ESTIMATION OF THE COMPLEX AMPLITUDES ASSOCIATED TO THE COMMON POLES IN A MULTICHANNEL SIGNAL

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ABSTRACT

Recently, certain solutions have been proposed to solve the common poles estimation problem in a multichannel Exponentially Damped Sinusoidal (EDS) signal. In this work, we tackle the closely related problem consisting of the estimation of the complex amplitudes associated to the common poles of a multichannel EDS signal. Our approach is based on the estimation of an oblique projector on the space of the estimated/common poles along the space of the unknown/non-common poles. We derive three projection/estimation schemes. The first one is based on a sequential (channel by channel) processing of the data while the others are based on joint estimation of the complex amplitudes.

1. INTRODUCTION

Very few works have been proposed to solve the problem of the estimation of the common poles of a multichannel EDS signal with more than two channels. Nevertheless, this problem is a typical Signal Processing problem and is at the heart of biomedical signal analysis. Recently, two approaches have been proposed. The first one is based on shift-invariance of the signal subspace [3] and the second one is based on a root-version of a multichannel MUSIC algorithm [4]. Here, we propose several algorithms to estimate the complex amplitudes associated *only* to the common poles. Our approach is based on oblique projection [1, 2] of the noisy channel output. So, assume that we have estimated the common poles by one of the referenced techniques, we can compute the noise subspace associated to these poles. This quantity is sufficient to compute an orthogonal projector, but not oblique projector since the latter depends also on the to *unknown* non-common poles in each channel. We propose here a methodology to estimate an oblique projection and we derive three estimation schemes. In addition, we discuss the effect of an oblique projector on the noise and we compare the derived algorithms in the context of monte-carlo simulations.

2. MULTI-CHANNEL EDS MODEL

Define the noise-free N-sample EDS signal in the k-th channel according to

$$x_k^{(N)} = \begin{bmatrix} x_k(0) & \dots & x_k(N-1) \end{bmatrix}^T = \bar{Z}^{(N)} a_k + Z_k^{(N)} b_k \quad (1)$$

where $x_k(n)$ denotes the n-th sample of the k-th channel and $a_k = [a_{1,k} \dots a_{\bar{M},k}]^T$ and $b_k = [b_{1,k} \dots b_{M_k,k}]^T$ are the complex amplitude vectors constituted by the common complex amplitudes $a_{m,k} = s_{m,k} e^{i\phi_{m,k}}$ and the "non-common" complex amplitudes $b_{m,k} = g_{m,k} e^{i\varphi_{m,k}}$ and

$$\bar{Z}^{(N)} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ \bar{z}_1^{N-1} & \bar{z}_2^{N-1} & \dots & \bar{z}_{\bar{M}}^{N-1} \end{bmatrix}_{N \times \bar{M}} \quad (2)$$

contains the common poles $\bar{z}_m = e^{\bar{d}_m + i\bar{\omega}_m^{(c)}}$ while

$$Z_k^{(N)} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ z_{1,k}^{N-1} & z_{2,k}^{N-1} & \dots & z_{M_k,k}^{N-1} \end{bmatrix}_{N \times M_k} \quad (3)$$

is the Vandermonde matrix containing the non-common poles of the k-th channel. A well-known Vandermonde-type decomposition of the Hankel channel matrix

$$H_k = \begin{bmatrix} x_k(0) & x_k(1) & \dots & x_k(L-1) \\ x_k(1) & x_k(2) & \dots & x_k(L) \\ \vdots & \vdots & & \vdots \\ x_k(N-L-1) & x_k(N-L) & \dots & x_k(N-2) \end{bmatrix}_{(N-L) \times L} \quad (4)$$

associated to the k-th channel is given by

$$H_k = \bar{Z}^{(N-L)} A_k \bar{Z}^{(L)T} + Z_k^{(N-L)} B_k {Z_k^{(L)}}^T \quad (5)$$

with $A_k = \mathrm{diag}(a_k)$ and $B_k = \mathrm{diag}(b_k)$. L represents here a window parameter chosen according to $L \leq N/2$.

3. THE COMMON POLE ESTIMATION PROBLEM

The common poles estimation problem can be algebraically described in the following manner. Assume that $\forall k$, $\mathcal{R}(\bar{Z}^{(N)})$ and $\mathcal{R}(Z_k^{(N)})$ intersect trivially ($= \{0\}$), then we look for the intersection of K spaces defined by

$$\bigcap_{k=1}^{K} \mathcal{R}\left(\begin{bmatrix} \bar{Z}^{(N)} & Z_k^{(N)} \end{bmatrix} \right) = \mathcal{R}\left(\bar{Z}^{(N)} \right). \quad (6)$$

where $\mathcal{R}(A)$ denotes the column range space of matrix A.

Assume that, we have solved this problem (*cf.* references [3, 4]) and one needs to estimate the complex amplitudes associated to the set of the common poles. To answer this question, we propose in the following section three algorithms.

4. ESTIMATION ALGORITHMS

4.1. Sequential-algorithm

The Sequential-algorithm, denoted in short Seq-Algo, can be described in the following manner.

1) For each channel, compute the channel matrix H_k defined in expression (4).
2) Assume that the common poles have been estimated and form the orthogonal projector $P_c^\perp = I - \bar{Z}^{(L)}\bar{Z}^{(L)\dagger}$ where symbol † denotes the Moore-Penrose pseudo-inverse. Next, define the weighted channel matrix according to
$$\bar{H}_k = P_c^\perp H_k \tag{7}$$
3) Now, decompose matrix $\bar{H}_k$ through the Singular Value Decomposition (SVD) according to
$$\bar{H}_k = U_k \Sigma_k \begin{bmatrix} V_k & \bar{V}_k \end{bmatrix}^H \tag{8}$$
where V_k (*resp.* $\bar{V}_k$) is an $L \times M_k$ (*resp.* $L \times (L - M_k)$) unitary matrix.
4) Compute the following $(L \times L)$ oblique projector
$$\Psi_k = \bar{Z}^{(L)} \left(\hat{P}_k^\perp \bar{Z}^{(L)} \right)^\dagger \tag{9}$$
where $\hat{P}_k^\perp = \bar{V}_k^* \bar{V}_k^T$.
5) Minimize the Least-Squares (LS) criterion
$$\min_{A_k} \left\| \Psi_k x_k^{(L)} - \bar{Z}^{(L)} a_k \right\|^2 \tag{10}$$
by considering the following minimum norm solution:
$$a_k = \mathcal{Z}_k^{(L)\sharp} x_k^{(L)}. \tag{11}$$
where $\mathcal{Z}_k^{(L)\sharp} = \bar{Z}^{(L)\dagger}\Psi_k$ is the oblique pseudo-inverse [1, 2]. Note that a fast computation, *ie.*, without pseudo-inversion of matrix $\bar{Z}^{(L)}$, can be $a_k = \left(\hat{P}_k^\perp \bar{Z}^{(L)} \right)^\dagger x_k^{(L)}$.

We can make the following remarks on this algorithm.

- First, observe that at the first step, the orthogonal projection of the channel matrix is given by
$$\bar{H}_k = P_c^\perp H_k = \left(P_c^\perp Z_k^{(L)} \right) B_k Z_k^{(L)^T}. \tag{12}$$
As the left Vandermonde basis is corrupted by the projection, the right dominant singular basis provides a basis of $\mathcal{R}(Z_k^{(L)})$. Note that in the noisy case, we can easily show that the orthogonal projection does not destroy the whiteness of the additive noise.
- At the last step, we proceed vector $x_k^{(L)}$ by an oblique projector which leaves unchanged the desired quantities while the unknown quantities are rejected according to
$$\begin{aligned} \Psi_k x_k^{(L)} &= \bar{Z}^{(L)} \left(\hat{P}_k^\perp \bar{Z}^{(L)} \right)^\dagger \left(\bar{Z}^{(L)} a_k + Z_k^{(L)} b_k \right) \\ &= \bar{Z}^{(L)} a_k. \end{aligned}$$
since by expanding term $\left(\hat{P}_k^\perp \bar{Z}^{(L)} \right)^\dagger$, it comes
$$\begin{aligned} \bar{Z}^{(L)} \underbrace{\left(\bar{Z}^{(L)H} \hat{P}_k^\perp \bar{Z}^{(L)} \right)^{-1} \bar{Z}^{(L)H} \hat{P}_k^\perp \bar{Z}^{(L)}}_{I} a_k &= \bar{Z}^{(L)} a_k, \\ \bar{Z}^{(L)} \left(\bar{Z}^{(L)H} \hat{P}_k^\perp \bar{Z}^{(L)} \right)^{-1} \bar{Z}^{(L)H} \underbrace{\hat{P}_k^\perp Z_k^{(L)}}_{0} b_k &= 0. \end{aligned}$$

The number of poles that can be estimated has to satisfy: $rank(\bar{H}_k) = M_k \leq L$.

4.2. Block-algorithm

In this section, we introduce a second algorithm, called Block-Algo which is based on the joint decomposition of all the channel matrices. Its algorithmic description is given below.

1) Let H be the Block-Hankel channel matrix containing the K channel matrices, defined by
$$H = \begin{bmatrix} H_1 \\ \vdots \\ H_K \end{bmatrix}_{K(N-L)\times L}. \tag{13}$$
2) Assume that the common poles have been estimated and form the orthogonal projector $P_c^\perp = I - \bar{Z}^{(L)}\bar{Z}^{(L)\dagger}$. Next, define the weighted channel matrix according to
$$\bar{H} = (I \otimes P_c^\perp) H \tag{14}$$
where $\otimes$ defines the Kronecker product.
3) Now, decompose the weighted channel matrix through the SVD according to
$$\bar{H} = U \Sigma \begin{bmatrix} V & \bar{V} \end{bmatrix}^H \tag{15}$$
where V (*resp.* $\bar{V}$) is an $L \times \left(\sum_k M_k\right)$ (*resp.* $L \times (L - \left(\sum_k M_k\right))$) unitary matrix.
4) Compute the following oblique projector
$$\Theta = \bar{Z}^{(L)} \left(\hat{P}^\perp \bar{Z}^{(L)} \right)^\dagger \tag{16}$$
where $\hat{P}^\perp = \bar{V}^* \bar{V}^T$.

5) Minimize the LS criterion $\min_{\{a_1 \dots a_K\}} \left\| \Theta X - \bar{Z}^{(L)} A \right\|^2$ where $A = [a_1 \dots a_K]$, by considering the following minimal norm solution:

$$A = \bar{Z}^{(L)\dagger} \Theta X \tag{17}$$

with $X = [x_1^{(L)} \dots x_K^{(L)}]$. Note that a fast computation, *ie.*, without pseudo-inversion of matrix $\bar{Z}^{(L)}$, can be $A = \left(\hat{P}^{\perp} \bar{Z}^{(L)} \right)^{\dagger} X$.

The number of poles that can be estimated has to satisfy: $rank(\bar{H}) = \sum_{k=1}^{K} M_k \leq L$.

4.3. Sum-Algorithm

We propose a last algorithm, called Sum-Algo, which is based on the computation of the sum of the channel matrices over the channel index. Its algorithmic description is given below.

1) Compute matrices H_k for $k \in [1 : K]$. Based on these set of matrices, compute
$$\mathcal{H} = \sum_{k=1}^{K} H_k. \tag{18}$$
2) Assume that the common poles have been estimated and form the orthogonal projector $P_c^{\perp} = I - \bar{Z}\bar{Z}^{\dagger}$. Next, define the weighted channel matrix given by
$$\bar{\mathcal{H}} = P_c^{\perp} \mathcal{H} \tag{19}$$
3) Follow Step 3-5 of the Block-Algorithm

The number of poles that can be estimated has to satisfy: $rank(\bar{\mathcal{H}}) = \sum_{k=1}^{K} M_k \leq L$. Clearly, this algorithm is less expensive than the two previous one.

5. EFFECT OF AN OBLIQUE PROJECTOR

Consider the noisy signal defined by

$$y_k^{(L)} = x_k^{(L)} + \sigma n_k^{(L)} \tag{20}$$

where $\sigma \in \mathbb{R}^+$, $n_k^{(L)}$ is a zero-mean Gaussian white vector noise. Apply an oblique projector on signal (20) according to $\Psi_k y_k^{(L)}$. Then, define the noise gain [1] associated to the previous model according to

$$\rho = \frac{1}{L} tr(\Gamma_k) \tag{21}$$

where Γ_k is the noise covariance. For model (20), we have $\rho = \sigma^2$ and for signal $P_c y_k^{(L)}$, obtained by orthogonal (instead of oblique) projection onto $\mathcal{R}(\bar{Z})$, it comes

$$\rho_{\mathtt{ortho}} = \frac{\sigma^2}{L} tr(P_c) = \sigma^2 \frac{\bar{M}}{L}. \tag{22}$$

As $\bar{M} < L$, we have $\rho_{\mathtt{ortho}} < \rho$. This means that for finite L, it is beneficial to work in a subspace of reduced dimension. For an oblique projector the noise covariance is given by $\Gamma_k = \sigma^2 \Psi_k \Psi_k^H$. As an oblique projector is a deficient matrix, we can consider the truncated SVD of the Hermitian matrix $\Psi_k \Psi_k^H$ according to

$$\Psi_k \Psi_k^H = \begin{bmatrix} Q & \times \end{bmatrix} \begin{bmatrix} D^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q^H \\ \times \end{bmatrix} \tag{23}$$

then the noise gain is given by

$$\rho_{\mathtt{obli}} = \frac{\sigma^2}{L} tr(\Psi_k \Psi_k^H) = \frac{\sigma^2}{L} tr(Q^H \Psi_k \Psi_k^H Q) \tag{24}$$

$$= \frac{\sigma^2}{L} tr(D^2) = \frac{\sigma^2}{L} \sum_{\ell=1}^{\bar{M}} \frac{1}{\sin(\theta_\ell)^2} \geq \rho_{\mathrm{ortho}}. \tag{25}$$

These expressions are obtained by remarking that the trace operator is invariant under unitary transformations and the singular values of an oblique projector are relied to the canonical angles $\theta_\ell \in [0 : \pi/2]$ between $\mathcal{R}(\bar{Z})$ and $\mathcal{R}(Z_k)$ [1, 2]. If these spaces are mutually orthogonal then $\rho_{\mathtt{obli}}$ is near to its lower bound, *ie.*, $\rho_{\mathtt{obli}} \approx \rho_{\mathtt{ortho}}$. If $\mathcal{R}(\bar{Z})$ and $\mathcal{R}(Z_k)$ are close then $\rho_{\mathtt{obli}}$ is large and far from its minimal bound. In this case, the noise term may increased significantly. In other words, one observes that, contrary to the orthogonal projection, the oblique projection mitigates compplectly the interference due to the non-common poles of the channel but may increase the additive noise term effect, depending on the canonical angle between $\mathcal{R}(\bar{Z})$ and $\mathcal{R}(Z_k)$.

6. NUMERICAL SIMULATIONS

We consider the following two-channels case $y_1(n) = c_1(n) + z_{1,1}^n + z_{2,1}^n$ and $y_2(n) = c_2(n) + z_{1,2}^n$ where $c_1(n) = s_{1,1} e^{i\phi_{1,1}} \bar{z}_1^n + \sigma n_1(n)$ and $c_2(n) = s_{1,2} e^{i\phi_{1,2}} \bar{z}_1^n + \sigma n_2(n)$. To compare the derived methods, we compute the "ideal" estimation of the common amplitudes according to $a_{1,1} = s_{1,1} e^{i\phi_{1,1}} = \bar{Z}^{(N)\dagger} \begin{bmatrix} c_1(0) & \dots & c_1(N-1) \end{bmatrix}^T$ and $a_{1,2} = s_{1,2} e^{i\phi_{1,2}} = \bar{Z}^{(N)\dagger} \begin{bmatrix} c_2(0) & \dots & c_2(N-1) \end{bmatrix}^T$. So, in the "ideal" case, we assume that there is no non-common poles which disturb the estimation of the common poles. Inversely, we consider a "naive" solution where the non-common poles are simply ignored, given by $a_{1,1} = \bar{Z}^{(N)\dagger} \begin{bmatrix} y_1(0) & \dots & y_1(N-1) \end{bmatrix}^T$ and $a_{1,2} = \bar{Z}^{(N)\dagger} \begin{bmatrix} y_2(0) & \dots & y_2(N-1) \end{bmatrix}^T$. The performance criterion is the Mean Squares Error (MSE) averaged over 500 experiments and the desired parameters are the real amplitudes and the initial phases associated to the common pole: $(s_{1,1}, \phi_{1,1})$, in the first channel and $(s_{1,2}, \phi_{1,2})$, in the second channel. In practice, the common poles are estimated by one of the methods presented in [3, 4]. However, in this simulation, we prefer to assume that the common poles are error-free to focus this study only on the performance of the proposed schemes. The analysis window size is $N = 50$ samples. We consider three scenarios.

6.1. Scenario 1: all the poles are largely spaced

In the first scenario, we consider that all the poles are largely spaced. On Fig. 1, we have reported the MSE for the complex amplitudes associated to the common poles in the two channels versus the SNR. We can see that all the three algorithms are equivalent in this context. As the Sum-Algo has the lowest complexity cost, this method is preferable for this scenario. Note that the "ideal" method has a lower variance which shows that our techniques are not optimal in this sense. However, they provide satisfactory results in comparison to the "naive" approach which is lower-bounded at high SNR, due to the interference of the non-common poles.

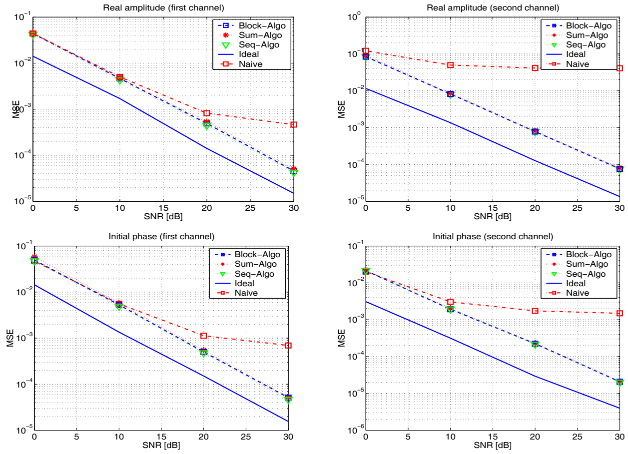

Fig. 1. MSE Vs. SNR [dB] for $\bar{z}_1 = e^{i-0.02}$, $a_{1,1} = 1$, $a_{1,2} = 2e^{i\frac{\pi}{3}}$, $z_{1,1} = e^{1.4i-0.015}$, $z_{2,1} = e^{1.7i-0.02}$ and $z_{1,2} = e^{0.5i-0.01}$.

6.2. Scenario 2: the non-common poles are closely spaced

In the second scenario, we consider that the non-common poles are closely spaced. According to Fig. 2, the Block-Algo and Seq-Algo have an equivalent accuracy but the Sum-Algo is less efficient since the summation of closely spaced sinusoids can be problematic. Here again, the "ideal" method has a lower variance and the "naive" one is lower-bounded at high SNR.

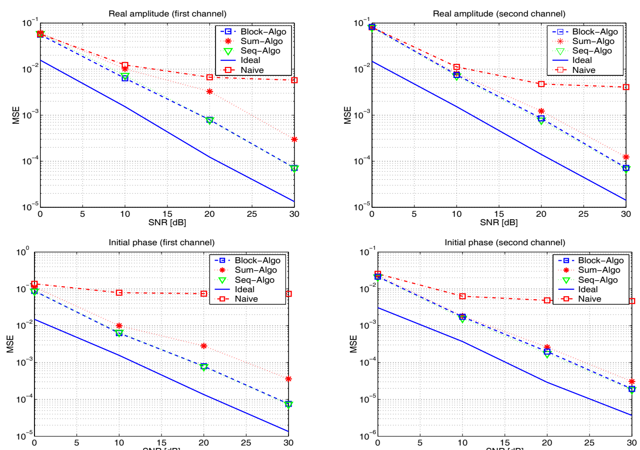

Fig. 2. MSE Vs. SNR [dB] for $\bar{z}_1 = e^{i-0.02}$, $a_{1,1} = 1$, $a_{1,2} = 2e^{i\frac{\pi}{3}}$, $z_{1,1} = e^{0.5i-0.01}$, $z_{2,1} = e^{0.6i-0.01}$ and $z_{1,2} = e^{0.7i-0.01}$.

6.3. Scenario 3: one non-common pole and the common pole are closely spaced

In this last scenario, we consider that one non-common pole and the common pole are closely spaced. We have reported the MSE with respect to the SNR on Fig. 3. As we can note the Sum-Algo is the less efficient method. The Block-Algo and the Seq-Algo show similar accuracy for the first channel but we can note that the Seq-Algo is a better choice for the second channel.

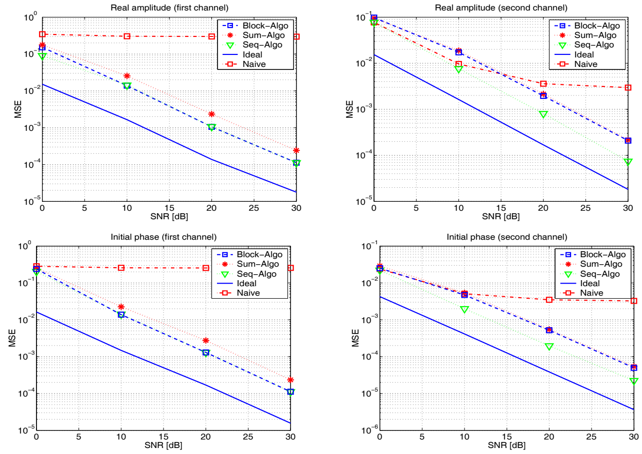

Fig. 3. MSE Vs. SNR [dB] for $\bar{z}_1 = e^{i-0.01}$, $a_{1,1} = 1$, $a_{1,2} = 2e^{i\frac{\pi}{3}}$, $z_{1,1} = e^{1.1i-0.01}$, $z_{2,1} = e^{1.7i-0.02}$ and $z_{1,2} = e^{0.4i-0.015}$.

In conclusion, we can see that the "naive" approach where we simply ignore the interference is not practicable. Consequently, we have derived three algorithms. In a computational point of view, the less complex is the Sum-Algo since its complexity cost is independent of the number of channel. This algorithm has the lower accuracy. The two other algorithms have similar complexity costs (the first one being slightly less expensive than the second) but the Seq-Algo seems a little bit more efficient than the Block-Algo, especially in difficult scenario as for closely spaced poles. In addition, with the Seq-Algo, the maximal number of poles that can be estimated is higher than with the two other algorithms.

7. CONCLUSION

In this paper, we have designed three algorithms which allow the estimation of the complex amplitudes associated to the common poles of a multichannel EDS model. Our solution is based on oblique projection of the noisy observation in each channel or in a block fashion. At our best knowledge, there is no concurrent method to solve this problem.

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