

Case 1: If $\mathbf{c}(z) = 0$, set

$$\mathbf{r}_N(z) = \frac{\mathbf{d}(z)}{1 + \mathbf{d}_N(z)\mathbf{d}^H(z)}[\mathbf{E}^{\dagger}_{N-1}(z)]^H;$$

Case 2: If $\mathbf{c}(z) \neq 0$, set

$$\mathbf{r}_N(z) = \frac{\mathbf{c}(z)}{\mathbf{c}(z)\mathbf{c}^H(z)},$$

then

$$\mathbf{E}^{\dagger}_N(z) = \begin{bmatrix} \mathbf{E}^{\dagger}_{N-1}(z) & \mathbf{0} \end{bmatrix} + \mathbf{r}^H_N(z)\begin{bmatrix} -\mathbf{d}_N(z) & 1 \end{bmatrix} \tag{4}$$

Remarks: For Case 1, $\mathbf{e}_N(z)$ is in the range of $\mathbf{E}_{N-1}(z)$ and for case 2, it is not.

2.2. The backward Greville Formula

In this subsection, we aim to derive the reverse problem. That is, suppose we know $\mathbf{E}_N^\dagger(z)$, how to calculate $\mathbf{E}_{N-1}^\dagger(z)$? The answer is presented in Theorem 1.

Theorem 1. *(Backward Greville Formula) Suppose that the $N \times M$ matrix $\mathbf{E}(z)$ is as defined in (3). Partition its dual frame $\mathbf{E}_N^\dagger(z)$ into the form of*

$$\mathbf{E}_N^\dagger(z) = \begin{bmatrix}\mathbf{R}_{N-1}(z) & \mathbf{r}_N^H(z)\end{bmatrix}, \quad (5)$$

where $\mathbf{R}_{N-1}(z)$ is the $M \times (N-1)$ submatrix and $\mathbf{r}_N^H(z)$ the $M \times 1$ column vector. Then, the dual frame of $\mathbf{E}_{N-1}(z)$ is given by

$$\mathbf{E}_{N-1}^\dagger(z) = \mathbf{R}_{N-1}(z) + \mathbf{r}_N^H(z)\mathbf{d}_N(z),$$

where $\mathbf{d}_N(z)$ can be expressed as

$$\mathbf{d}_N(z) = \begin{cases} \dfrac{\mathbf{e}_N(z)\mathbf{R}_{N-1}(z)}{1-\mathbf{e}_N(z)\mathbf{r}_N^H(z)}, & \text{if } \mathbf{e}_N(z)\mathbf{r}_N^H(z) < 1 \quad (6) \\ -\dfrac{\mathbf{r}_N(z)\mathbf{R}_{N-1}(z)}{\mathbf{r}_N(z)\mathbf{r}_N^H(z)}, & \text{if } \mathbf{e}_N(z)\mathbf{r}_N^H(z) = 1 \quad (7) \end{cases}$$

Remarks

1. In the backward Greville formula, (6) and (7) correspond to Case 1 and Case 2 of the forward Greville formula, respectively.

2. In the above Theorem, we only consider the cases when $\mathbf{e}_N(z)\mathbf{r}_N^H(z) \leq 1$. Interested readers may ask whether it is possible to have $\mathbf{e}_N(z)\mathbf{r}_N^H(z) > 1$? The answer is no. Detailed derivations will be presented in the journal version.

3. The backward Greville formula was also investigated in [8]. But the derivations there are limited to the case when the row vectors in $\mathbf{E}_N(z)$ are independent. In other words, our derivations are more general.

3. DUAL FRAME FOR LAPLACIAN PYRAMIDS

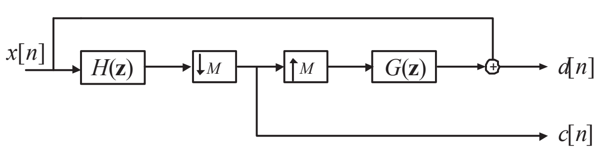

Fig. 1. Implementation of the LP frame.

To demonstrate the application of the Greville formula, this section considers the computation of the dual frame for the *laplacian pyramid* (LP) [6], which has been proved to be a useful tool for image processing and computer vision. Fig. 1 shows its implementation diagram, where $H(z)$ and $G(z)$ represent, respectively, the decimation and the interpolation low-pass filters. The output signal is made up of two components: the coarse signal $c[n]$ represents the low-frequency components of the original input, while the details (with band-pass and high-pass frequency components) are contained in $d[n]$.

As $x[n]$ can be always reconstructed from $c[n]$ and $d[n]$, the LP realizes a frame expansion [9]. From the FB point of view, the LP can be implemented through an $(M+1)$-channel PR-OFB with the sampling factor of M, whose polyphase matrix is

$$\mathbf{E}_{lp}(z) = \begin{bmatrix}\mathbf{I} - \mathbf{g}^H(z)\mathbf{h}(z) \\ \mathbf{h}(z)\end{bmatrix}, \quad (8)$$

where the $M \times 1$ vectors $\mathbf{h}(z)$ and $\mathbf{g}(z)$ represent the Type-I polyphase matrices [10] of the low-pass filters of $H(z)$ and $G(z)$, respectively. Although PR can be achieved for *any* pair of $H(z)$ and $G(z)$, a typical choice is to set $H(z)$ and $G(z)$ as biorthogonal filters. Under this restriction, the corresponding polyphase matrices satisfy

$$\mathbf{h}(z)\mathbf{g}^H(z) = 1. \quad (9)$$

In the special case when $H(z)$ is an orthogonal filter and $G(z) = H(z)$ [9], (9) is reduced to $\mathbf{h}(z)\mathbf{h}^H(z) = 1$. Accordingly, $\mathbf{E}_{lp}(z)$ is a paraunitary matrix [10] and its dual frame is its Hermitian transpose $\mathbf{E}_{lp}^\dagger(z) = \mathbf{E}_{lp}^H(z)$

Our purpose here is to derive a *closed-form* solution for $\mathbf{E}_{lp}^\dagger(z)$ when (9) holds. First, note that when $\mathbf{h}(z)$ and $\mathbf{g}^H(z)$ satisfy (9), $\mathbf{D}(z) = \mathbf{I} - \mathbf{g}^H(z)\mathbf{h}(z)$ is a rank-deficient matrix as one of its eigen-value is zero [9]. Using the Woodbury formula, we can show that the para-pseudo inverse of $\mathbf{D}(z)$ is [11]

$$\begin{aligned}\mathbf{D}^\dagger(z) = \mathbf{I}_M &- \frac{\mathbf{g}^H(z)\mathbf{g}(z)}{\mathbf{g}(z)\mathbf{g}^H(z)} - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)} \\ &+ \frac{\mathbf{g}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)\mathbf{g}(z)\mathbf{g}^H(z)}.\end{aligned} \quad (10)$$

Next, note that $\mathbf{E}_{lp}(z)$ satisfies the PR property [9]. Hence, it is of full rank on the unit circle, i.e., $rank(\mathbf{E}_{lp}(e^{j\omega}) = M$. As $\mathbf{D}(z)$ is a rank deficient matrix, the row vector $\mathbf{h}(z)$ is *not* in the range of $\mathbf{D}(z)$. Accordingly, we can use Case 2 of the Greville formula to get $\mathbf{E}_{lp}^\dagger(z)$, as presented in the following theorem:

Theorem 2. *For the LP frame shown in Fig. 1, let its $(M+1) \times M$ polyphase matrix $\mathbf{E}_{lp}(z)$ be given as in (8). Suppose that $H(z)$ and $G(z)$ are biorthogonal filters with their polyphase matrices satisfying (9). Then, its dual frame can be expressed as*

$$\mathbf{E}_{lp}^\dagger(z) = \begin{bmatrix}\mathbf{I}_M - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)} & \mathbf{g}^H(z)\end{bmatrix} \quad (11)$$

Proof. The derivation of (11) is quite lengthy. To prove that (11) is indeed the dual frame, we will show that it satisfies the Moorse-Penrose equations [11]

$$\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z)\mathbf{E}_{lp}(z) = \mathbf{E}_{lp}(z); \quad (12)$$

$$\mathbf{E}_{lp}^\dagger(z)\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z) = \mathbf{E}_{lp}^\dagger(z); \quad (13)$$

$$\left(\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z)\right)^H = \mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z); \quad (14)$$

$$\left(\mathbf{E}_{lp}^\dagger(z)\mathbf{E}_{lp}(z)\right)^H = \mathbf{E}_{lp}^\dagger(z)\mathbf{E}_{lp}(z); \quad (15)$$

First, it is easy to verify that $\mathbf{E}_{lp}^\dagger(z)\mathbf{E}_{lp}(z) = \mathbf{I}_M$ when (9) holds. Hence, Eqs. (12), (13) and (15) are met. Also, through simple math manipulations, we can get $\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z) = \begin{bmatrix}\mathbf{I}_M - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)} & 0 \\ 0 & 1\end{bmatrix}$, which indicates that $\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^\dagger(z)$ is a Hermitian matrix. Therefore, (14) also holds. □

Remarks:

1. Note that in general, $\mathbf{E}_{lp}^{\dagger}(z)$ given by (11) corresponds to an IIR filterbank. To get an FIR implementation, we can approximate $\mathbf{h}(z)\mathbf{h}^H(z)$ by a positive constant k [1]. That is, at the reconstruction side, we can replace $\mathbf{E}_{lp}^{\dagger}(z)$ by

$$\mathbf{R}(z) = \begin{bmatrix}\mathbf{I}_M - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{k} & \mathbf{g}^H(z)\end{bmatrix} \tag{16}$$

It is interesting to note that with such approximation, the PR property in (1) is retained.

2. When $H(z)$ is an orthogonal filter and $G(z) = H(z)$, we have $\mathbf{h}(z)\mathbf{h}^H(z) = 1$. Under these constraints, it is easy to show that (11) boils down to $\mathbf{E}_{lp}^{\dagger}(z) = \begin{bmatrix}\mathbf{I}_M - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)} & \mathbf{h}^H(z)\end{bmatrix}$, which is exactly $\mathbf{E}_{lp}^H(z)$ as expected. In this special case, the synthesis bank is FIR.

4. OFBS FOR ERASURE CHANNELS

Due to the redundancy introduced in frame expansion, OFBs can be used as joint source-channel codes to provide robustness to erasures [7]. In this Section, we examine the resilience of OFBs to one erasure channel, based on Case 1 of the backward Greville formula. To this aim, we introduce the following definition:

Definition 1. *Let $\mathbf{E}_N(z)$ denote the $N \times M$ analysis polyphase matrix of a PR-OFB. Denote by $\mathbf{E}_{\{i\}}(z)$ the polyphase matrix obtained by deleting the i-th row of $\mathbf{E}_N(z)$. $\mathbf{E}_N(z)$ is said to be robust to one erasure if $\mathbf{E}_{\{i\}}(e^{j\omega})$ is of full rank on the unit circle [7].*

4.1. OFBs robust to one erasure

This subsection studies the necessary and sufficient conditions for OFBs to be robust to one erasure channel. Theorem 3 discusses the case of general PR-OFBs.

Theorem 3. *Suppose that $\mathbf{E}_N(z)$ is the polyphase matrix of an N-channel PR-OFB with its dual frame given by $\mathbf{E}_N^{\dagger}(z)$. Let $\mathbf{e}_i(z)$ and $\mathbf{r}_i^H(z)$ (for $i = 1, \cdots, N$) denote the i-th row vector of $\mathbf{E}(z)$ and the i-th column vector of $\mathbf{E}_N^{\dagger}(z)$, respectively. Then, $\mathbf{E}_N(z)$ is robust to one erasure if and only if*

$$\mathbf{e}_i(e^{j\omega})\mathbf{r}_i^H(e^{j\omega}) < 1 \tag{17}$$

for $i = 1, \cdots, N$ and for all $\omega \in [0, 2\pi)$.

Proof. Note that when there is only one erasure channel, through row permutation, we can assume that erasure occurs in the N-th channel. Let $\mathbf{E}_N(z)$ be written as in (3), hence $\mathbf{E}_{\{N\}}(z) = \mathbf{E}_{N-1}(z)$. Without loss of generality, the proof is equivalent to showing that $\mathbf{E}_{N-1}(e^{j\omega})$ defined in (3) has full rank if and only if

$$\mathbf{e}_N(e^{j\omega})\mathbf{r}_N^H(e^{j\omega}) < 1. \tag{18}$$

The "only if" part is obvious as when $\mathbf{E}_{N-1}(e^{j\omega})$ is of full rank, $\mathbf{e}_N(e^{j\omega})$ is in the range of $\mathbf{E}_{N-1}(e^{j\omega})$. Hence, by the backward Greville formula, (18) holds. For the "if" part, according to the backward Greville algorithm, when (18) holds, $\mathbf{e}_N(e^{j\omega})$ is in the range of $\mathbf{E}_{N-1}(e^{j\omega})$. Hence, there exists a vector $\mathbf{d}(e^{j\omega})$ so that

$$\mathbf{e}_N(e^{j\omega}) = \mathbf{d}(e^{j\omega})\mathbf{E}_{N-1}(e^{j\omega}).$$

and accordingly, we can re-write $\mathbf{E}_N(e^{j\omega})$ into

$$\mathbf{E}_N(e^{j\omega}) = \begin{bmatrix}\mathbf{I}_M \\ \mathbf{d}(e^{j\omega})\end{bmatrix}\mathbf{E}_{N-1}(e^{j\omega})$$

Recall that for any two matrices $\mathbf{X}$ and $\mathbf{Y}$, the inequality $rank(\mathbf{Y}) \geq rank(\mathbf{XY})$ holds. Hence, we have

$$rank(\mathbf{E}_{N-1}(e^{j\omega})) \geq rank(\mathbf{E}_N(e^{j\omega})) = M,$$

where we have used the fact that $\mathbf{E}_N(z)$ satisfies the PR property. Also, the size of $\mathbf{E}_{N-1}(e^{j\omega})$ is $(N-1) \times M$, which indicates that

$$rank(\mathbf{E}_{N-1}(e^{j\omega})) \leq M.$$

Combining the above two inequalities, we know that $\mathbf{E}_{N-1}(e^{j\omega})$ is of full rank on the unit circle. □

Example 1. *For the LP frame depicted in Fig. 1, we have shown in the proof of Theorem 2 that if (9) is satisfied, $\mathbf{E}_{lp}(z)\mathbf{E}_{lp}^{\dagger}(z) = \begin{bmatrix}\mathbf{I}_M - \frac{\mathbf{h}^H(z)\mathbf{h}(z)}{\mathbf{h}(z)\mathbf{h}^H(z)} & 0 \\ 0 & 1\end{bmatrix}$. This indicates that $\mathbf{e}_N(z)\mathbf{r}_N^H(z) = 1$. By Theorem 3, the LP frame is* not *robust to one erasure channel by using biorthogonal filters.*

For the special case when $\mathbf{E}_N(z)$ implements a tight frame, i.e., when $\mathbf{E}_N^H(z)\mathbf{E}_N(z) = A\mathbf{I}_M$ with $A > 0$, its dual frame can be simply written into $\mathbf{E}_N^{\dagger}(z) = \frac{1}{A}\mathbf{E}^H(z)$. A consequence of Theorem 3 is as follows:

Corollary 1. *If $\mathbf{E}_N(z)$ corresponds to a tight frame, i.e., when it satisfies $\mathbf{E}_N^H(z)\mathbf{E}_N(z) = A\mathbf{I}_M$ with $A > 0$, it is robust to one erasure channel if and only if its i-th row vector $\mathbf{e}_i(z)$ satisfies*

$$\mathbf{e}_i(e^{j\omega})\mathbf{e}_i^H(e^{j\omega}) < A \tag{19}$$

for all $i = 1, \cdots, N$ and for all $\omega \in [0, 2\pi)$.

Remarks:

Note that [7] also investigated the case where there is only one erasure. But the discussions there are focused on the uniform tight frame (UTF), a special class of tight frames with equal norm for each analysis filter. Our derivations are for general frames implemented via PR and PU OFBs. It can be shown that when $A = N/M$, Corollary 1 boils down to Theorem 5 in [7]. Although theoretically, UTFs provide optimal performance, their design is very difficult. On the other hand, several works have reported simple design methods and fast implementations for cosine modulated OFBs [12] and linear-phase OFBs, which are attractive in practical applications like othorgonal frequency-division multiplexing (OFDM) and image coding. The theory developed here can be used for those FBs which do not generate UTFs.

4.2. Implementation structure

Suppose that $\mathbf{E}_N(z)$ is robust to one erasure channel. In this section, we consider the following problem: when the erasure occurs in one channel, how to implement the dual frame of the remaining analysis bank? Again, without loss of generality, let us assume that the subband coefficients in the N-th channel are completely lost. As (18) holds, by using the backward Greville algorithm, we have

$$\mathbf{E}_{N-1}^{\dagger}(z) = \mathbf{R}_{N-1}(z) + \frac{\mathbf{r}_N^H(z)\mathbf{e}_N(z)}{1 - \mathbf{e}_N(z)\mathbf{r}_N^H(z)}\mathbf{R}_{N-1}(z), \tag{20}$$

[1] k can be determined by calculate the average value of $|\mathbf{h}(e^{j\omega})|^2$ for $\omega \in [0, 2\pi)$

where all the notations here follow those of Section 2.2. By (5), $\mathbf{R}_{N-1}(z)$ can be expressed as $\mathbf{R}_{N-1}(z) = \mathbf{E}_N^\dagger(z)\begin{bmatrix}\mathbf{I}_{N-1}\\ \mathbf{0}\end{bmatrix}$. Substituting it into (20), we arrive at the following expression of $\mathbf{E}_{N-1}^\dagger(z)$

$$\mathbf{E}_{N-1}^\dagger(z) = \left(\mathbf{I}_M + \frac{\mathbf{r}_N^H(z)\mathbf{e}_N(z)}{1-\mathbf{e}_N(z)\mathbf{r}_N^H(z)}\right)\mathbf{E}_N^\dagger(z)\begin{bmatrix}\mathbf{I}_{N-1}\\ \mathbf{0}\end{bmatrix}. \quad (21)$$

Example 2. *To have a quick check of (21), let us consider the* 3×2 Mercedes-Benz *(MB) frame [7] with the analysis polyphase matrix* $\mathbf{E}_3(z) = \mathbf{E}_3 = \begin{bmatrix}0 & 1\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$. *As* $\mathbf{E}_3$ *is a uniform tight frame in* $\mathbb{R}^2$ *[7], it is robust to one erasure channel with the dual frame given by* $\mathbf{E}_3^\dagger(z) = \frac{2}{3}\mathbf{E}_3^T$. *Suppose that the erasure occurs in the last channel and we aim to reconstruct the signal using the dual frame of* $\mathbf{E}_2(z) = \begin{bmatrix}0 & 1\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$. *By definitions of (3) and (5), we know that* $\mathbf{e}_3(z) = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$ *and* $\mathbf{r}_3^H(z) = \frac{2}{3}\mathbf{e}_3^H(z)$. *Substituting* $\mathbf{E}_3^\dagger(z)$, $\mathbf{e}_3(z)$ *and* $\mathbf{r}_3^H(z)$ *into (21) yields* $\mathbf{E}_2^\dagger(z) = \begin{bmatrix}-\frac{\sqrt{3}}{3} & -\frac{2\sqrt{3}}{3}\\ 1 & 0\end{bmatrix}$, *which is exactly the inverse of* $\mathbf{E}_2^\dagger(z)$.

Fig. 2 shows the corresponding implementation structure of (21). The process can be described as follows. First, in the frequency domain, all the subband coefficients in the erasure channel (i.e., the N-th channel in the diagram) are set to zeros. Then, the original dual frame $\mathbf{E}_N^\dagger(z)$ is applied, followed by a *time-domain* post-filter $\mathbf{P}(z) = \mathbf{I}_M + \frac{\mathbf{r}_N^H(z)\mathbf{e}_N(z)}{1-\mathbf{e}_N(z)\mathbf{r}_N^H(z)}$ to yield the reconstructed signal. In particular, the filter $\frac{\mathbf{r}_N^H(z)\mathbf{e}_N(z)}{1-\mathbf{e}_N(z)\mathbf{r}_N^H(z)}$ is used to compensate for the erasure in the N-th channel.

From the above description, one can see that our proposed structure in Fig. 2 is based on *time-domain post-processing* through $\mathbf{P}(z)$. An alternative way to implement $\mathbf{E}_{N-1}^\dagger(z)$ is through the method proposed in [13], where the lost subband coefficients are first predicted in *the frequency domain* before being reconstructed by $\mathbf{E}_N^\dagger(z)$. In other words, the structure in [13] is based on *frequency domain pre-filtering*. It can be easily shown that the implementation complexities of these two structures are about the same. One attractive property of our proposed structure is that it can be combined with time-domain over-sampled lapped transform [14], which adds time-domain oversampled pre-/post-filters outside the DCT and the IDCT. In this way, exiting DCT-based standards (like the JPEG) can be intact. One of our on-going works is to investigate the theoretical result presented here for robust transmission of images/videos that are compressed by the DCT-based codecs.

5. CONCLUSIONS

In this paper, we have studied the computation of the dual frame via Greville's formulas. We derived the backward Greville formula, which iteratively computes the pseudo inverse of a matrix when a row is deleted. The applications of the formulas were then demonstrated. In particular, the Greville formula leads to a closed-form solution of the dual frame for Laplacian pyramids. Based on the backward Greville formula, we derived the necessary and sufficient conditions for PR-OFBs robust to one erasure channel. We also proposed a post-filtering structure to implement the dual frame in the presence one erasure channel. Detailed proofs of all the theorems along with more examples and their practical applications will be reported in the journal version.

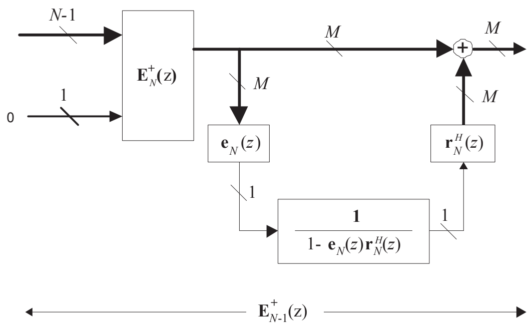

Fig. 2. Post-filtering based implementation structure for $\mathbf{E}_{N-1}^\dagger(z)$

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