RESOLUTION THRESHOLD FOR CLOSELY SPACED NONCIRCULAR EMITTERS

H. Abeida and J.P. Delmas

GET/INT, Département CITI, UMR-CNRS 5157, Evry, France

ABSTRACT

This paper addresses the resolution of the standard and noncircular MUSIC algorithms for arbitrary distribution and noncircularity of two closely spaced transmitters. Using an analysis based on perturbations of the noise projector instead of those of the eigenvectors, interpretable closed-form expressions of the threshold array signal to noise ratios (ASNR) at which these two algorithms are able to resolve the transmitters along the Cox and the Sharman and Durrani criteria are given. We prove in particular that the threshold ASNRs given by the noncircular MUSIC algorithm are sensitive to the noncircularity phase separation of the sources and are comfortably smaller that those given by the standard MUSIC algorithm. Numerical examples illustrate these results.

Index Terms— direction of arrival, statistical performances, resolution, noncircular sources, MUSIC algorithm

1. INTRODUCTION

Deducing array resolution limits is a very old problem that has been studied extensively in the literature, first in astronomy and subsequently in signal processing. Based on the classical beamformer, different resolution criteria have been defined from the main lobe of the array spectrum as the celebrated Rayleigh resolution that depends solely on the antenna geometry. Then for specific so-called high resolution algorithms based on the search for two local minima of directional spectra such as different MUSIC-like algorithms, two main criteria based on the mean spectrum have been defined. For the first, introduced by Cox [1] and then studied by Kaveh and Barabell [2], two sources are resolved if the midpoint mean spectrum is greater than the mean spectrum in the two true source DOAs and for the second one, introduced by Sharman and Durrani [3] and studied by Forster and Villier [4], they are resolved if the second derivative of the mean spectrum at the midpoint is negative. Moreover, several authors have considered (e.g., [5, 6]) the resolution probability to circumvent the possible misleading results given by these two criteria.

We note that all these studies have been obtained under a circular Gaussian distribution of signals. The aim of this paper is to extend some of these previous results under arbitrary second-order distributions, with a particular attention to noncircular signals often used in digital communications. More precisely, we consider the two resolution criteria based on mean spectra associated with the standard MUSIC algorithm and with a MUSIC-like (denoted noncircular MUSIC) algorithm introduced and studied in [7] which is an extension of a root MUSIC-like algorithm devised in [8] to an arbitrary array that benefits from the second-order noncircularity of the sources.

The paper is organized as follows. The array signal model and the statement of the problem are given in Section II. Using an analysis based on perturbations of the noise projector instead of those of the eigenvectors, we prove in section III that the resolution threshold expressions given by the standard MUSIC algorithm for circular Gaussian sources impinging on a uniform linear array (ULA) in [2] and [4] extend to arbitrary circular or noncircular source distributions and arbitrary arrays. This analysis is applied in Section IV, to derive closed-form expressions of the resolution thresholds associated with the mean spectrum of the noncircular MUSIC algorithm associated with ULAs. These expressions confirm that the noncircular MUSIC algorithm largely outperforms the standard MUSIC one from the resolution point of view. Finally, numerical illustrations and Monte Carlo simulations are given in Section V with particular attention paid to the noncircularity phase separation.

2. STATEMENT OF THE PROBLEM

Let an arbitrary array of M sensors receive the signals transmitted by two equipowered narrowband independent sources of power σ_s^2. The observation vectors are modelled as

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \qquad t = 1, \ldots, T,$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ are independent and identically distributed. $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2]$ is the steering matrix where each vector $\mathbf{a}_k = \mathbf{a}(\theta_k)$ is parameterized by the real scalar parameter θ_k. $\mathbf{x}_t = (x_{t,1}, x_{t,2})^T$ and $\mathbf{n}_t$ model signals transmitted by sources and additive measurement noise, respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are independent, zero-mean, $\mathbf{n}_t$ is assumed to be Gaussian complex circular, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2 \mathbf{I}_M$; while $\mathbf{x}_t$ is complex noncircular not necessarily Gaussian with covariance matrices $\mathbf{R}_x \stackrel{\mathrm{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x \stackrel{\mathrm{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^T) \neq \mathbf{O}$. Consequently, this leads to two covariance matrices of $\mathbf{y}_t$ that contain information about (θ_1, θ_2)

and
$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \stackrel{\mathrm{def}}{=} \mathbf{S} + \sigma_n^2\mathbf{I}_M$$
$$\mathbf{R}'_y = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T \neq \mathbf{O}.$$

These covariance matrices are classically estimated by $\mathbf{R}_{y,T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t\mathbf{y}_t^H$ and $\mathbf{R}'_{y,T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t\mathbf{y}_t^T$, respectively.

For performance analysis, we suppose that the signal waveforms have finite fourth-order moments $\kappa_k \stackrel{\text{def}}{=} \text{Cum}(x_{t,k}, x_{t,k}^*, x_{t,k}, x_{t,k}^*)$. The non-circularity rate ρ_k of the kth source is defined by $\text{E}(x_{t,k}^2) = \rho_k e^{i\phi_k}\text{E}|x_{t,k}^2| = \rho_k e^{i\phi_k}\sigma_s^2$ where ϕ_k is its noncircularity phase. Note that $\rho_k = 1$ in the particular case of rectilinear signals. The ASNR is defined by $M\sigma_s^2/\sigma_n^2$ with $\|\mathbf{a}_k\|^2 = M$.

The problem addressed in this paper is to derive in these conditions, resolution threshold expressions associated with the standard and noncircular MUSIC algorithms. The DOA estimated by the standard MUSIC algorithm are given by the 2 smallest minima of the following so-called spectrum $g_T^{\text{Alg}_\text{C}}(\theta)$:

$$\widehat{\theta}_{k,T}^{\text{Alg}_\text{C}} = \arg\min_\theta g_T^{\text{Alg}_\text{C}}(\theta)$$

with

$$g_T^{\text{Alg}_\text{C}}(\theta) \stackrel{\text{def}}{=} \mathbf{a}^H(\theta)\mathbf{\Pi}_T\mathbf{a}(\theta),$$

where $\mathbf{\Pi}_T$ denotes the projector matrix associated with the noise subspace of $\mathbf{R}_{y,T}$. Then, for the noncircular MUSIC algorithms devised for rectilinear signals[1], the estimated DOA are given by the 2 smallest minima of the following so-called spectrum $g_T^{\text{Alg}_\text{NC}}(\theta)$:

$$\widehat{\theta}_{k,T}^{\text{Alg}_\text{NC}} = \arg\min_\theta g_T^{\text{Alg}_\text{NC}}(\theta)$$

with [7]

$$g_T^{\text{Alg}_\text{NC}}(\theta) \stackrel{\text{def}}{=} \left(\mathbf{a}^H(\theta)\mathbf{\Pi}_{1,T}\mathbf{a}(\theta)\right)^2 - |\mathbf{a}^T(\theta)\mathbf{\Pi}_{2,T}\mathbf{a}(\theta)|^2, \quad (1)$$

where $\mathbf{\Pi}_{1,T}$ and $\mathbf{\Pi}_{2,T}$ are Hermitian and complex symmetric respectively, given by the projector matrix

$$\tilde{\mathbf{\Pi}}_T = \begin{pmatrix} \mathbf{\Pi}_{1,T} & \mathbf{\Pi}_{2,T} \\ \mathbf{\Pi}_{2,T}^* & \mathbf{\Pi}_{1,T}^* \end{pmatrix}$$

associated with the noise subspace of $\mathbf{R}_{\tilde{y},T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H$ with $\tilde{\mathbf{y}}_t$ is the extended observation $\begin{pmatrix} \mathbf{y}_t \\ \mathbf{y}_t^* \end{pmatrix}$ for which

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \text{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H) = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}_{2M} \stackrel{\text{def}}{=} \tilde{\mathbf{S}} + \sigma_n^2\mathbf{I}_{2M}$$

with $\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix}$ and $\mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{R}_x & \mathbf{R}_x^{'} \\ \mathbf{R}_x^{'*} & \mathbf{R}_x^* \end{pmatrix}$.

3. RESOLVING POWER OF STANDARD MUSIC

Since it is proved in [2] that the bias $\text{E}[g_T^{\text{Alg}_\text{C}}(\theta)]$ is substantially larger that the standard deviation $\sqrt{\text{Var}[g_T^{\text{Alg}_\text{C}}(\theta)]}$, it is reasonable to use the Cox [1] and Sharman and Durrani criteria [3] for which two closely spaced sources are resolved if the following respective conditions are satisfied:

$$\text{E}[g_T^{\text{Alg}}(\theta_1)] = \text{E}[g_T^{\text{Alg}}(\theta_2)] \;\leq\; \text{E}[g_T^{\text{Alg}}(\theta_m)] \quad (2)$$

[1] noncircular with unit rate of noncircularity, i.e., $\rho_k = 1$.

$$\frac{d^2\text{E}[g_T^{\text{Alg}}(\theta)]}{d\theta^2}_{|\theta=\theta_m} \leq 0 \quad (3)$$

where $\theta_m \stackrel{\text{def}}{=} \frac{\theta_1+\theta_2}{2}$. Approximations to the resolution threshold are deduced from equalities in (2) and (3). Consequently, the key point to derive these resolution thresholds depends on the expectation of the random variable $g_T^{\text{Alg}_\text{C}}(\theta)$. To obtain this expectation, we resort to an analysis based on perturbations of the noise projector [9] instead of those of the eigenvectors. Therefore, we consider the following second-order expansion of $\delta\mathbf{\Pi}_T \stackrel{\text{def}}{=} \mathbf{\Pi}_T - \mathbf{\Pi}$ w.r.t. $\delta\mathbf{R}_{y,T} \stackrel{\text{def}}{=} \mathbf{R}_{y,T} - \mathbf{R}_y$ proved in [9]

$$\begin{aligned} \delta\mathbf{\Pi}_T &= -(\mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{S}^\# + \mathbf{S}^\#\delta\mathbf{R}_{y,T}\mathbf{\Pi}) \\ &+ \mathbf{S}^\#\delta\mathbf{R}_{y,T}\mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{S}^\# - \mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{S}^{\#2}\delta\mathbf{R}_{y,T}\mathbf{\Pi} \\ &+ \mathbf{S}^\#\delta\mathbf{R}_{y,T}\mathbf{S}^\#\delta\mathbf{R}_{y,T}\mathbf{\Pi} + \mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{S}^\#\delta\mathbf{R}_{y,T}\mathbf{S}^\# \\ &- \mathbf{S}^{\#2}\delta\mathbf{R}_{y,T}\mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{\Pi} - \mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{\Pi}\delta\mathbf{R}_{y,T}\mathbf{S}^{\#2} \\ &+ o(\delta\mathbf{R}_{y,T}^2). \end{aligned} \quad (4)$$

To proceed, we need the expression of $\text{E}(\delta\mathbf{R}_{y,T}\mathbf{B}\delta\mathbf{R}_{y,T})$ for arbitrary $M\times M$ matrices $\mathbf{B}$. Using simple algebraic manipulations of $\text{E}(\delta\mathbf{R}_{y,T}^T\otimes\delta\mathbf{R}_{y,T})$ with $\text{E}[\text{vec}(\mathbf{y}_t\mathbf{y}_t^H)\text{vec}^H(\mathbf{y}_t\mathbf{y}_t^H)] - \text{vec}(\mathbf{R}_y)\text{vec}^H(\mathbf{R}_y) = \mathbf{R}_y^*\otimes\mathbf{R}_y + \mathbf{K}(\mathbf{R}_y^{'}\otimes\mathbf{R}_y^{'*}) + (\mathbf{A}^*\otimes\mathbf{A})(\sum_{k=1}^{2}\kappa_k(\mathbf{e}_{2,k}\otimes\mathbf{e}_{2,k})(\mathbf{e}_{2,k}^T\otimes\mathbf{e}_{2,k}^T))(\mathbf{A}^T\otimes\mathbf{A}^H)$ where $\mathbf{K}$ is the vec-permutation matrix for which $\text{vec}(\mathbf{C}^T) = \mathbf{K}\text{vec}(\mathbf{C})$, $\mathbf{e}_{2,1} \stackrel{\text{def}}{=} (1,0)^T$ and $\mathbf{e}_{2,2} \stackrel{\text{def}}{=} (0,1)^T$, we obtain the following lemma:

Lemma 1 *For independent arbitrary noncircular, possibly non Gaussian sources, we have:*

$$\begin{aligned} \text{E}(\delta\mathbf{R}_{y,T}\mathbf{B}\delta\mathbf{R}_{y,T}) &= \frac{1}{T}\Big(\text{Tr}(\mathbf{B}\mathbf{R}_y)\mathbf{R}_y + \mathbf{R}'_y\mathbf{B}^T\mathbf{R}'^*_y \\ &+ \sum_{k=1}^{K}\kappa_k\mathbf{a}_k\mathbf{a}_k^H\mathbf{B}\mathbf{a}_k\mathbf{a}_k^H\Big) + o(\frac{1}{T}), \end{aligned}$$

that allows us to prove [11] from (4)

$$\text{E}(\delta\mathbf{\Pi}_T) = \frac{1}{T}\left(\text{Tr}(\mathbf{\Pi})\mathbf{U} - \text{Tr}(\mathbf{U})\mathbf{\Pi}\right) + o(\frac{1}{T}),$$

with $\mathbf{U} \stackrel{\text{def}}{=} \sigma_n^2\mathbf{S}^\#\mathbf{R}_y\mathbf{S}^\#$. Consequently, we have

$$\begin{aligned} \text{E}(g_T^{\text{Alg}_\text{C}}(\theta)) &= g^{\text{Alg}_\text{C}}(\theta) + \frac{1}{T}\Big((M-2)\mathbf{a}^H(\theta)\mathbf{U}\mathbf{a}(\theta) \\ &- \text{Tr}(\mathbf{U})g^{\text{Alg}_\text{C}}(\theta)\Big) + o(\frac{1}{T}) \end{aligned}$$

where $g^{\text{Alg}_\text{C}}(\theta) \stackrel{\text{def}}{=} \mathbf{a}^H(\theta)\mathbf{\Pi}\mathbf{a}(\theta)$. This expression of the mean spectrum coincides with those given in the circular Gaussian assumption [2, 4]. Therefore we can conclude the following result:

Result 1 *The threshold ASNRs deduced from the Cox* (2) *and Sharman and Durrani criteria* (3) *given for the standard MUSIC algorithm do not depend on the distribution and on the noncircularity of the sources.*

Consequently, expressions [2, (rel.(35)], [4, rel.(18)] and [10, rels.(91)(93)] of the threshold ASNRs remain valid for arbitrary distributions of the sources. The first two expressions[2] of this threshold ASNR are given in the following to be compared to those derived in the next section.

$$\xi_1 = \frac{1}{T}\frac{\alpha_{1,M}}{(\Delta\theta)^4}\left(1+\sqrt{1+\frac{T(\Delta\theta)^2}{\beta_{1,M}}}\right) \quad (5)$$

$$\xi_2 = \frac{1}{T}\frac{\alpha_{2,M}}{(\Delta\theta)^4}\left(1+\sqrt{1+\frac{T(\Delta\theta)^2}{\beta_{2,M}}}\right) \quad (6)$$

with $\alpha_{1,M} \stackrel{\text{def}}{=} \frac{20M^2}{(M^2-1)(M+2)}$, $\beta_{1,M} \stackrel{\text{def}}{=} \frac{5}{M+2}$, $\alpha_{2,M} \stackrel{\text{def}}{=} \frac{10M^4}{(M+2)(M^2-1)}$, $\beta_{2,M} \stackrel{\text{def}}{=} \frac{5M^2}{2(M+2)}$, for which $\Delta\theta \stackrel{\text{def}}{=} M(\theta_1-\theta_2)/2\sqrt{3}$[3] associated with the steering vectors $\mathbf{a}_k = (1, e^{i\theta_k}, ..., e^{i(M-1)\theta_k})^T$.

4. RESOLVING POWER OF NONCIRCULAR MUSIC

The previous approach applies to the noncircular MUSIC algorithm by replacing $\mathbf{\Pi}$, $\mathbf{R}_{y,T}$ and $\mathbf{S}$ by $\tilde{\mathbf{\Pi}}$, $\mathbf{R}_{\tilde{y},T}$ and $\tilde{\mathbf{S}}$ in (4) respectively. Using the following lemma proved in the same way as lemma 1

Lemma 2 *For independent rectilinear, possibly non Gaussian sources, we have:*

$$\begin{aligned}\mathrm{E}(\delta\mathbf{R}_{\tilde{y},T}\mathbf{B}\delta\mathbf{R}_{\tilde{y},T}) &= \frac{1}{T}\Big(\mathrm{Tr}(\mathbf{B}\mathbf{R}_{\tilde{y}})\mathbf{R}_{\tilde{y}} + \mathbf{R}_{\tilde{y}}\mathbf{J}\mathbf{B}^T\mathbf{J}\mathbf{R}_{\tilde{y}} \\ &+ \sum_{k=1}^{K}\kappa_k\tilde{\mathbf{a}}_k\tilde{\mathbf{a}}_k^H\mathbf{B}\tilde{\mathbf{a}}_k\tilde{\mathbf{a}}_k^H\Big) + o(\frac{1}{T})\end{aligned}$$

with $\tilde{\mathbf{a}}_k \stackrel{\text{def}}{=} \begin{pmatrix}\mathbf{a}_k \\ \mathbf{a}_k^* e^{-i\phi_k}\end{pmatrix}$ *and* $\mathbf{J} \stackrel{\text{def}}{=} \begin{pmatrix}\mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O}\end{pmatrix}$,

we prove [11] the following expression

$$\mathrm{E}(\delta\tilde{\mathbf{\Pi}}_T) = \frac{1}{T}\left(\mathrm{Tr}(\tilde{\mathbf{\Pi}})\tilde{\mathbf{U}} - \mathrm{Tr}(\tilde{\mathbf{U}})\tilde{\mathbf{\Pi}}\right) + o(\frac{1}{T})$$

with $\tilde{\mathbf{U}} \stackrel{\text{def}}{=} \sigma_n^2\tilde{\mathbf{S}}^{\#}\mathbf{R}_{\tilde{y}}\tilde{\mathbf{S}}^{\#} = \begin{pmatrix}\mathbf{U}_1 & \mathbf{U}_2 \\ \mathbf{U}_2^* & \mathbf{U}_1^*\end{pmatrix}$ which gives with $\tilde{\mathbf{\Pi}} = \begin{pmatrix}\mathbf{\Pi}_1 & \mathbf{\Pi}_2 \\ \mathbf{\Pi}_2^* & \mathbf{\Pi}_1^*\end{pmatrix}$

$$\begin{aligned}\mathrm{E}(\delta\mathbf{\Pi}_{1,T}) &= \frac{1}{T}\left(\mathrm{Tr}(\tilde{\mathbf{\Pi}})\mathbf{U}_1 - 2\mathrm{Tr}(\mathbf{U}_1)\mathbf{\Pi}_1\right) + o(\frac{1}{T}) \\ \mathrm{E}(\delta\mathbf{\Pi}_{2,T}) &= \frac{1}{T}\left(\mathrm{Tr}(\tilde{\mathbf{\Pi}})\mathbf{U}_2 - 2\mathrm{Tr}(\mathbf{U}_1)\mathbf{\Pi}_2\right) + o(\frac{1}{T}).\end{aligned}$$

This allows us to derive the mean spectrum associated with the noncircular MUSIC algorithm (1). After simple but tedious algebraic manipulations [11], we obtain under the assumptions of lemma 2

$$\begin{aligned}\mathrm{E}(g_T^{\mathrm{Alg_{NC}}}(\theta)) &= g^{\mathrm{Alg_{NC}}}(\theta) \\ &+\frac{2}{T}\Big((2M-3)[(\mathbf{a}^H(\theta)\mathbf{U}_1\mathbf{a}(\theta))(\mathbf{a}^H(\theta)\mathbf{\Pi}_1\mathbf{a}(\theta)) \\ &- \Re[(\mathbf{a}^H(\theta)\mathbf{U}_2\mathbf{a}^*(\theta))(\mathbf{a}^T(\theta)\mathbf{\Pi}_2^*\mathbf{a}(\theta))]] \\ &- 2\mathrm{Tr}(\mathbf{U}_1)g^{\mathrm{Alg_{NC}}}(\theta)\Big) + o(\frac{1}{T})\end{aligned}$$

with $g^{\mathrm{Alg_{NC}}}(\theta) \stackrel{\text{def}}{=} (\mathbf{a}^H(\theta)\mathbf{\Pi}_1\mathbf{a}(\theta))^2 - |\mathbf{a}^T(\theta)\mathbf{\Pi}_2\mathbf{a}(\theta)|^2$. Since the expression of this mean spectrum depends on the second-order statistics only, it is the same for the different threshold ASNRs deduced from it. This allows us to prove [11] the following result, after tedious algebraic manipulations obtained from closed-form expressions of $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$.

Result 2 *The threshold ASNRs deduced from the Cox* (2) *and Sharman and Durrani criteria* (3) *given for the noncircular MUSIC algorithm and a ULA depend only on the second-order statistics of the sources and are respectively given by*

$$\xi_1 = \frac{1}{T}\alpha_{1,M}^{\Delta\theta,\Delta\phi}\left(1+\sqrt{1+\frac{T}{\beta_{1,M}^{\Delta\theta,\Delta\phi}}}\right) \quad (7)$$

$$\xi_2 = \frac{1}{T}\alpha_{2,M}^{\Delta\theta,\Delta\phi}\left(1+\sqrt{1+\frac{T}{\beta_{2,M}^{\Delta\theta,\Delta\phi}}}\right) \quad (8)$$

with $\Delta\phi \stackrel{\text{def}}{=} (\phi_1-\phi_2)$ *the noncircularity phase separation and where* $\alpha_{1,M}^{\Delta\theta,\Delta\phi}$, $\beta_{1,M}^{\Delta\theta,\Delta\phi}$, $\alpha_{2,M}^{\Delta\theta,\Delta\phi}$ *and* $\beta_{2,M}^{\Delta\theta,\Delta\phi}$ *are second order expansions in* $(\Delta\theta)^2$ *without constant term, whose coefficients depend on* M *and* $\Delta\phi$.

We note that these threshold ASNRs (7),(8) depend not only on $\Delta\theta$, T, M, but also on $\Delta\phi$ contrary to the threshold ASNRs obtained for the standard MUSIC algorithm. These intricate expressions reduce to simple interpretable expressions for weak and large noncircularity phase separation $\Delta\phi$.

More precisely for $\sin(\Delta\phi) \ll (M-1)\frac{\Delta\theta}{2}$, we prove [11] that

$$\begin{aligned}\alpha_{1,M}^{\Delta\theta,\Delta\phi} &\approx \frac{10M^4(2M-3)}{(2M-1)(M^2-1)(8M-11)(\Delta\theta)^4} \\ \beta_{1,M}^{\Delta\theta,\Delta\phi} &\approx \frac{10M^2(2M-3)}{(M+1)(8M-11)(\Delta\theta)^2} \\ \alpha_{2,M}^{\Delta\theta,\Delta\phi} &\approx \frac{5M^4(2M-3)}{(M^2-1)(M^2-4)(\Delta\theta)^4} \\ \beta_{2,M}^{\Delta\theta,\Delta\phi} &\approx \frac{5M^2(2M-3)}{2(M^2-4)(\Delta\theta)^2}\end{aligned}$$

and the behavior of the standard and noncircular MUSIC algorithms are similar due to the similarity of the dependence in $\Delta\theta$ of the expressions (5), (6), (7) and (8). In the opposite case for $\tan(\frac{\Delta\phi}{2}) \gg (M-1)\frac{\Delta\theta}{2}$, we prove that

$$\begin{aligned}\alpha_{1,M}^{\Delta\theta,\Delta\phi} &\approx \frac{2M^2(2M-3)}{(M^2-1)\sin^2(\frac{\Delta\phi}{2})(\Delta\theta)^2} \\ \beta_{1,M}^{\Delta\theta,\Delta\phi} &\approx \frac{2M^2(2M-3)}{(M^2-1)(1+\cos^2(\frac{\Delta\phi}{2}))(\Delta\theta)^2}\end{aligned}$$

[2] Note that a mistake in [2, (rel.(35)] has been corrected.

[3] Most of the papers dealing with this topic use this normalization, so we also use it in order to simplify comparisons with the literature.

$$\alpha_{2,M}^{\Delta\theta,\Delta\phi} \approx \frac{M^2(2M-3)}{(M^2-1)\sin^2(\frac{\Delta\phi}{2})(\Delta\theta)^2}$$

$$\beta_{2,M}^{\Delta\theta,\Delta\phi} \approx \frac{M^2(2M-3)}{(M^2-1)(1+\cos^2(\frac{\Delta\phi}{2}))(\Delta\theta)^2}$$

and the noncircular MUSIC algorithm largely outperforms the standard algorithm due to the proportionality in $1/(\Delta\theta)^2$ in the place of $1/(\Delta\theta)^4$ given in Result 1 for the MUSIC algorithm. Consequently the noncircularity phase separation between the two sources plays an important role in the behavior of the noncircular MUSIC algorithm.

5. ILLUSTRATIVE EXAMPLES

To illustrate Result 2, we consider two uncorrelated equipowered BPSK modulated signals impinging on a ULA with $M = 6$ and $T = 500$. We clearly see in Figs.1 and 2 that the noncircular MUSIC algorithm outperforms the standard MUSIC algorithm for all values of the noncircularity phase separation. We note that this difference of behavior in resolution is connected to the best accuracy of the DOA estimate given by the noncircular MUSIC algorithm [7] compared to those of the standard MUSIC algorithm. This is illustrated in Fig.3. Furthermore, by comparing Fig.1 and Fig.2 we note that the threshold ASNRs given by the two criteria are relatively similar.

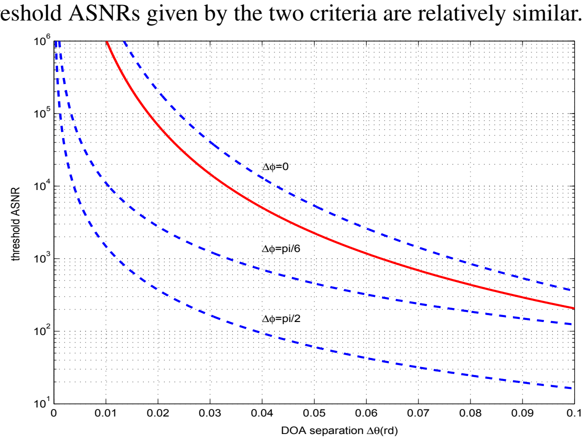

Fig.1 Comparison of the threshold ASNRs given by the Cox criterion as a function of the DOA separation $\Delta\theta$ associated with the standard MUSIC (—) and noncircular MUSIC algorithms (- -) for txo values of the noncircularity phase separation $\Delta\phi$.

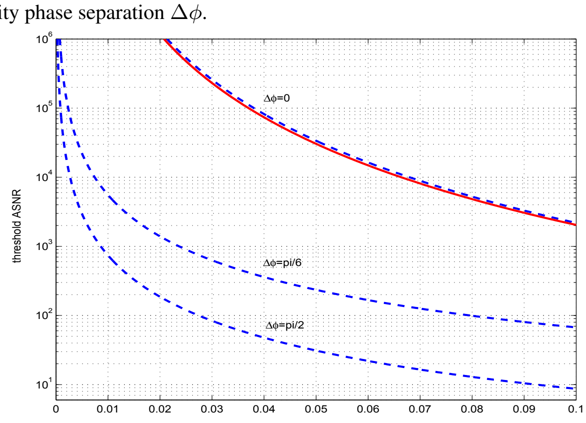

Fig.2 Comparison of the ASNR thresholds given by the Sharman and Durrani criterion as a function of the DOA separation $\Delta\theta$ associated with the standard MUSIC (—) and noncircular MUSIC algorithms (- -) for two values of the noncircularity phase separation $\Delta\phi$.

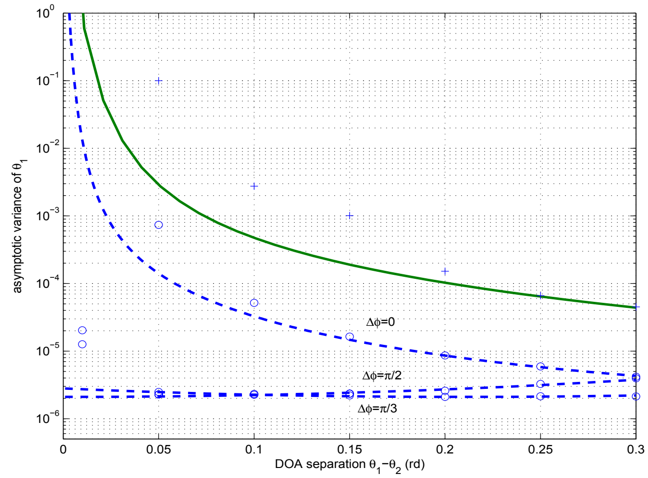

Fig.3 Theoretical and empirical (with 1000 Monte Carlo runs) asymptotic variance given by the standard MUSIC (—) and the noncircular MUSIC (- -) algorithms for two values of the noncircularity phase separation $\Delta\phi$ as a function of the DOA separation $\Delta\theta$, for $SNR = 20dB$.

6. REFERENCES

[1] H. Cox, "Resolving power and sensitivity to mismatch of optimum array processors," *J. Acoust. Soc. America*, vol. 54, no. 3, pp. 771-785, 1973.

[2] M. Kaveh, A.J. Barabell, "The statistical performance of the MUSIC and the Minimum-Norm algorithms in resolving plane waves in noise," *IEEE Trans. on ASSP*, vol. 34, no. 2, pp. 331-341, April 1986.

[3] K.C. Sharman, S.T. Durrani, "Resolving power of signal subspace methods for finie data lengths," in *Proc. ICASSP*, Tampa, Florida, April 1985.

[4] P. Foster, E. Villier, "Simplified formulas for performance analysis of MUSIC and Min Norm," in *Proc. OCEANS'98*, Nice, September 1998.

[5] H.B. Lee, M.S. Wengrovitz, "Statistical characterization of the MUSIC algorithm null spectrum," *IEEE Trans. on ASSP*, vol. 39, no. 6, pp. 1333-1347, June 1991.

[6] Q.T. Zhang, "A statistical resolution theory of the beamformer-based spatial spectrum for determining the directions of signals in white noise," *IEEE Trans. on ASSP*, vol. 43, no. 8, pp. 1867-1873, August 1995.

[7] H. Abeida, J.P. Delmas, "MUSIC-like estimation of direction of arrival for non-circular sources," *IEEE Trans. on Signal Processing*, vol. 54, no. 7, pp. 2678-2690, July 2006.

[8] P. Chargé, Y. Wang and J. Saillard, "A non-circular sources direction finding method using polynomial rooting," *Signal Processing*, vol. 81, pp. 1765-1770, 2001.

[9] H. Krim, P. Forster and G. Proakis, "Operator approach to performance analysis of root-MUSIC and root-min-norm," *IEEE Trans. on Signal Processing*, vol. 40, no. 7, pp. 1687-1696, July 1992.

[10] H.B. Lee, M.S. Wengrovitz, "Resolution threshold of beamspace MUSIC for two closely spaced emitters," *IEEE Trans. on Signal Processing*, vol. 38, no. 9, pp. 1445-1559, September 1990.

[11] J.P. Delmas, H. Abeida, "The statistical performance of MUSIC-like algorithms in resolving noncircular sources," in preparation to *IEEE Trans. on Signal Processing*.