REGULAR WAVELETS USING A THREE-STEP LIFTING SCHEME

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ABSTRACT

We propose a structural design of multidimensional two-channel filter banks with desirable numbers of vanishing moments for the analysis and synthesis banks. We use a three-step lifting scheme as opposed to the conventional two-step lifting method in order to provide more symmetry between the analysis and synthesis filters. Design examples are also provided.

Index Terms**—** Filter banks, lifting scheme, wavelets

1. INTRODUCTION

Lifting scheme is a representation of wavelets where it provides a fast algorithm for computing the wavelet transform [3], [8]. Lifting can be interpreted as representation of a filter bank in the polyphase domain using ladder structure [1], [6]. Since perfect reconstruction is guaranteed for any kind of selection of lifting steps, the filter design for this approach can be accomplished by only imposing the desired characteristics such as vanishing moments into the filters.

Kovačević and Sweldens [4] proposed a structural design of multidimensional wavelets of any order and any number of analysis and synthesis vanishing moments. For two-channel schemes they used a two-step lifting technique. This approach could be considered as a generalization of the design method proposed by Phoong *et al.* [6], where they proposed a ladder structure for *one-dimensional* (1-D) filer banks and also they designed *quincunx filter bank* (QFB) using transformation.

Ansari *et al.* [1] proposed a two-channel filter bank using a triplet of halfband filters (equivalent to a three-step lifting), where they could address the restrictions in double-halfband filter bank (equivalent to a two-step lifting) [6]. While the magnitudes of the normalized analysis filter pair for the double-halfband filter bank in [6] have to be 1 and 0.5 at $\omega = \pi/2$, they can be set to an equal value of $\sqrt{2}/2$ in the triple-halfband design.

In this work, we generalize Ansari's method [1] to a multidimensional filter bank design with any number of analysis and synthesis vanishing moments using a structural approach based on Kovačević's method [4]. In our design we only consider FIR filters. Below, we introduce the notations that are used in this paper.

<u>Notations</u>

A d-dimensional discrete signal $x[\mathbf{n}]$ with $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$ has z-transform $X(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} x[\mathbf{n}]\mathbf{z}^{-\mathbf{n}}$. Here $\mathbf{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$ and we denote $\mathbf{z}^{\mathbf{n}} = \prod_{i=1}^{d} z_i^{n_i}$. We also define $\mathbf{z}^M$ as $\mathbf{z}^M = (\mathbf{z}^{M_1}, \mathbf{z}^{M_2}, \ldots, \mathbf{z}^{M_d})$, where $M = [M_1 \;\; M_2 \;\; \cdots \;\; M_d]$ is a $d \times d$ integer matrix with M_i as its ith column. We use asterisk to denote time reverse as $x^*[\mathbf{n}] = x[-\mathbf{n}]$ and $X^*(\mathbf{z}) = X(\mathbf{z}^{-1})$.

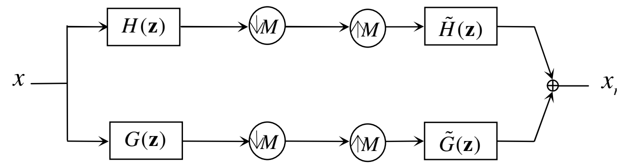

Fig. 1. A two-channel multidimensional filter bank.

We denote a multivariate polynomial in continuous time as $p(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbb{Z}_+^d} a_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$ where $\mathbf{t} \in \mathbb{R}^d$, $\mathbf{m} = (m_1, m_2, \ldots, m_d)$, and $\mathbb{Z}_+^d = \{\mathbf{m} \in \mathbb{Z}^d \,|\, m_i \geq 0, 1 \leq i \leq d\}$. Similarly, we denote a discrete-time polynomial as $p[\mathbf{n}] = \sum_{\mathbf{m}} a_{\mathbf{m}} \mathbf{n}^{\mathbf{m}}$. The degree of a monomial $\mathbf{t}^{\mathbf{m}}$ or $\mathbf{n}^{\mathbf{m}}$ is denoted as $|\mathbf{m}| = \sum_{i=1}^{d} m_i$ and we use $\mathcal{P}_N$ for the set of polynomials with degree less than N.

2. BACKGROUND

We consider a multidimensional two-channel critically-sampled filter bank as depicted in Fig. 1. Here M is the sampling matrix with size $d \times d$, $H(\mathbf{z})$ and $\tilde{H}(\mathbf{z})$ are lowpass analysis and synthesis filter pair and likewise, $G(\mathbf{z})$ and $\tilde{G}(\mathbf{z})$ are the highpass filter pair. We can express the filter bank in the polyphase domain using the polyphase matrices

$$\mathbf{A}_p(\mathbf{z}) = \begin{bmatrix} H_0(\mathbf{z}) & H_1(\mathbf{z}) \\ G_0(\mathbf{z}) & G_1(\mathbf{z}) \end{bmatrix} \text{ and } \mathbf{S}_p(\mathbf{z}) = \begin{bmatrix} \tilde{H}_0(\mathbf{z}) & \tilde{H}_1(\mathbf{z}) \\ \tilde{G}_0(\mathbf{z}) & \tilde{G}_1(\mathbf{z}) \end{bmatrix}, \quad (1)$$

where for perfect reconstruction we have $\mathbf{A}_p(\mathbf{z})\mathbf{S}_p^T(\mathbf{z}) = I$. Further, the analysis and synthesis lowpass filters are expressed as $H(\mathbf{z}) = \sum_{i=0}^{1} \mathbf{z}^{\mathbf{c}_i} H_i(\mathbf{z}^M)$ and $\tilde{H}(\mathbf{z}) = \sum_{i=0}^{1} \mathbf{z}^{-\mathbf{c}_i} \tilde{H}_i(\mathbf{z}^M)$, and similarly the highpass filters as $G(\mathbf{z}) = \sum_{i=0}^{1} \mathbf{z}^{\mathbf{c}_i} G_i(\mathbf{z}^M)$ and $\tilde{G}(\mathbf{z}) = \sum_{i=0}^{1} \mathbf{z}^{-\mathbf{c}_i} \tilde{G}_i(\mathbf{z}^M)$ where $\{\mathbf{c}_0, \mathbf{c}_1\}$ are cosets of the sampling matrix M assuming $\mathbf{c}_0 = \mathbf{0}$.

By iterating the filter bank, we obtain the analysis scaling and wavelet functions obeying [5]:

$$\varphi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} h^*[\mathbf{n}]\varphi(M\mathbf{t} - \mathbf{n}),$$

and

$$\psi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} g^*[\mathbf{n}]\varphi(M\mathbf{t} - \mathbf{n}). \quad (2)$$

And also we have $\varphi_{j,\mathbf{n}}(\mathbf{t}) = d_M^{-j/2}\varphi(M^{-j}\mathbf{t} - \mathbf{n})$, where $j \in \mathbb{Z}$ is the scale and we define $d_M \triangleq \det(M)$. Similarly, we have synthesis scaling and wavelet functions as

$$\tilde{\varphi}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \tilde{h}[\mathbf{n}]\tilde{\varphi}(M\mathbf{t} - \mathbf{n}), \text{ and } \tilde{\psi}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \tilde{g}[\mathbf{n}]\tilde{\varphi}(M\mathbf{t} - \mathbf{n}). \quad (3)$$

Vanishing moments in wavelets are critical regarding the regularity and also approximation power of wavelets [5], [7]. If the

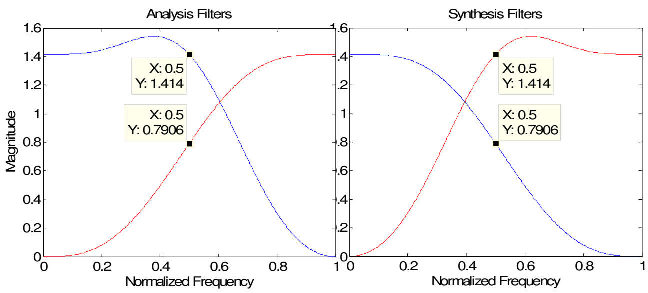

Fig. 2. The frequency response of the analysis filters in Example 1.

wavelet analysis and synthesis stages have N and $\tilde{N}$ vanishing moments, respectively, we have

$\int \mathbf{t}^{\mathbf{m}} \psi(\mathbf{t}) d\mathbf{t} = 0$, for $|\mathbf{m}| < N$, and $\int \mathbf{t}^{\mathbf{m}} \tilde{\psi}(\mathbf{t}) d\mathbf{t} = 0$, for $|\mathbf{m}| < \tilde{N}$.

These conditions are expressed for the highpass filters and polynomial $P_{oly}(\mathbf{z})$ as

$$\left(\downarrow M\right) G^{*}(\mathbf{z}) P_{oly}(\mathbf{z}) = 0, \quad \text{for } P_{oly}(\mathbf{z}) \in \mathcal{P}_N, \tag{4}$$

and

$$\left(\downarrow M\right) \tilde{G}(\mathbf{z}) P_{oly}(\mathbf{z}) = 0, \quad \text{for } P_{oly}(\mathbf{z}) \in \mathcal{P}_{\tilde{N}}. \tag{5}$$

(Recall that wavelet analysis highpass filter is $G^{*}(\mathbf{z}) = G(\mathbf{z}^{-1})$.) As a result, $G(\mathbf{z})$ has N zeros and $\tilde{G}(\mathbf{z})$ has $\tilde{N}$ zeros at $\boldsymbol{\omega} = \mathbf{0}$, but $H(\mathbf{z})$ has $\tilde{N}$ zeros and $\tilde{H}(\mathbf{z})$ has N zeros at $\boldsymbol{\omega} = \boldsymbol{\pi}$. Therefore, if $N > \tilde{N}$, the wavelet transform has more approximation power in the decomposition section and is more regular in reconstruction. This setting is more desirable in compression since more regular filters are required to compensate for the quantization in signal reconstruction [5].

Neville filters is a notion introduced and referred to a class of filters related to the interpolation of polynomials [4], as described below.

Definition 1: A filter $L(\mathbf{z})$ is a Neville filter of order N and shift $\boldsymbol{\tau} \in \mathbb{R}^{d}$ if the following condition is satisfied:

$$l[\mathbf{n}] * p_{oly}[\mathbf{n}] = p_{oly}[\mathbf{n} + \boldsymbol{\tau}], \quad \text{for} \quad p_{oly}[\mathbf{n}] \in \mathcal{P}_N. \tag{6}$$

This condition is equivalent to having $\sum_{\mathbf{k}} l[\mathbf{k}](-\mathbf{k})^{\mathbf{m}} = \boldsymbol{\tau}^{\mathbf{m}}$, for $|\mathbf{m}| < N$.

Remark 1: Here we should point out some of the important properties of the Neville filters we need later. 1) The filter $l^{*}[\mathbf{n}] = l[-\mathbf{n}]$ will be a Neville filter of order N and shift $-\boldsymbol{\tau}$ if $l[\mathbf{n}]$ is a Neville filter of order N and shift $\boldsymbol{\tau}$. 2) The upsampled version of a Neville filter of order N with shift $\boldsymbol{\tau}$ (i.e. $Q(\mathbf{z}) = L(\mathbf{z}^{M})$) is also a Neville filter of order N but with shift $M\boldsymbol{\tau}$. 3) If $L_1(\mathbf{z})$ and $L_2(\mathbf{z})$ are Neville filters of order N_1 and N_2, and shift $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, then $L_1(\mathbf{z})L_2(\mathbf{z})$ is a Neville filter of order $\min(N_1, N_2)$ and shift $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$.

An important class of filters are Deslauriers-Dubuc filters. These filters are Neville filters with shift $-1/2$ [4]; thus, they can interpolate polynomials at half integers. The coefficients of these filters when the order N is *even* are obtained through $R_{(N)}(z) = \sum_{k=1}^{N/2} a_k \left(z^{-k} + z^{k-1}\right)$, where

$$a_k = \frac{(-1)^{k+N/2-1} \prod_{i=1}^{N} (N/2 + 0.5 - i)}{(N/2 - k)!(N/2 - 1 + k)!(k - 0.5)}.$$

Hence, for even N the filter $R_{(N)}(z)$ is linear phase. Table I shows a few Deslauriers-Dubuc filters. In the next section we show how we can benefit from Neville filters in order to design two-channel filter banks.

TABLE I
DESLAURIERS-DUBUC FILTERS $R_{(N)}(z)$

N	$R_{(N)}(z)$
1	1
2	$(1 + z^{-1})/2$
3	$(-z + 6 + 3z^{-1})/2^{3}$
4	$(-z + 9 + 9z^{-1} - z^{-2})/2^{4}$
5	$(3z^{2} - 20z + 90 + 60z^{-1} - 5z^{-2})/2^{7}$
6	$(3z^{2} - 25z + 150 + 150z^{-1} - 25z^{-2} + 3z^{-3})/2^{8}$

3. TWO-CHANNEL FILTER BANK DESIGN

3.1. Kovačević's Design

A two-channel filter bank using a lifting framework provides perfect reconstruction for any kind of lifting steps. Therefore, one can design filters by simply imposing vanishing moments on the filters and derive the appropriate lifting steps. This has been done by Kovačević and Sweldens in [4] for filter banks with two lifting steps (*see* Fig. 3). In this case if we express the filter bank in the polyphase domain we have

$$\mathbf{A}_p = \begin{bmatrix} 1 - Q(\mathbf{z})L(\mathbf{z}) & Q(\mathbf{z}) \\ -L(\mathbf{z}) & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_p = \begin{bmatrix} 1 & L(\mathbf{z}) \\ -Q(\mathbf{z}) & 1 - Q(\mathbf{z})L(\mathbf{z}) \end{bmatrix}. \tag{7}$$

The two-channel filter bank designed by Phoong *et al.* [6] is similar to the 1-D design of [4], where they used a ladder network in the polyphase domain. This setting as pointed out by Ansari *et al.* [1] has a drawback that the values of lowpass (and highpass) filter pair cannot be made the same at $\omega = \pi/2$ (one is 1 and the other is 0.5 in the normalized 1-D filters of [6]). Here we show that we have the same problem for the general multidimensional design of two-channel filter banks proposed in [4].

Kovačević and Sweldens showed that by imposing N and $\tilde{N}$ vanishing moments to the wavelet analysis and synthesis banks shown in Fig. 1 when using the two-step lifting step depicted in Fig. 3 results in the following solution:

$L(\mathbf{z})$ should be a Neville filter of order N with shift $\boldsymbol{\tau}_0 = M^{-1}\mathbf{c}_1$, And $2Q^{*}(\mathbf{z})$ should be a Neville filter of order $\tilde{N}$ with shift $\boldsymbol{\tau}_0$, assuming that $N \geq \tilde{N}$.

The constants K_0 and K_1 in Fig. 3 are for the appropriate normalization of the filters. Two examples of the above framework are as follows.

Example 1: Consider a 1-D case where $M = 2$ and thus $\tau = 1/2$ where we can use time-revesed version of Deslauriers-Dubuc filters. Suppose that $N = 3$ and $\tilde{N} = 2$. Therefore, we can obtain the analysis filters using $L(z) = R_{(3)}(z^{-1})$ and $Q(z) = (1/2)R_{(2)}(z)$ from Table I as

$$H(z) = \left(\sqrt{2}/32\right)(z+1)^{2}(z^{2} - 2z - 2 + 14z^{-1} - 3z^{-2})$$

and

$$G(z) = \left(\sqrt{2}/16\right) z(1 - z^{-1})^{3}(z + 3).$$

We see that at $\omega = \pi/2$ or $z = j$ the magnitude of H and G are $\sqrt{2}$ and $\sqrt{10}/4$. Fig. 2 shows the frequency responses of the filters.

Example 2: In this example we consider the *quincunx filter bank* (QFB) design example in [4] with $N = 4$ and $\tilde{N} = 2$. Here, we can find that the filters evaluated at $\mathbf{z} = (j, j)$ give $H(j, j) = \sqrt{2}$ and $G(j, j) = \sqrt{2}/2$.

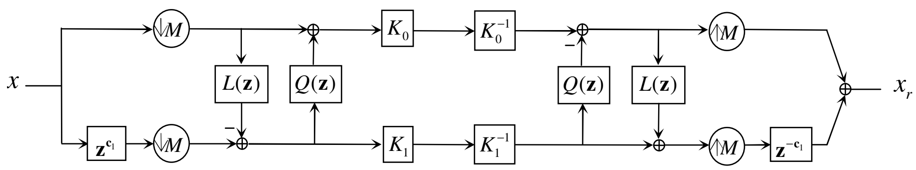

Fig. 3. A two-channel multidimensional filter bank using a two-step lifting scheme.

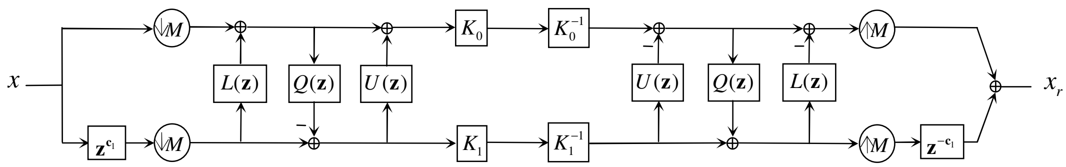

Fig. 4. A two-channel multidimensional filter bank using a three-step lifting scheme.

3.2. A Three-Step Lifting Design

In this section we address the drawback of the Kovačević's Design by adding an additional lifting step to the framework of Fig. 3. The proposed scheme is depicted in Fig. 4. We can write the analysis and synthesis polyphase matrices in terms of the lifting steps as

$$\mathbf{A}_p = \begin{bmatrix} 1 & U(\mathbf{z}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -Q(\mathbf{z}) & 1 \end{bmatrix} \begin{bmatrix} 1 & L(\mathbf{z}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-Q(\mathbf{z})U(\mathbf{z}) & L(\mathbf{z})+U(\mathbf{z})-L(\mathbf{z})Q(\mathbf{z})U(\mathbf{z}) \\ -Q(\mathbf{z}) & 1-L(\mathbf{z})Q(\mathbf{z}) \end{bmatrix}, \tag{8}$$

and

$$\mathbf{S}_p = (\mathbf{A}_p^{-1})^T = \begin{bmatrix} 1-L(\mathbf{z})Q(\mathbf{z}) & Q(\mathbf{z}) \\ -L(\mathbf{z})-U(\mathbf{z})+L(\mathbf{z})Q(\mathbf{z})U(\mathbf{z}) & 1-Q(\mathbf{z})U(\mathbf{z}) \end{bmatrix}. \tag{9}$$

Now we apply N vanishing moments to the wavelet analysis branch. Using (1), and (8) we have

$$G(\mathbf{z}) = -Q(\mathbf{z}^M) + \mathbf{z}^{\mathbf{c}_1}\left(1 - L(\mathbf{z}^M)Q(\mathbf{z}^M)\right).$$

Thus, by employing (4) we obtain

$$\begin{aligned} &-q^*[\mathbf{n}] * p_{oly}[\mathbf{n}M] + p_{oly}[\mathbf{n}M - \mathbf{c}_1] \\ &\quad - q^*[\mathbf{n}] * l^*[\mathbf{n}] * p_{oly}[\mathbf{n}M - \mathbf{c}_1] = 0 \end{aligned} \quad (p_{oly} \in \mathcal{P}_N). \tag{10}$$

Suppose that $k_L L^*(\mathbf{z})$, where k_L is a constant, is a Neville filter of order N and shift $\boldsymbol{\tau}_0 = M^{-1}\mathbf{c}_1$, then we have $k_L l^*[\mathbf{n}] * p_{oly}[\mathbf{n}M - \mathbf{c}_1] = p_{oly}[\mathbf{n}M]$. Hence, we can rewrite (10) as

$$(1 + 1/k_L)q^*[\mathbf{n}] * p_{oly}[\mathbf{n}M] = p_{oly}[\mathbf{n}M - \mathbf{c}_1] \quad (p_{oly} \in \mathcal{P}_N),$$

which implies that $k_Q Q^*(\mathbf{z})$ with

$$k_Q = 1 + 1/k_L, \tag{11}$$

is a Neville filter of order N and shift $-\boldsymbol{\tau}_0$, or $k_Q Q(\mathbf{z})$ is a Neville filter of order N and shift $\boldsymbol{\tau}_0$.

Now we impose $\tilde{N}$ vanishing moments on the analysis bank. We can obtain analysis highpass filter by using (1) and (9) as

$$\begin{aligned} \tilde{G}(\mathbf{z}) &= L(\mathbf{z}^M)Q(\mathbf{z}^M)U(\mathbf{z}^M) - L(\mathbf{z}^M) - U(\mathbf{z}^M) + \\ &\quad \mathbf{z}^{-\mathbf{c}_1}\left(1 - Q(\mathbf{z}^M)U(\mathbf{z}^M)\right) \end{aligned}.$$

Using (5) on the obtained $\tilde{G}(\mathbf{z})$ we have

$$\begin{aligned} &u[\mathbf{n}] * q[\mathbf{n}] * l[\mathbf{n}] * p_{oly}[\mathbf{n}M] - l[\mathbf{n}] * p_{oly}[\mathbf{n}M] - u[\mathbf{n}] * p_{oly}[\mathbf{n}M] + \\ &\quad p_{oly}[\mathbf{n}M - \mathbf{c}_1] - u[\mathbf{n}] * q[\mathbf{n}] * p_{oly}[\mathbf{n}M - \mathbf{c}_1] = 0 \quad (p_{oly} \in \mathcal{P}_{\tilde{N}}). \end{aligned} \tag{12}$$

If we assume $N \ge \tilde{N}$, since $k_L L^*(\mathbf{z})$ and $k_Q Q(\mathbf{z})$ are Neville filters with shift $\boldsymbol{\tau}_0$, we can reduce (12) to

$$\begin{aligned} &\frac{1}{k_L k_Q} u[\mathbf{n}] * p_{oly}[\mathbf{n}M] - \frac{1}{k_L} p_{oly}[\mathbf{n}M - \mathbf{c}_1] - u[\mathbf{n}] * p_{oly}[\mathbf{n}M] + \\ &\quad p_{oly}[\mathbf{n}M - \mathbf{c}_1] - \frac{1}{k_Q} u[\mathbf{n}] * p_{oly}[\mathbf{n}M] = 0 \quad (p_{oly} \in \mathcal{P}_{\tilde{N}}), \end{aligned}$$

or

$$k_U u[\mathbf{n}] * p_{oly}[\mathbf{n}M] = p_{oly}[\mathbf{n}M - \mathbf{c}_1] \quad (p_{oly} \in \mathcal{P}_{\tilde{N}}),$$

where by using (11) we have

$$k_U = 2k_L^2 \Big/ \left(k_L^2 - 1\right). \tag{13}$$

As a result, $k_U U^*(\mathbf{z})$ is a Neville filter of order N and shift $\boldsymbol{\tau}_0 = M^{-1}\mathbf{c}_1$. As seen, in the proposed design we have a free parameter where we can adjust the filters to have more symmetry. For the above setting we assumed $N \ge \tilde{N}$, which is more useful in compression as described before. If we need to set $\tilde{N} \ge N$ vanishing moments, we can reverse the directions and also signs of the lifting steps $L(\mathbf{z})$, $Q(\mathbf{z})$, and $U(\mathbf{z})$ in the filter bank shown in Fig. 4.

In the next section we present a few design examples using the proposed scheme.

3.3. Design Examples

3.3.1. 1-D Filter Banks

In 1-D case, since $M = 2$ we have $\tau_0 = 1/2$, and hence, we can use the time-reversed version of Deslauriers-Dubuc filters to change the shift from $-1/2$ to $\tau = 1/2$. Thus, for a filter bank with N vanishing moments in the analysis bank and $\tilde{N}$ vanishing moments in the synthesis stage assuming $N \ge \tilde{N}$, we have $k_L L^*(z) = R^*_{(N)}(z)$ or $k_L L(z) = R_{(N)}(z)$, and $k_Q Q(z) = R^*_{(N)}(z) = R_{(N)}(z^{-1})$, and finally $k_U U(z) = R_{(\tilde{N})}(z)$. Using (8), (11), and (13), we derive the highpass analysis filter as

$$G(z) = K_1\left(-\left(1/k_Q\right)R_{(N)}(z^{-2}) + z\left(1 - \left(1/(k_L k_Q)\right)R_{(N)}(z^2)R_{(N)}(z^{-2})\right)\right),$$

or

$$G(z) = \frac{K_1}{1 + k_L}\left(-k_L R_{(N)}(z^{-2}) + z\left(1 + k_L - R_{(N)}(z^2)R_{(N)}(z^{-2})\right)\right), \tag{14}$$

and the lowpass analysis filter is derived as

$$H(z) = K_0\left(1 + U(z^2)G(z)/K_1 + zL(z^2)\right),$$

or

$$H(z) = \frac{K_0}{2k_L^2}\left(2k_L^2 + (k_L^2 - 1)R_{(\tilde{N})}(z^2)G(z)/K_1 + z2k_L R_{(N)}(z^2)\right). \tag{15}$$

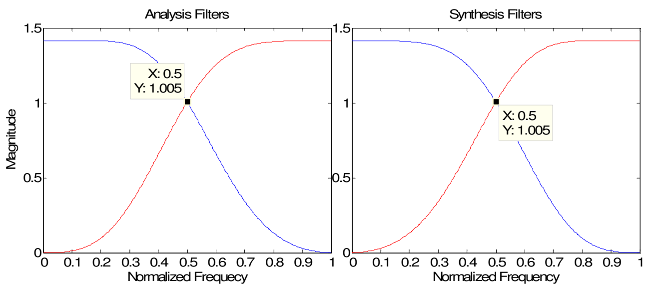

Fig. 5. The frequency responses of the filters in Example 3.

Note that we included positive constants K_0 and K_1 for proper normalization (*see* Fig. 4). From Table I, since $R_{(N)}(1)=1$, we have

$H(1)=K_0(k_L+1)/k_L=\sqrt{2}$, and

$$|G(-1)|=2K_1k_L/(k_L+1)=\sqrt{2}. \quad (16)$$

Now we can find the parameters by evaluating

$$|H(j)|=|G(j)|. \quad (17)$$

In the case when both vanishing moments are of even order, we have $R_{(N)}(-1)=R_{(\tilde{N})}(-1)=0$, hence, (17) yields $K_0=K_1$ and from (16) we have $K_0=K_1=1$, and $k_L=1/\left(\sqrt{2}-1\right)$ where this solution gives $|H(j)|=|G(j)|=1$. If N or $\tilde{N}$, or both are odd then we should solve (16) and (17) to find the parameters. We can also obtain the synthesis filters using the following equations

$$\tilde{H}(z)=-z^{-1}G(-z), \quad \text{and} \quad \tilde{G}(z)=z^{-1}H(-z).$$

Here we should point out that the design method of Ansari *et al.* [1] for regular filters is similar to the proposed 1-D three-step lifting design when $N=\tilde{N}$.

Below we provide an example for 1-D filter bank design using the proposed method.

Example 3: Consider the design criteria in Example 1, that is, we desire $N=3$ and $\tilde{N}=2$ vanishing moments. By solving (16) and (17) we obtain the parameters as $k_L=2.2686$, $K_0=0.9816$, and $K_1=1.0188$ where we have $|H(j)|=|G(j)|=1.005$. Fig. 5 shows the frequency responses of the filters designed in this example. The symmetry of these filter pairs is clear when compared with the design using two lifting steps and same number of vanishing moments (*see* Example 1 and Fig. 2).

Note that to avoid finding a solution for the parameters, we can set $K_0=K_1=1$ and $k_L=1/\left(\sqrt{2}-1\right)=2.414$ where they satisfy (16), but we obtain $|H(j)|=1.021$ and $|G(j)|=0.9919$, which are very close together.

3.3.2. *Quincunx Filter Banks*

Quincunx filter banks (QFB) are 2-D two-channel filter banks where they can provide multiresolution analysis under certain regularity conditions. Suppose that we choose the sampling matrix as

$$M=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},$$

where we have the coset vector as $\mathbf{c}_1=(1,0)^T$ and hence $\boldsymbol{\tau}_0=M^{-1}\mathbf{c}_1=(1/2,1/2)^T$. As a result, we should use Neville filters with shift $\boldsymbol{\tau}_0$. The Neville filters reported in [4] for QFB provide a shift equal to $-\boldsymbol{\tau}_0$. The filters with order $N=2$ and $N=4$ are

$$R_{(2)}(z_1,z_2)=(1+z_1^{-1}+z_2^{-1}+z_1^{-1}z_2^{-1})/2^2,$$

and

$$R_{(4)}(z_1,z_2)=\Big(10(1+z_1^{-1}+z_2^{-1}+z_1^{-1}z_2^{-1}) - (z_1^{-2}+z_2^{-2}+z_1^{-2}z_2^{-1}+z_1^{-1}z_2^{-2}+z_1+z_2+z_1z_2^{-1}+z_1^{-1}z_2)\Big)/2^5.$$

Similar to the 1-D case we choose $k_LL(\mathbf{z})=R_{(N)}(\mathbf{z})$, $k_QQ(\mathbf{z})=R^*_{(N)}(\mathbf{z})=R_{(N)}(z_1^{-1},z_2^{-1})$, and $k_UU(\mathbf{z})=R_{(\tilde{N})}(\mathbf{z})$. Therefore, the analysis filters are derived as

$$G(\mathbf{z})=\frac{K_1}{1+k_L}\Big(-k_LR_{(N)}(\mathbf{z}^{-M})+z_1\left(1+k_L-R_{(N)}(\mathbf{z}^{M})R_{(N)}(\mathbf{z}^{-M})\right)\Big), \quad (18)$$

and

$$H(\mathbf{z})=\frac{K_0}{2k_L^2}\Big(2k_L^2+(k_L^2-1)R_{(\tilde{N})}(\mathbf{z}^{M})G(\mathbf{z})/K_1+z_12k_LR_{(N)}(\mathbf{z}^{M})\Big). \quad (19)$$

We can find the parameters through $H(1,1)=G(-1,-1)=\sqrt{2}$ and $|H(j,j)|=|G(j,j)|$.

In the next example we use these filters to design the quincunx filters.

Example 4: Suppose that we desire a QFB with $N=4$ and $\tilde{N}=2$ vanishing moments. To find the proper parameters we first evaluate $H(1,1)=G(-1,-1)=\sqrt{2}$, which leads to

$$K_0(k_L+1)/k_L=2K_1k_L/(k_L+1)=\sqrt{2},$$

and the next requirement $|H(j,j)|=|G(j,j)|$ yields $K_0=K_1$ (note that $\mathbf{z}^M=(z_1z_2, z_1z_2^{-1})$). Therefore, we have similar result to the one in the 1-D case when we have even orders of vanishing moments, that is, $K_0=K_1=1$, and $k_L=1/\left(\sqrt{2}-1\right)$. (see Example 2 for comparison.)

4. CONCLUSION

We proposed a structural design of two-channel multidimensional filter banks with any number of vanishing moments using a three-step lifting scheme. The proposed scheme provides more symmetry between analysis and synthesis banks (compared with the two-step lifting design) and therefore, they have more regularity. Further analysis of the proposed scheme is under investigation.

5. REFERENCES

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