FURTHER RESULTS OF THE ANALYSIS OF THE MUSIC FOR CLOSELY SPACED, NON-EQUAL POWER PLANE WAVES

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ABSTRACT

In this paper, a detailed performance analysis of the MUltiple Signal Classification (MUSIC) algorithm for non-resolvable sources with non-equal power is carried out. The signals considered consist of clusters of sources, in which the directions of arrival (DOAs) of the sources are close in each cluster. Only one of the source is of interest, while the others are treated as interferences. In this scenario, the estimation accuracy is influenced by both the finite sample effect and the perturbation caused by the interferences, the latter of which is the focus of this paper. By using the first order of the Taylor series expansion of the perturbation caused by the interferences, the bias of the DOAs are derived in a closed form. It is shown that if the closely spaced signals exist, the MUSIC algorithm become biased, and the bias depends on their power and the distance between their DOAs. Simulation results are also conducted to verify the derived analytic expressions.

***Index Terms*—** MUSIC, DOA estimation, Error analysis, Antenna arrays

1. INTRODUCTION

The DOA estimation of the sources impinging on the arrays has been one of the central problems in statistical signal processing over the past few decades [1]. The subspace based algorithms, most notably the MUSIC [2], which makes full use of the characteristics of the data model to render high-resolution estimates, strike a good balance between the computational complexity and performance, and has been in particular of great research interest.

To get further insight into the MUSIC algorithm, the statistical analysis of which has received considerable attention. For example, Stoica *et al* derived the performance of the MUSIC in a closed form and analyzed its statistical efficiency [3]. It is observed in [3] that the Mean Square Error (MSE) is proportional to the inverse of the Signal to noise ratio (SNR), and that the MUSIC algorithm is a unbiased estimator. However, if there are closely spaced sources, which are not resolvable by the MUSIC algorithm, in the observed data such as the local scattering [4], the estimated value will approach the average of the true parameters of the sources [5, 6] and the MUSIC algorithm yields a biased estimator and thus the analysis in [3] is no longer applicable. For this, Xu *et al* [7] exploited the second order Taylor series expression of the derivative of the null spectrum to establish a rigorous bias analysis of the MUSIC location estimator. However, when SNR is below the resolution threshold of the MUSIC, their analytical results are not applicable.

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In this paper, a detailed error analysis of the MUSIC algorithm for non-resolvable sources with non-equal power is carried out. The signals considered consist of clusters of sources, in which the directions of arrival (DOAs) of the sources are close in each cluster. Only one of the source is of interest, while the others are treated as interferences which perturb the estimation accuracy of the desired source. The analysis of the finite sample effect has been well studied in the literatures, say [8]-[10], and in this paper we simply modify the derivations given in [3]. As for the analysis of the perturbation effect, by using the first order of the Taylor series expansion of the perturbation caused by the interferences, the bias of the DOA estimates is derived in a closed form. It is shown that the MUSIC algorithm is an unbiased estimator when the DOAs are well separated. However, if there are some closely spaced sources, the MUSIC becomes biased, and the bias depends on their power and the distance between their DOAs. Compared with previous works, the developed expressions are applicable to much more general scenarios and their mutual relationships are also highlighted. Simulation results are conducted to verify the derived analytic expressions.

2. STATEMENT OF THE PROBLEM

Assume that K clusters of sources impinge on the array with P antennas ($K < P$) and the numbers of the uncorrelated narrow band sources in each cluster are $L_i,\ i = 1, ..., K$. Let $\theta_{i,j}$ be the DOA of the j^{th} source in the i^{th} cluster, then observed signal at the antenna array, $\mathbf{x}(t)$, can be expressed as

$$\mathbf{x}(t) = \sum_{i=1}^{K}\sum_{j=1}^{L_i} \mathbf{a}(\theta_{i,j})\mathbf{s}_{i,j}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{s}_{i,j}(t)$ is the transmitted signal and $\mathbf{a}(\theta_{i,j})$=[$e^{-j2\pi\phi_1(\theta_{i,j})}$,. . .,$e^{-j2\pi\phi_P(\theta_{i,j})}]^T$ denotes the steering vector of the j^{th} source in the i^{th} cluster, in which $\phi_k(\theta_{i,j})$ is determined by the array geometric patterns and $(\cdot)^T$ denotes the transposition. $\mathbf{n}(t)$ is the additive white Gaussian noise with zero mean and variances σ_{n}^2.

Based on (1), the covariance matrix, $\mathbf{R} \triangleq E[\mathbf{x}(t)\mathbf{x}^H(t)]$, where $(\cdot)^H$ denotes the Hermitain operation, can be expressed as

$$\mathbf{R} = \sum_{i=1}^{K}\sum_{j=1}^{L_i} \sigma_{i,j}^2 \mathbf{a}(\theta_{i,j})\mathbf{a}^H(\theta_{i,j}) + \sigma_n^2\mathbf{I} \tag{2}$$

where $\sigma_{i,j}^2 = E[s_{i,j}(t)s_{i,j}^*(t)]$ is the power of the signal $\mathbf{s}_{i,j}(t)$ with $(\cdot)^*$ being the complex conjugation operation.

Without loss of generality, the discussion to follow will focus on the estimation accuracy of source 1 in the i^{th} cluster, $i = 1, \ldots, K$, while treating other sources as interferences. The analysis can be readily extended to other sources as well. Since the DOAs of the sources in the same clusters are close ($\theta_{i,1} \approx \theta_{i,2} \approx \cdots \approx \theta_{i,L_i}$, $i = 1, ..., K$), using the Taylor series expansion $\mathbf{a}(\theta_{i,j}) \approx \mathbf{a}(\theta_{i,1}) + \Delta\theta_{i,j}\mathbf{d}(\theta_{i,1})$, in which $\Delta\theta_{i,j} = \theta_{i,j} - \theta_{i,1}$ and $\mathbf{d}(\theta_{i,1}) = \frac{d\mathbf{a}(\theta)}{d\theta}|_{\theta=\theta_{i,1}}$, we have

$$\begin{aligned}\mathbf{a}(\theta_{i,j})\mathbf{a}^H(\theta_{i,j}) &\approx \mathbf{a}(\theta_{i,1})\mathbf{a}^H(\theta_{i,1}) + \Delta\theta_{i,j}\mathbf{E}_i, \\ & \qquad i = 1, ..., K,\ j = 2, ..., L_i\end{aligned} \quad (3)$$

where the second-order term of $\Delta\theta_{i,j}$ has been neglected and

$$\mathbf{E}_i = \mathbf{a}(\theta_{i,1})\mathbf{d}^H(\theta_{i,1}) + \mathbf{d}(\theta_{i,1})\mathbf{a}^H(\theta_{i,1}) \quad (4)$$

By substituting (3) into (1), the covariance matrix $\mathbf{R}$ can be rewritten as

$$\begin{aligned}\mathbf{R} &= \sum_{i=1}^{K}\sigma_i^2\mathbf{a}(\theta_{i,1})\mathbf{a}^H(\theta_{i,1}) + \sum_{i=1}^{K}\epsilon_i\mathbf{E}_i + \sigma_n^2\mathbf{I} \\ &= \mathbf{A}_1\mathbf{B}_1\mathbf{A}_1^H + \mathbf{E} + \sigma_n^2\mathbf{I}\end{aligned} \quad (5)$$

where $\mathbf{A}_1 = [\mathbf{a}(\theta_{1,1}), ..., \mathbf{a}(\theta_{K,1})]$, $\mathbf{B}_1 = \mathrm{diag}\{\sigma_1^2, ..., \sigma_K^2\}$, $\mathbf{E} = \sum_{i=1}^{K}\epsilon_i\mathbf{E}_i$, in which

$$\sigma_i^2 = \sum_{j=1}^{L_i}\sigma_{i,j}^2 \ \text{ and } \ \epsilon_i = \sum_{j=2}^{L_i}\sigma_{i,j}^2\Delta\theta_{i,j} \quad (6)$$

Since $\Delta\theta_{i,j}$ is very small, $\sigma_i^2 \gg \epsilon_i$. Therefore, $\mathbf{E}_i$ and ϵ_i can be regarded as a perturbation matrix and power, respectively. Eq. (5) implies that the signal subspace of $\mathbf{R}$ is spanned by K desired signals but perturbed by $\mathbf{E}_i$ with the perturbation power depending on the power of the interferences and the distance between the DOAs of the desired source and interferences.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_P$ and $\mathbf{v}_1, ..., \mathbf{v}_P$ denote the eigenvalues and the associated eigenvectors of $\mathbf{R}$. Since there are K sources in the received data, we can obtain that $\lambda_i > \sigma_n^2$ for $i = 1, ..., K$ and $\lambda_i \approx \sigma_n^2$ for $i = K+1, ..., P$. Also, the signal and noise subspaces of $\mathbf{R}$ are spanned by $\mathbf{V}_s = \{\mathbf{v}_1, ..., \mathbf{v}_K\}$ and $\mathbf{V}_n = \{\mathbf{v}_{K+1}, ..., \mathbf{v}_P\}$, respectively. Furthermore, the steering vectors, $\mathbf{a}(\theta_{i,1})$, $i = 1, ..., K$, lie on the signal subspace, so those steering vectors and the noise subspace are orthogonal.

In practice, $\mathbf{R}$ is unknown and can be estimated by

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t) \quad (7)$$

where N is the number of snapshots. By taking the eigendecomposition of $\hat{\mathbf{R}}$ renders

$$\hat{\mathbf{R}} = \hat{\mathbf{V}}_\mathbf{s}\hat{\mathbf{\Lambda}}_\mathbf{s}\hat{\mathbf{V}}_\mathbf{s}^H + \hat{\mathbf{V}}_\mathbf{n}\hat{\mathbf{\Lambda}}_\mathbf{n}\hat{\mathbf{V}}_\mathbf{n}^H \quad (8)$$

where the column vectors of $\hat{\mathbf{V}}_s$ and $\hat{\mathbf{V}}_n$ are respectively the eigenvectors which span the signal subspace and the noise subspace of $\hat{\mathbf{R}}$ with the associated eigenvalues on the diagonals of $\hat{\mathbf{\Lambda}}_\mathbf{s}$ and $\hat{\mathbf{\Lambda}}_\mathbf{n}$. Furthermore, the pseudospectrum of the MUSIC algorithm can be defined as

$$f(\theta) = \mathbf{a}^H(\theta)\left(\mathbf{I} - \hat{\mathbf{V}}_\mathbf{s}\hat{\mathbf{V}}_\mathbf{s}^H\right)\mathbf{a}(\theta) \quad (9)$$

The estimates of $\theta_{i,1}$ are obtained by selecting the minima of $f(\theta)$ [2].

3. PERFORMANCE ANALYSIS

Based on the estimated covariance matrix, $\hat{\mathbf{R}}$ given in (7), we consider the errors of the DOA estimates caused by the finite sample effect and the perturbation of the interferences. We refer to the former as $\Delta\theta_{\mathrm{n},i,1}$ and the latter as $\Delta\theta_{\mathrm{r},i,1}$.

First, we consider the errors of the DOA estimates caused by the finite sample effect. By neglecting the perturbation (the second term in (5)), the MUSIC estimation errors, $\Delta\theta_{\mathrm{n},i,1}$, is asymptotically Gaussian distributed with zero mean ($E[\Delta\theta_{\mathrm{n},i,1}] = 0$) and variance given by [3]

$$\begin{aligned}E[(\Delta\theta_{\mathrm{n},i,1})^2] &\approx \frac{1}{2N\cdot\mathrm{SNR}_i}\left[1 + \frac{(\mathbf{A}_1^H\mathbf{A}_1)_{i,i}^{-1}}{\mathrm{SNR}_i}\right]/h(\theta_{i,1}) \\ & \qquad i = 1, ..., K\end{aligned} \quad (10)$$

where $(.)_{i,i}$ denotes the $(i,i)^{th}$ element of the embraced matrix, $\mathrm{SNR}_i = \sigma_i^2/\sigma_n^2$, and

$$h(\theta_{i,1}) = \mathbf{d}^H(\theta_{i,1})[\mathbf{I} - \mathbf{A}_1(\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H]\mathbf{d}(\theta_{i,1}) \quad (11)$$

Since $\theta_{i,1}{}'s$ are well separated, $|\mathbf{a}_{i,1}^H\mathbf{a}_{j,1}| \ll P,\ i,j = 1, ..., K$. Hence, the determinant of $\mathbf{A}_1^H\mathbf{A}_1$ and the cofactor of the $(i,i)^{th}$ element of $\mathbf{A}_1^H\mathbf{A}_1$ are approximately equal to P^K and P^{K-1}, respectively. Therefore, $(\mathbf{A}_1^H\mathbf{A}_1)_{i,i}^{-1} \approx 1/P$ and $E[(\Delta\theta_{\mathrm{n},i,1})^2]$ can be simplified as

$$E[(\Delta\theta_{\mathrm{n},i,1})^2] \approx \frac{1}{2N\cdot\mathrm{SNR}_i}\left[1 + \frac{1}{P\cdot\mathrm{SNR}_i}\right]/h(\theta_{i,1}) \quad (12)$$

Next, we derive the errors of the DOA estimates caused by the perturbation of the interferences, $\Delta\theta_{\mathrm{r},i,1}$. Taking the eigendecomposition of $\mathbf{A}_1\mathbf{B}_1\mathbf{A}_1^H$ given in (5) renders

$$\mathbf{A}_1\mathbf{B}_1\mathbf{A}_1^H = \begin{bmatrix}\mathbf{V}_{\mathbf{ss},1} & \mathbf{V}_{\mathbf{sn},1}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}_{\mathbf{ss},1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{\mathbf{ss},1}^H \\ \mathbf{V}_{\mathbf{sn},1}^H\end{bmatrix} \quad (13)$$

where the column space of $\mathbf{V}_{\mathbf{ss},1} = [\mathbf{v}_{\mathbf{ss},1}, ..., \mathbf{v}_{\mathbf{ss},K}]$ is the signal subspace of $\mathbf{A}_1\mathbf{B}_1\mathbf{A}_1^H$, the column space of $\mathbf{V}_{\mathbf{sn},1} = [\mathbf{v}_{\mathbf{sn},1}, ..., \mathbf{v}_{\mathbf{sn},(P-K)}]$ is the orthogonal complement of $\mathbf{V}_{\mathbf{ss},1}$, and $\mathbf{\Lambda}_{\mathbf{ss},1} = \mathrm{diag}\{\bar{\lambda}_1, ..., \bar{\lambda}_K\}$ is composed of the eigenvalues corresponding to $\mathbf{v}_{\mathbf{ss},1}, ..., \mathbf{v}_{\mathbf{ss},K}$, respectively. If the perturbation $\mathbf{E}$ is taken into account, we denote the perturbed $\mathbf{V}_{\mathbf{ss},1}$ by $\mathbf{V}_{\mathbf{ss},1}(\mathbf{E}) = [\mathbf{v}_{\mathbf{ss},1}(\mathbf{E}), ..., \mathbf{v}_{\mathbf{ss},K}(\mathbf{E})]$. After taking the Taylor series expansion of each column of $\mathbf{V}_{\mathbf{ss},i}(\mathbf{E})$ and neglecting the higher-order terms, we can obtain [11]

$$\mathbf{v}_{\mathbf{ss},i}(\mathbf{E}) \approx \mathbf{v}_{\mathbf{ss},i} + \left(\sum_{j=1,j\neq i}^{K}\alpha_{i,j}\mathbf{v}_{\mathbf{ss},j} + \sum_{k=1}^{P-K}\beta_{i,k}\mathbf{v}_{\mathbf{sn},k}\right) \quad (14)$$

for $i = 1, ..., K$, where

$$\alpha_{i,j} = \frac{\mathbf{v}_{\mathbf{ss},j}^H\mathbf{E}\mathbf{v}_{\mathbf{ss},i}}{\bar{\lambda}_i - \bar{\lambda}_j} \ \text{ and } \ \beta_{i,k} = \frac{\mathbf{v}_{\mathbf{sn},k}^H\mathbf{E}\mathbf{v}_{\mathbf{ss},i}}{\bar{\lambda}_i} \quad (15)$$

By substituting (4) into (15), $\beta_{i,k}$ can be re-expressed as

$$\beta_{i,k} = \frac{\mathbf{v}_{\mathbf{sn},k}^H\sum_{j=1}^{K}\epsilon_j\mathbf{d}(\theta_{j,1})\mathbf{a}^H(\theta_{j,1})\mathbf{v}_{\mathbf{ss},i}}{\bar{\lambda}_i} \quad (16)$$

where we have used the fact that $\mathbf{v}_{\mathbf{sn},k}^H\mathbf{a}(\theta_{j,1}) = 0$.

We can note from (5) that the perturbation power ϵ_i depends on the distance between $\theta_{i,1}$ and $\theta_{i,j},\ j = 2, ..., L_i$ and the power of the interferences, $\sigma_{i,j}^2,\ j = 2, ..., L_i$. Hence, the closer the DOAs of the desired source and the interferences, the smaller the perturbation power. When $\Delta\theta_{i,j}$ equals zero ($\theta_{i,1} = \theta_{i,j}$), the perturbation power ϵ_i becomes zero, and then the DOA estimation of the desired source will not be influenced by the interferences.

Also, taking the derivative of (9) and setting it to zero yields

$$2\mathrm{Re}\left\{\mathbf{a}^H(\hat{\theta}_{i,1})\left(\mathbf{I} - \mathbf{V}_{\mathbf{ss},1}(\mathbf{E})\mathbf{V}_{\mathbf{ss},1}^H(\mathbf{E})\right)\mathbf{d}(\hat{\theta}_{i,1})\right\} = 0 \quad (17)$$

Using the Taylor series expansion $\mathbf{a}(\hat{\theta}_{i,1}) = \mathbf{a}(\theta_{i,1}) + \Delta\theta_{\mathrm{r},i,1}\mathbf{d}(\theta_{i,1})$ and the approximation that $\mathbf{d}(\hat{\theta}_{i,1}) \approx \mathbf{d}(\theta_{i,1})$ [3], we can obtain from (17) that

$$\begin{aligned}\Delta\theta_{\mathrm{r},i,1} &\approx -\frac{\mathrm{Re}\left\{\mathbf{a}(\theta_{i,1})^H\left(\mathbf{I} - \mathbf{V}_{\mathbf{ss},1}(\mathbf{E})\mathbf{V}_{\mathbf{ss},1}^H(\mathbf{E})\right)\mathbf{d}(\theta_{i,1})\right\}}{\mathbf{d}^H(\theta_{i,1})\left(\mathbf{I} - \mathbf{V}_{\mathbf{ss},1}(\mathbf{E})\mathbf{V}_{\mathbf{ss},1}^H(\mathbf{E})\right)\mathbf{d}(\theta_{i,1})} \\ &\approx -\frac{\mathrm{Re}\left\{\mathbf{a}(\theta_{i,1})^H\left(\mathbf{I} - \mathbf{V}_{\mathbf{ss},1}(\mathbf{E})\mathbf{V}_{\mathbf{ss},1}^H(\mathbf{E})\right)\mathbf{d}(\theta_{i,1})\right\}}{h(\theta_{i,1})} \\ &\qquad i = 1, ..., K \end{aligned} \quad (18)$$

where $h(\theta_{i,1})$ is given in (11) and we have used the fact that $\mathbf{I} - \mathbf{V}_{\mathbf{ss},1}(\mathbf{E})\mathbf{V}_{\mathbf{ss},1}^H(\mathbf{E}) \approx \mathbf{I} - \mathbf{A}_1(\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H$.

Based on the fact that $\mathbf{a}(\theta_{i,1})$ and $\mathbf{v}_{\mathbf{sn},k}$ are orthogonal, by substituting (14) into (18) and after some manipulations $\Delta\theta_{\mathrm{r},i,1}$ can be expressed as

$$\Delta\theta_{\mathrm{r},i,1} = \frac{\mathrm{Re}\left\{\mathbf{a}^H(\theta_{i,1})\left(\sum_{j=1}^{K}\sum_{k=1}^{P-K}\beta_{j,k}^H\mathbf{v}_{\mathbf{ss},j}\mathbf{v}_{\mathbf{sn},k}^H\right)\mathbf{d}(\theta_{i,1})\right\}}{h(\theta_{i,1})} \quad (19)$$

By substituting $\beta_{j,k}$ given in (16) into (19), (19) can be reduced to

$$\Delta\theta_{\mathrm{r},i,1} = \frac{\sum_{j=1}^{K}\epsilon_j\mu_{i,j}\nu_{i,j}}{h(\theta_{i,1})} \quad (20)$$

where

$$\mu_{i,j} \triangleq \mathbf{a}^H(\theta_{i,1})\mathbf{V}_{\mathbf{ss},1}\mathbf{\Lambda}_{\mathbf{ss},1}^{-1}\mathbf{V}_{\mathbf{ss},1}^H\mathbf{a}(\theta_{j,1}) \quad (21)$$

and

$$\nu_{i,j} \triangleq \mathbf{d}^H(\theta_{j,1})[\mathbf{I} - \mathbf{A}_1(\mathbf{A}_1^H\mathbf{A}_1)^{-1}\mathbf{A}_1^H]\mathbf{d}(\theta_{i,1}) \quad (22)$$

Based on the fact that $(\mathbf{A}_1^H\mathbf{V}_{\mathbf{ss},1})\mathbf{\Lambda}_{\mathbf{ss},1}^{-1}(\mathbf{V}_{\mathbf{ss},1}^H\mathbf{A}_1) = \mathbf{B}_1^{-1}$ [1], where $\mathbf{B}_1$ is as given in (5), $\mu_{i,j}$ becomes

$$\mu_{i,j} \approx \begin{cases} (\sigma_i^2)^{-1} & , \quad i = j \\ 0 & , \quad i \neq j \end{cases} \quad (23)$$

By substituting (23) into (20), (20) can be re-written as

$$\Delta\theta_{\mathrm{r},i,1} = \frac{\epsilon_i}{\sigma_i^2} \quad (24)$$

Finally, by substituting (6) into (24), $\Delta\theta_{\mathrm{r},i,1}$ can be re-expressed as

$$\Delta\theta_{\mathrm{r},i,1} = \frac{\sum_{j=2}^{L_i}\Delta\theta_{i,j}\sigma_{i,j}^2}{\sum_{j=1}^{L_i}\sigma_{i,j}^2} \quad (25)$$

Summarizing the above, we have the following proposition:

Proposition :

The DOA estimates of the MUSIC for closely spaced signals are biased, and their bias and MSE are given, respectively, by

$$\mathrm{BIAS}(\hat{\theta}_{i,1}) \approx \frac{\sum_{j=2}^{L_i}\Delta\theta_{i,j}\sigma_{i,j}^2}{\sum_{j=1}^{L_i}\sigma_{i,j}^2}, \quad i = 1, \ldots, K \quad (26)$$

and

$$\mathrm{MSE}(\hat{\theta}_{i,1}) \approx \frac{1}{2N\cdot\mathrm{SNR}_i}\left[1 + \frac{1}{P\cdot\mathrm{SNR}_i}\right]/h(\theta_{i,1}) + \left(\frac{\sum_{j=2}^{L_i}\Delta\theta_{i,j}\sigma_{i,j}^2}{\sum_{j=1}^{L_i}\sigma_{i,j}^2}\right)^2, \quad i = 1, \ldots, K \quad (27)$$

where N, P, and L_i are the numbers of the snapshots, the antennas, and the source(s) in the i^{th} cluster, respectively, SNR_i is the ratio of the total signal power in the i^{th} cluster to noise power, $\Delta\theta_{i,j}$ is the distance between the DOAs of the desired source and the j^{th} interference in the i^{th} cluster, and $\sigma_{i,j}^2$ is the power of the j^{th} source in the i^{th} cluster.

Based on the above, we have the following observations:

1) If there is only one source in each cluster, the bias caused by the perturbation is negligible, and the estimator is unbiased and Eq. (27) is reduced to Eq. (7.7a) in [3].

2) If there is more than one source in the cluster, the estimator will become biased. We can note from (26) that the bias of the DOA estimate of the desired source only depend on the perturbation caused by the interferences in the same cluster, but is independent of those in the other clusters. Furthermore, the bias is proportional to the inverse of the total power of the sources in the same cluster, the product of the power of the interferences, and the distance between the DOAs of the desired source and the interference, as shown in (26). Also, we can note from (26) that the bias is independent of the number of the antennas P. In the equal-power scenario, the bias is reduced to

$$\mathrm{BIAS}(\hat{\theta}_{i,1}) \approx \frac{L_i - 1}{L_i}\sum_{j=2}^{L_i}\Delta\theta_{i,j} \quad (28)$$

which implies that the bias increases as L_i increases, and is proportional to the summation of the distance between the DOAs of the desired source and the interferences. In particular, if there are two close sources ($L_i = 2$), the bias is equal to half of their distance, which is consistent with the observation in [5, 6].

3) We can note from (27) that the MSE caused by the finite samples, $E[(\Delta\theta_{\mathrm{n},i,1})^2]$, is proportional to both the inverse of the number of snapshots N and the SNR_1. In contrast, the MSE caused by the perturbation, $E[(\Delta\theta_{\mathrm{r},i,1})^2]$ is independent of the number of snapshots N and the SNR_1. Therefore, $E[(\Delta\theta_{\mathrm{n},i,1})^2]$ can be neglected in high-SNR or large-M scenarios.

4. SIMULATIONS AND DISCUSSION

Simulations are conducted in this section to verify the derived analytic expressions. Assume that there are three clusters of sources with first cluster consisting of two sources, the second cluster three sources, and the third cluster one source. The impinging signals are received by a six-element uniform linear array which spaced a half wavelength apart. The DOAs of the first cluster are $[10, 10 + \Delta\theta]^o$, those of the second cluster is $[40,\ 41,\ 42]^o$, and that of the third cluster is 80^o, where $\Delta\theta = 0.2,\ 0.4,\ 0.6,\ 0.8$. The average power of all sources is equal. 32 snapshots are employed to estimate the covariance matrix. For each specific SNR, 200 Monte Carlo trials are carried out. For a clear illustration, only the bias and root-MSE (*RMSE*) of the DOA estimates of the first source in the first cluster are provided, as shown in Figs. 1 and 2, respectively.

We can note from Fig. 1 that the bias of the DOA estimates equals half of the $\Delta\theta$ in high-SNR scenarios, but is irregular in low-SNR scenarios, as the noise is more pronounced then. Also, we can note from Fig. 2 that the RMSE is proportional to the inverse of the SNR and matches the theoretical value derived above and in

[3], which decreases as the SNR increases in low-SNR scenarios. Also, the RMSE approach a constant and equal the theoretical value desired above in high-SNR scenarios. This is due to the fact that the first term in (27), which is caused by the additive noise, is larger than the second term, which is caused by the perturbation, in low-SNR scenarios. However, the first term decreases as SNR increases, while the second term remains to be constant for all SNRs. Therefore, the second term in (27) can be neglected in low-SNR, but not so in high-SNR scenarios. The expressions developed in [3] only account for the first term in (27) and thus are not applicable in high-SNR scenarios, as shown in Fig. 2.

Finally, we consider a non-equal power scenario, which has the same settings as the above except that $\Delta\theta = 1$ and the power ratio of the sources 2 to 1 in the first cluster is varied from 0 to 1. Figs. 3 compares the bias of the DOA estimates by simulations and the analytic expressions with SNR=10 dB. We can observe from Fig. 3 that the bias increases as the power of source 2 increases. When the power of source 2 equals that of source 1, the bias will be equal to half of the distance between the DOAs of sources 1 and 2, which is consistent with (28).

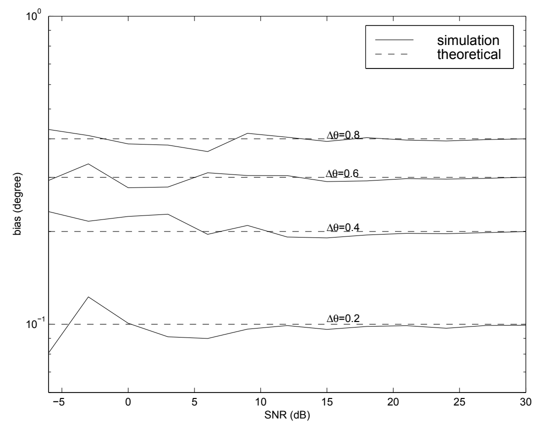

Fig. 1. Comparison of the bias of the DOA estimates based on simulation and theoretical values of (26).

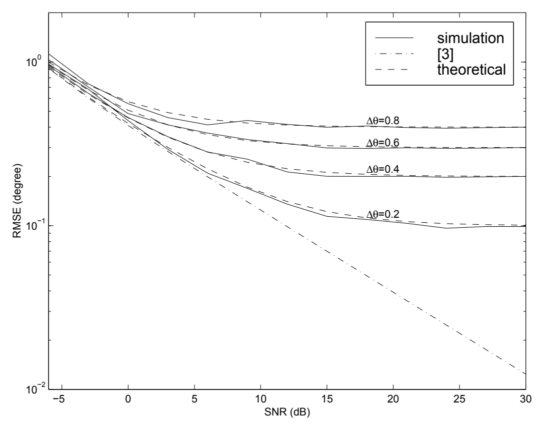

Fig. 2. Comparison of the RMSE of the DOA estimates based on simulation and theoretical values of (27).

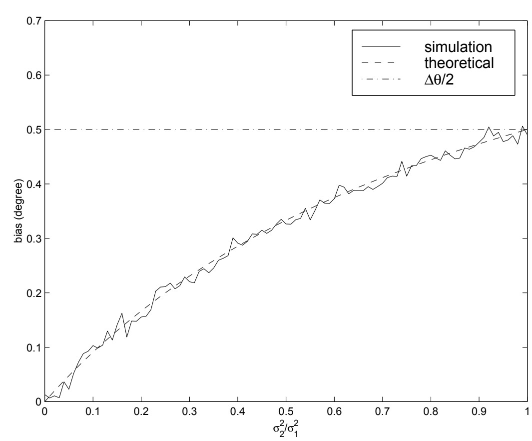

Fig. 3. Comparison of the bias of the DOA estimates based on simulation and theoretical values of (26) in non-equal power scenario.

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