ON THE ROBUSTNESS OF FILTER BANK FRAME TO QUANTIZATION AND ERASURES

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ABSTRACT

This paper studies the robustness of filter bank (FB) frame in $l^2(\mathbb{Z})$ to quantization and erasures from the perspective of both frame and FB theory. It is shown that the equal-norm property, imposed frequently in previous papers, is *not* necessary for optimization of frames under certain conditions and not desirable from the viewpoint of designing FB frames. We introduce a novel notion, named frame energy, which actually dictates the quantization performance of FB frames. Then, the erasure effects on the structures and quantization performance of FB frames are investigated. The design freedom obtained from removing the equal-norm property is explained and illustrated with examples.

***Index Terms*—** Digital filters, Error recovery, Quantization, Redundancy, Signal representations

1. INTRODUCTION

Digital FB actually implements a class of discrete-time signal expansions, which finds many applications in signal processing and communications [1]. The relation between critically sampled FBs and signal expansions has been extensively studied in [1, 2]. Recently, attention has been drawn to oversampled FBs (OFB) and corresponding redundant signal expansions. It was studied in [3, 4] that OFBs correspond to a class of frames, named FB frames, which extend finite frames in space $\mathbb{C}^M$ to more general space $l^2(\mathbb{Z})$. OFBs have been of great interest recently because they can offer some desired advantages over traditional critically sampled FBs, such as structural redundancy left in subband signals for error resilient transmission. Thus, they are appropriate for multiple description coding and design of space-time coding for multiple antenna wireless systems.

The motivation for our research on FB frames is due to their structural redundancy which can enhance resilience to erasures for information transmission over erasure networks like Internet. Considering a P-channel OFB with subsampling factor M ($M < P$), after the analysis FB which transforms the input signal into subbands, the P subband signals are scalar quantized separately by uniform scalar quantizers and sent over P independent channels. Each channel is lossless or totally lost during transmission. The decoder receives $P-e$ channel information, where e is the number of erasures, and tries to reconstruct the source by a linear method, i.e., synthesis FB. In this paper, we do not assume any prior information on the input signal, and only use the popular uncorrelated quantization noise model and assume equal probability of erasures. Thus, the reconstruction performance only depends on the characteristics of the FB. We investigate the FB properties induced by being optimally robust to quantization and erasures.

Previous work on frame expansion with quantization and erasures has been focused on finite frames in space $\mathbb{C}^M$ or $\mathbb{R}^M$ [5, 6]. Recently, it has been extended to FB frame in $l^2(\mathbb{Z})$ [7]. However, an unnecessary condition of equal-norm for FB frames is always imposed in [5]-[7] which leads to the *restriction* of the optimality of tight frames over equal-norm FB frames. In this paper, we show that the equal-norm property is not always necessary and tight FB frames are optimal over more general FB frames. In addition, the equal-norm property is not desirable in the perspective of designing FB frames. Even for finite frames, it is difficult to design an equal-norm tight frame, i.e., a tall rectangular orthogonal matrix with orthogonal columns and equal-norm of rows, with some freedom for given design specifications P and M. Furthermore, it is shown that it is the *frame energy*, not the equal-norm property, that dictates the quantization performance of FB frames under uncorrelated noise model.

Notations: Bold-faced quantities denote matrices and vectors. $\mathbf{I}_M$ denotes the identity matrix with size M. Any P-channel OFB with subsampling factor M is denoted by its $P\times M$ analysis polyphase matrix $\mathbf{E}(z)$. For arbitrary $P \times M$ constant matrix $\mathbf{A}$ and polynomial matrix $\mathbf{A}(z)$, we say $\mathbf{A}$ and $\mathbf{A}(z)$ are orthogonal and paraunitary (PU), respectively, if the $M \times P$ matrix $\mathbf{A}^T$ and $\tilde{\mathbf{A}}(z)$ satisfy $\mathbf{A}^T\mathbf{A} = c\mathbf{I}_M$ and $\tilde{\mathbf{A}}(z)\mathbf{A}(z) = c\mathbf{I}_M$ for some positive constant c, where $\tilde{\mathbf{A}}(z) = \mathbf{A}^*(1/z^*)$. The superscript $*$ denotes the Hermitian transpose; when used with scalars, it denotes only complex conjugation.

2. PRELIMINARIES

2.1. Fundamentals of Frames

In this paper, we mainly study the frames in the Hilbert space $l^2(\mathbb{Z})$ implemented by a FB and specialize them to finite dimensional space $\mathbb{H}^M$ ($\mathbb{H} = \mathbb{C}$ or $\mathbb{R}$). A set of sequences (or vectors) $\Phi = \{\phi_k\}_{k\in I}$ with index set I in a Hilbert space $\mathcal{H}$ (with finite or infinite dimension) of square summable sequences is a frame if for any $x \in \mathcal{H}$,

$$0 < A\|x\|^2 \leq \sum_{k\in I} |\langle\phi_k, x\rangle|^2 \leq B\|x\|^2 < +\infty \qquad (1)$$

where the constants A, B are called frame bounds. For finite frames in space $\mathbb{H}^M$, the cardinality of the index set I is finite. In this paper, a finite frame is defined as a collection of P vectors with length M satisfying (1), which can be equivalently represented by a $P \times M$ matrix $\mathbf{F}$ with the P vectors as the rows.

However, for the frame in $l^2(\mathbb{Z})$ implemented by an OFB with P channels and subsampling factor M, the elements constituting a frame correspond to the translated version of P elementary waveforms $\Phi = \{\phi_{ij}|\phi_{ij}[n] = \phi_i[n - jM], 0 \leq i \leq P - 1, j \in \mathbb{Z}\}$, where the elementary waveforms are related to the filter impulse responses as $\phi_i[n] = h_i^*[-n], 0 \leq i \leq P - 1$. The finite frame can be seen as a special case of FB frame if we constrain the filter length to be the subsampling factor M, i.e., the polyphase matrix $\mathbf{E}(z)$ degenerating into constant matrix $\mathbf{F}$. Thus, all results for FB frame in

$l^2(\mathbb{Z})$ is also applicable to finite frames in $\mathbb{H}^M$. The frame condition (1) on a FB can also be expressed in terms of the properties of the polyphase matrix $\mathbf{E}(z)$, which establishes a bridge between the frame and FB theory [3, 4].

Proposition 1. *[3, 4] A FB implements a frame expansion if and only if its analysis polyphase matrix $\mathbf{E}(z)$ is of full rank on the unit circle. Moreover, a FB implements a tight frame expansion if and only if it is paraunitary, i.e., $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = A\mathbf{I}_M$.*

A very important concept of FB frames is the frame operator $\mathbf{S}(z)$ which is an $M \times M$ matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\mathbf{E}(z)$. The eigenvalues $\lambda_i(\omega)$ of the frame operator indexed by frequency ω are called spectral eigenvalues. Another operator of FB frame is the Gram matrix which is a $P \times P$ matrix and defined as $\mathbf{G}(z) = \mathbf{E}(z)\tilde{\mathbf{E}}(z)$. Both the above two matrices are Hermitian and positive semidefinite. In addition, they have the same nonzero spectral eigenvalues. For a given FB frame with polyphase matrix $\mathbf{E}(z)$, its pseudo-inverse $\hat{\mathbf{E}}(z)$ is defined as $\hat{\mathbf{E}}(z) = [\tilde{\mathbf{E}}(z)\mathbf{E}(z)]^{-1}\tilde{\mathbf{E}}(z)$. The existence of inversion is guaranteed due to the full rank property of $\mathbf{E}(z)$.

2.2. Classification and Eigenstructures of Frames

To prevent confusion of various terms used in the literature, the standard notations established by [6] are adopted and expanded in this paper.

- λ-tight frame (λ-TF): Tight frame with frame bound λ.
- Equal-norm frame (ENF): Frame where all the elements have the same norm, $\|\phi_m\| = \|\phi_n\|$ for all m, n.
- Unit-norm frame (UNF): Frame where all the elements have norm 1, $\|\phi_m\| = 1$ for all m.
- Uniform frame (UF): FB Frame where the norm of the polyphase vector of each filter is independent of frequency, i.e., $P_i(\omega) = \sum_{j=1}^{M} |\mathbf{E}_{ij}(\omega)|^2 = \|h_i[n]\|^2$ [7].

In the following sections, we will see that the eigenstructure of the frame operator $\mathbf{S}(\omega)$ plays an important role in analysis of FB frame. There is a fundamental equality on the integral sum of the spectral eigenvalues,

$$\int_{-\pi}^{\pi} \sum_{i=1}^{M} \lambda_i(\omega) \frac{d\omega}{2\pi} = \int_{-\pi}^{\pi} \sum_{i=1}^{P} P_i(\omega) \frac{d\omega}{2\pi} = \sum_{i=1}^{P} \|h_i[n]\|^2 \quad (2)$$

For tight FB frames, i.e., paraunitary FBs, spectral eigenvalues are all equal to frame bound A and constant over the unit circle. Thus $MA = \sum_{i=1}^{P} \|h_i[n]\|^2$. In addition, tight FB frames have a very nice property that the pseudo-inverse of $\mathbf{E}(z)$ can be easily obtained by $\hat{\mathbf{E}}(z) = A^{-1}\tilde{\mathbf{E}}(z)$.

3. QUANTIZED FILTER BANK FRAME

In this section, we investigate the sensitivity of FB frame expansion to quantization noise added to the subband signals in absence of erasures. The reconstruction method from subbands is constrained to be linear, which is equivalent to using a synthesis FB. For mathematical tractability, we assume the quantization noise vector to be white and pairwise uncorrelated with identical variances σ_q^2. It is well known that the optimal PR synthesis FB is the pseudo-inverse $\hat{\mathbf{E}}(z)$ and the minimum mean square error (MSE) is [8],

$$\sigma_e^2 = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \mathrm{tr}[\hat{\mathbf{E}}(\omega)\hat{\mathbf{E}}^*(\omega)] d\frac{\omega}{2\pi} = \frac{\sigma_q^2}{M} \int_{-\pi}^{\pi} \sum_{i=1}^{M} \frac{1}{\lambda_i(\omega)} d\frac{\omega}{2\pi} \quad (3)$$

The above result reveals the optimal synthesis bank given a FB frame. In the rest of this paper, we will assume using the pseudo-inverse for reconstruction when quantization is involved. Here, we want to further investigate the optimal FB frame under the same quantization model and find what properties of FB frame are induced by such optimality. Although we don't impose any constraint on the underlying frames, there is a constraint as in (2) on the integral sum of the spectral eigenvalues $\lambda_i(\omega)$ of FB frames. This leads naturally to defining an important quantity of FB frames, namely frame energy.

Definition 1. *The frame energy T of a FB frame is defined as $T = \sum_{i=1}^{P} \|h_i[n]\|^2$. For finite frame, it is just the sum of square of norms of the frame vectors.*

Without loss of generality, we can always scale matrix $\mathbf{E}(\omega)$ to let all frame elements with norm $\|\phi_i\| \le 1$. Thus, the frame energy has an upper bound P. Now, our optimization problem becomes trying to minimize σ_e^2 given the constraint of upper bounded frame energy. This optimal FB frame in the sense of MSE, is the tight frame as stated in the following theorem.

Theorem 1. *When encoding with a FB frame in $l^2(\mathbb{Z})$ with frame energy T upper bounded by C, i.e., $T \le C (0 < C \le P)$, and decoding with the pseudo-inverse under the additive white and pairwise uncorrelated noise model, the reconstruction MSE is minimum if and only if the FB frame is tight and the minimum MSE is $\frac{M}{C}\sigma_q^2$. Furthermore, the optimal frame is generally not unique although the ENTF is always an optimal one.*

Proof. From the equality (2), we see easily that the optimal frame occurs when frame energy T attains its upper bound C. In addition, from the form of MSE in (3) and the Euler-Lagrange condition of the calculus of variations [9], we know that the minimization of MSE can be treated as a succession of ordinary minimization problems indexed by ω between 0 and 2π. Then we can see that the MSE is minimum when spectral eigenvalues $\lambda_i(\omega)$ are identical and constant over the unit circle, which is true if and only if the FB frame is tight. Thus, the sum of eigenvalues $\sum_{i=1}^{M} \lambda_i = M\lambda = C$, i.e., $\lambda_i(\omega) = C/M$. The minimum MSE then follows from (3). □

This theorem reveals the optimality of tight FB frames to quantization. Note that this result is also applicable to finite frames. Compared to the previous results in [5] and [7] for finite and FB frames, respectively, which *only* stated the optimality of TFs among UNFs, our result is more general since it states the optimality of tight frames over *all* frames and it can cover theirs as a special case of ours. Moreover, theorem 1 reveals that it is the frame energy, not the unit-norm property, that dictates the optimal quantization performance of FB frames. In addition, the equal-norm condition is not desirable from the perspective of designing tight frames since it is very difficult, if not impossible, to impose both the column-wise orthogonality condition (tight) and row-wise constant norm condition (equal-norm) *simultaneously* on frame $\mathbf{F}$, while still leaving some degree of *design freedom*. So far, the well known Harmonic frames are the only known equal-norm, tight finite frames [6, 10]. However, they are fixed when design specifications P and M are given. On the contrary, we can obtain some degree of design freedom by removing equal-norm condition without destroying optimality on MSE. We give two examples of optimal tight frames with the same minimum MSE to validate the *nonuniqueness* of the optimal frames with respect to quantization. For the purpose of simplicity, we use finite frames with frame energy $T \le P$ for illustration.

Example 1: *Optimal tight frame is an ENTF.*

For such frame, the frame vector norm $\|\phi_i\| = T/P$ for $i = 1, \cdots, P$. The eigenvalues of frame operator $\lambda_i = T/M$ for $i = 1, \cdots, P$. The minimum MSE is $\frac{M}{T}\sigma_q^2$. The existence of ENTF is shown by explicit construction in [6]. When $T = P$, it is just the UNTF studied in [5, 7].

Example 2: *Optimal tight frame is an unequal norm TF.*
We show this by an explicit construction. Starting with any $P \times P$ orthogonal matrix $\mathbf{A}$ (for example $P \times P$ DCT matrix), we scale $\mathbf{A}$ with factor $\sqrt{T/M}$ and then delete any $P - M$ columns, which results in a matrix $\mathbf{F}$. It can be shown easily that $\mathbf{F}^T\mathbf{F} = \frac{T}{M}\mathbf{I}_M$, i.e., $\mathbf{F}$ represents a TF with frame bound T/M. Its eigenvalues are same as those of Example 1, thus the same MSE performance. However, the norms of frame vectors usually may not be equal to T/P although their sum are still equal to frame energy T. However, in the extreme case of $T = P$, the optimal frame must be an UNTF.

4. ROBUSTNESS OF FB FRAME TO ERASURES

In this section, we study the robustness of FB frame to erasures, which is a typical phenomenon in information transmission over packet based networks like Internet. Assuming a FB frame with $P \times M$ polyphase matrix $\mathbf{E}(\omega)$ and an index set of erasures E, we denote the polyphase matrix after e erasures by $\mathbf{E}_E(\omega)$, where e is the cardinality of the set E, i.e., $e = |E|$. The matrix $\mathbf{E}_E(\omega)$ is a $(P - e) \times M$ matrix obtained by deleting the rows indexed by E from the original $\mathbf{E}(\omega)$. When $\mathbf{E}_E(\omega)$ is still of full rank on the unit circle, it is called a subframe. However, the subframe $\mathbf{E}_E(\omega)$ is usually dependent on the erasure set E, which means that a subframe may not exist for some erasure set. From the viewpoint of robust transmission over erasure network, it is desirable that the subframe after e erasures is independent of specific erasure set E. A FB frame is said to be robust to e erasures if a subframe exists for any erasure set E with $e = |E|$.

4.1. Erasure Effect on Tight Filter Bank Frames

TF has found many applications in signal processing and communications since they have many desirable properties. Here, we first derive an important property of tight FB frames unnoticed before.

Lemma 1. *The square of the norms of the polyphase vectors of any tight FB frame is upper bounded by its frame bound A, i.e., $P_i(\omega) \leq A$ on the unit circle for all $i = 1, 2, \cdots, P$.*

Proof. It can be seen easily that $P_i(\omega)$ is just the diagonal element of the Gram matrix of FB frame, i.e., $P_i(\omega) = \mathbf{e}_i^T\mathbf{G}(\omega)\mathbf{e}_i$ where $\mathbf{G}(\omega) = \mathbf{E}(\omega)\mathbf{E}^*(\omega)$ and $\mathbf{e}_i$ is the standard basis in $\mathbb{R}^P$. Since $\mathbf{G}(\omega)$ is a Hermitian matrix, it can be diagonalized by a unitary matrix. However, $\mathbf{G}(\omega)$ has the same nonzero eigenvalues as frame operator $\mathbf{S}(\omega) = \mathbf{E}^*(\omega)\mathbf{E}(\omega) = A\mathbf{I}_M$. Thus, we know that $\mathbf{G}(\omega)$ has M positive eigenvalues A and $P - M$ zero eigenvalues, which means there exists a unitary matrix $\mathbf{U}(\omega)$ such that $\mathbf{G}(\omega) = \mathbf{U}(\omega)\mathbf{\Sigma}\mathbf{U}^*(\omega)$, where $\mathbf{\Sigma} = \text{diag}(A\mathbf{I}_M, \mathbf{0}_{P-M})$. This leads to $P_i(\omega) = \mathbf{u}_i^T(\omega)\mathbf{\Sigma}\mathbf{u}_i(\omega)$, where $\mathbf{u}_i(\omega) = \mathbf{U}^*(\omega)\mathbf{e}_i$ and $\|\mathbf{u}_i(\omega)\| = \|\mathbf{e}_i\| = 1$ due to unitarity of $\mathbf{U}(\omega)$. Thus, $P_i(\omega) = A\sum_{j=1}^{M}\|u_{ij}(\omega)\|^2 \leq A\|\mathbf{u}_i(\omega)\|^2 = A$ for all ω and $i = 1, \cdots, P$. □

This lemma reveals a *novel* perspective on the frame bound of tight FB frames, which is an upper bound on the square of norm of the polyphase vector of each filter. For finite frames, it simplifies to the upper bound of square of norm of frame vectors, which is called the fundamental inequality of finite tight frames [11]. Since the tight FB frame is optimal in the sense of MSE as shown by Theorem 1, we attempt to find that under what condition, it is also robust to erasures. The following theorem gives the existence condition and reveals that the equal-norm condition is *not* necessary for robustness to one erasure, which is established in [7].

Theorem 2. *A TF implemented by a FIR OFB is robust to one erasure if and only if the square of the norm of the polyphase vector of each filter is strictly less than the frame bound A, i.e., $P_i(\omega) < A$ for all ω and $i = 1, 2, \cdots, P$. Furthermore, the subframe after one erasure generally cannot be a TF again.*

Proof. Assume that the erased channel is $H_i(\omega)$ whose polyphase vector is denoted by a row vector $\mathbf{h}_i(\omega)$. After one erasure, call the remaining $(P-1)\times P$ polyphase matrix $\mathbf{E}_i(\omega)$, then $\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega) = \mathbf{E}^*(\omega)\mathbf{E}(\omega) - \mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega) = A\mathbf{I}_M - \mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)$. The subframe $\mathbf{E}_i(\omega)$ exists if and only if $\mathbf{E}_i(\omega)$ is of full rank on the unit circle, which is equivalent to a nonzero determinant of $\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega)$. By the equality of determinant $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$, we can see $\det[\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega)] = A^M\det[\mathbf{I}_M - A^{-1}\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)] = A^M[1 - A^{-1}\mathbf{h}_i(\omega)\mathbf{h}_i^*(\omega)] = A^M[1 - A^{-1}P_i(\omega)]$. Thus, The subframe $\mathbf{E}_i(\omega)$ exists if and only if $1 - A^{-1}P_i(\omega) \neq 0$, i.e., $P_i(\omega) \neq A$ for all ω. Since the underlying FB is FIR and the fact that the frequency response of an FIR filter is continuous, the inequality $P_i(\omega) \neq A$ implies $P_i(\omega) > A$ or $P_i(\omega) < A$ for all ω. However, Lemma 1 has already excluded the possibility of $P_i(\omega) > A$. For a FB frame to be robust to one erasure, it must be robust to any possible channel erasure, thus $P_i(\omega) < A$ for all ω and $i = 1, 2, \cdots, P$. The subframe with $\mathbf{E}_i(\omega)$ is a TF if and only if $\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega) = c\mathbf{I}_M$ with nonzero constant c. This would demands that $\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)$ be zero since it is rank one matrix. This means the filter $h_i[n]$ is zero. □

Apparently, if the number of erasures $e > P - M$, the subframe cannot exist. Thus, a FB frame with polyphase matrix $\mathbf{E}(\omega)$ is called maximally robust (MR) to erasures, i.e., $e = P - M$, if every $M \times M$ submatrix of $\mathbf{E}(\omega)$ is invertible. Finite MR frames have been found and studied in [5, 6, 12, 13]. However, they have no design freedom when the design specifications of P and M are fixed, i.e., the finite MR frame is fixed. In the following lemma, we give a parameterized structure which has the MR property, but still has some degree of design freedom.

Lemma 2. *For any given $P \times P$ nonsingular Vandermonde matrix $\mathbf{V}$, we can obtain a maximally robust frame $\mathbf{F}$ represented by a $P \times M$ matrix after deleting the right $P - M$ columns of $\mathbf{V}$.*

Since every $M \times M$ submatrix of $\mathbf{F}$ is still a nonsingular Vandermonde matrix (due to the nonsingular matrix $\mathbf{V}$), $\mathbf{F}$ is a maximally robust frame. The lowpass DFT codes [12] and complex harmonic frames [5, 6] are special cases of this type of MR frames. In addition, we can construct MR FB frames by backward factorization of the polyphase matrix $\mathbf{E}(\omega)$.

Theorem 3. *A MR FB frame can be constructed by the pre-filtering structure, i.e., the analysis polyphase matrix $\mathbf{E}(\omega) = \mathbf{F}\mathbf{U}(\omega)$, where $\mathbf{F}$ is any MR finite frame in $\mathbb{H}^M$ and $\mathbf{U}(\omega)$ is an $M \times M$ polyphase matrix nonsingular over the unit circle.*

It can be shown easily that every $M \times M$ submatrix of $\mathbf{E}(\omega)$ is nonsingular over the unit circle since the erasures only affect $\mathbf{F}$. The detail is omitted due to the space limitation.

4.2. Erasure Effect on the Performance of Quantization

In the previous section, it is shown that is is possible to design FB frames which are robust to e $(0 < e \leq P-M)$ erasures. We assume such FB frames for the rest of this paper. We want to examine the effect of erasures on the MSE and try to find the optimal frame and its properties induced by this optimality in the sense of MSE.

Theorem 4. *When encoding with a uniform frame implemented by an FIR OFB with given frame energy $T \leq C$ $(0 < C \leq P)$, and decoding with the pseudo-inverse of the subframe under the additive white, pairwise uncorrelated noise model, the MSE averaged over all possible one channel erasures (assume equal probability of channel failure) is minimum if and only if the original frame is tight and equal-norm. The average MSE is $\overline{MSE}_1 = (1 + \frac{1}{P-M})MSE_0$, where $MSE_0 = \frac{M}{C}\sigma_q^2$.*

Proof. Assume the same setup as that in the proof of Theorem 2.

$$\overline{\text{MSE}}_1 = \frac{1}{P}\sum_{i=1}^{P}\frac{\sigma_q^2}{M}\frac{1}{2\pi}\int_{-\pi}^{\pi}\text{tr}\{[\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega)]^{-1}\}d\omega \tag{4}$$

By matrix inversion lemma $(\mathbf{A}-\mathbf{BCD})^{-1} = \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1}-\mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$, we can obtain $[\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega)]^{-1} = \mathbf{S}^{-1}(\omega) + \mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)[1-\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)]^{-1}\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)$, where $\mathbf{S}(\omega)$ is the frame operator with frame bounds A and B, i.e., $A\mathbf{I}_M \leq \mathbf{S}(\omega) \leq B\mathbf{I}_M$. Thus, we obtain $\text{tr}[\mathbf{E}_i^*(\omega)\mathbf{E}_i(\omega)]^{-1} = \text{tr}[\mathbf{S}^{-1}(\omega)] + [1-\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)]^{-1}\text{tr}[\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)]$ since scalars can be extracted out of trace operation. By the equality $\text{tr}(AB) = \text{tr}(BA)$, the term $\text{tr}[\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)]$ can be simplified to $\text{tr}[\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)]$ which is a scalar. The average MSE becomes

$$\begin{aligned}\overline{\text{MSE}}_1 \quad = \quad & \frac{\sigma_q^2}{M}\frac{1}{2\pi}\int_{-\pi}^{\pi}\text{tr}[\mathbf{S}^{-1}(\omega)]d\omega + \\ & \frac{1}{P}\sum_{i=1}^{P}\frac{\sigma_q^2}{M}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)}{1-\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)}d\omega\end{aligned} \tag{5}$$

The first term is minimized if and only if the frame is tight as seen from Theorem 1. Now we try to minimize the second term and investigate its lower bound. Since the FB frame is uniform, $P_i(\omega) = C_i$ independent of ω where C_i is a constant. By a similar derivation as that in Lemma 1, we can show $P_i(\omega) = C_i \leq B$. From the positive definiteness of $\mathbf{S}(\omega)$, we know that $B^{-1}\mathbf{I}_M \leq \mathbf{S}^{-1}(\omega) \leq A^{-1}\mathbf{I}_M$ which leads to the inequality $1-\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega) \leq 1-B^{-1}\mathbf{h}_i(\omega)\mathbf{h}_i^*(\omega) = 1-B^{-1}P_i(\omega) = 1-\frac{C_i}{B}$. Similarly, we get $\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega) \geq \frac{C_i}{B^2}$. Thus the critical part of the second term in (5) has a lower bound,

$$\sum_{i=1}^{P}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathbf{h}_i(\omega)\mathbf{S}^{-2}(\omega)\mathbf{h}_i^*(\omega)}{1-\mathbf{h}_i(\omega)\mathbf{S}^{-1}(\omega)\mathbf{h}_i^*(\omega)}d\omega \geq \sum_{i=1}^{P}\frac{B^{-2}C_i}{1-B^{-1}C_i}$$

Equality can be achieved if and only if the frame is tight, i.e., $\mathbf{S}(\omega) = A\mathbf{I}_M = B\mathbf{I}_M$. Let positive $z_i = C_i/B \leq 1$, then the lower bound becomes $B^{-1}\sum_{i=1}^{P}\frac{z_i}{1-z_i}$. Then, we have to minimize the central part of the lower bound $J = \sum_{i=1}^{P}\frac{z_i}{1-z_i}$ subject to the constraint on frame energy $\sum_{i=1}^{P}z_i = B^{-1}\sum_{i=1}^{P}C_i = T/B$. Since J is a convex function over z_i, we can show that J attains minimum if and only if all z_i are equal to T/PB, i.e., $C_i = T/P$ for all i. This demands the equal-norm property of the original frame $\mathbf{E}(\omega)$. When the FB frame is tight and equal-norm, the spectral eigenvalues $\lambda_i(\omega) = A = T/M$ for all ω and $i = 1, \cdots, P$. The average MSE follows from (4). □

Compared to previous result in [7], this theorem reveals the equal-norm property is *necessary* for optimality of MSE of FB frames.

5. CONCLUSION

This paper studies FB frames in $l^2(\mathbb{Z})$ from the perspective of both frame and FB theory. By this method, we can conduct theoretical analysis as well as consider real design. In particular, it is shown that the equal-norm property, imposed frequently in previous papers, is not necessary for optimization of frames under certain conditions. Moreover, this property is not desirable from the viewpoint of designing FB frames. In this paper, we introduce a novel and important notion, named frame energy, which actually dictates the quantization performance of FB frames. It is shown that tight frames are optimal for quantization with a constraint on frame energy. In case of erasures, a necessary and sufficient condition is given for FB frames robust to one erasure without the need of equal-norm property. Then we examine the effect of one erasure on the MSE for such FB frames. The design freedom obtained from removing the equal-norm property is explained and illustrated with examples.

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