THERE IS NO FREE LUNCH WITH CAUSAL APPROXIMATIONS

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ABSTRACT

This paper studies the approximation of continuous functions in subsets of all causal and stable transfer functions. Such approximations play a central roll in filter design, filter bank analysis, and in sampling, since any filtering can be considered as a kind of approximation in a space defined by the filters. The present paper studies in particular the consequences resulting from the causality and stability constrain imposed on the filter process. It is shown that there exists no linear approximation method which is also causal and stable. Only if either the causality or the stability constrain is left out, a linear approximation method may exist.

Index Terms— Approximation methods, Causality, Filtering, Stability

1. INTRODUCTION

The approximation by causal and stable transfer functions is closely related to any kind of filter problem and therefore it appears frequently in signal processing applications. We give two (out of many) well known examples:
1) Filterbank representation [1, 2]: Consider the approximation of a desired filter characteristic $f(e^{j\omega})$ by means of a filter bank $\{\varphi_k\}_{k=1}^{\infty}$ of the form

$$(\mathcal{S}_N f)(e^{j\omega}) = \sum_{k=1}^{N} c_k(f,N)\,\varphi_k(e^{j\omega}) \tag{1}$$

with certain numbers $c_k(f,N)$, $1 \leq k \leq N$ which are uniquely determined by the function f and where the normalized frequency is in the range $\omega \in [-\pi,\pi)$. From such an approximation, we require that the approximation error $\|f - \mathcal{S}_N f\|_\infty$ decreases as the approximation degree N is increased, and that all individual filterbank stages φ_k are causal and stable. Moreover, to obtain a simple representation of the filter bank, the coefficients $c_k(f,N)$ should depend linearly on f.
2) Sampling: Recent approaches [3, 4] to the representation of a signal f by means of a sequence of numbers show that the sampling procedure can be considered as an approximation of f in a certain (Hilbert) space spanned by a number of basis functions φ_k as in (1). Then the coefficients $c_k(f)$ are the "samples" of f. Thus, the sampling can be considered as a projection onto a certain subspace of the signal space. Also in this case, we require that the sampling $f \mapsto c_k(f)$ is a linear mapping and that the reconstruction of the signal f from the samples c_k has the form (1) with certain basis functions φ_k. Moreover, the reconstructed signal $\mathcal{S}_N f$ should be causal and bounded, i.e. $\sup_{N\in\mathbb{N}} \|\mathcal{S}_N f\|_\infty < \infty$.

Thus in both cases, we require the *linearity* of the coefficients $c_k(f,N)$ with respect to f. This is mainly due to practical constrains in order that a practical feasible algorithm is obtained. Moreover, the approximation $\mathcal{S}_N f$ should represent a *causal* function (due to physical considerations) and the maximum value $\|\mathcal{S}_N f\|_\infty$ ought to be uniformly bounded for all N (*stability*). However, this paper will show that there exists *no* approximation method (1) which has all three properties: stability, causality and linearity. However, it is also discussed that a) there exist stable and causal methods but which are non-linear, and b) that there exist stable and linear methods but which are non-causal. Thus, one can always have only two out of the three desired properties stability, causality, and linearity.

In [5] one special causal filterbank $\{\varphi_k\}_{k=1}^{\infty}$ was considered, and it was shown that the corresponding approximation $\mathcal{S}_N f$ diverges for some continuous transfer functions f as $N \to \infty$, i.e. the corresponding causal and linear filter bank (1) is not stable. Here, this result is generalized and it will be shown (cf. Theorem 3) that this property holds for *every* causal filter bank of the form (1) in which the coefficients $c_k(f,N)$ depend linear on the function f.

The outline of the paper is as follows. After a clarification of the notations, Section 2 gives a more detailed problem statement. Section 3 discusses some positive results on non-linear and causal approximations and on linear and non-causal methods. Afterwards, Section 4 investigates causal approximation methods. The paper closes with a discussion of the results in Section 5.

2. PROBLEM STATEMENT AND MOTIVATION

2.1. Notations

The set of all trigonometric polynomials on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with a degree of at most N is denoted by $\mathcal{T}_N$ and $\mathcal{C}(\mathbb{T})$ is the space of continuous functions on $\mathbb{T}$ equipped with the supremum norm $\|\cdot\|_\infty$. As usual, $L^p = L^p(\mathbb{T})$ with $1 \leq p \leq \infty$ denotes the set of all p-integrable functions on $\mathbb{T}$ with the common norm $\|\cdot\|_p$ [6]. Every $f \in L^p$ with $1 < p < \infty$ can be represented by its *Fourier series*

$$f(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \hat{f}_k\, e^{jk\omega} \tag{2}$$

with the *Fourier coefficients*

$$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{j\omega}) e^{-jk\omega}\, \mathrm{d}\omega\,. \tag{3}$$

Let $\mathcal{B}$ be a subspace of L^1, then $\mathcal{B}_+ = \{f \in \mathcal{B} : \hat{f}_k = 0 \text{ for all } k < 0\}$ denotes the (causal) subspace of all $f \in \mathcal{B}$ for which all Fourier coefficients with negative index are equal to zero. Every $f \in \mathcal{B}_+$ can be identified with a function

$$f(z) := \sum_{k=0}^{\infty} \hat{f}_k\, z^k$$

which is analytic inside the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

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For all $1 \leq p \leq \infty$ the spaces $(L^p)_+$ are also known as the *Hardy spaces* H^p with the usual norms [7], which are also denoted by $\|\cdot\|_p$. The subspace $H_0^p \subset H^p$ is the set of all functions $f \in H^p$ for which $f(0) = 0$. The space $\mathcal{A} := (\mathcal{C}(\mathbb{T}))_+$ is the so called *disk algebra*. It is equal to the set of all $f \in H^\infty$ which are continuous in the closure $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$ of the unit disk. Finally, $\mathcal{P}_N = (\mathcal{T}_N)_+$ is the set of all (complex) polynomials of a degree of at most N.

2.2. System theory

Let $\mathcal{L}$ be a causal linear system $y = \mathcal{L}x$ with a transfer function f which maps every input signal $x \in L^2$ of finite energy onto an output signal $y \in L^2$ of finite energy. Such a linear system is said to be (energy) *stable*, and it is clear that the operator norm of $\mathcal{L}$ is finite in this case: $\|\mathcal{L}\|_{L^2 \to L^2} < \infty$. Moreover, it is well known that this operator norm is equal to the supremum norm of its transfer function:

$$\|\mathcal{L}\|_{L^2 \to L^2} = \sup_{x \in L^2} \frac{\|\mathcal{L}x\|_2}{\|x\|_2} = \sup_{|z|<1} |f(z)| = \|f\|_\infty .$$

Moreover, the Fourier coefficients $\{\hat{f}_k\}_{k=-\infty}^{\infty}$ of any transfer function f can be interpreted as the *impulse response* of $\mathcal{L}$. Then, it is clear that $\mathcal{B}_+$ contains all causal transfer functions from a certain set $\mathcal{B}$ of transfer functions.

Therewith, it is clear that L^∞ can be identified with the set of all (energy) stable transfer functions, whereas H^∞ contains all causal and stable transfer functions. The disk algebra $\mathcal{A}$, on the other hand, comprises all causal and stable transfer functions which can be approximated by a finite impulse response (FIR) system (i.e. by an polynomial). Moreover, since we are always interested in stable approximations (1), it is clear that the approximation error $f - \mathcal{S}_N f$ has to be measured in the stability norm $\|\cdot\|_\infty$.

2.3. Problem statement

Let f be a given transfer function. As an example, assume that f should be approximated by a causal FIR system of degree N. Thus, we look for a certain polynomial $g \in \mathcal{P}_N$ such that the approximation error $\|f - g\|_\infty$ can be controlled.

One possibility for the determination of such an g is the truncated Fourier series: $g^{(1)}(e^{j\omega}) = (\mathcal{S}_N^{(1)} f)(e^{j\omega}) = \sum_{k=0}^{N} c_k^{(1)}(f)\, e^{j\omega k}$ in which $c_k^{(1)}(f) = \hat{f}_k$ are the Fourier coefficients (3) of f. This approximation is very simple, since the coefficients c_k depend linear on the given f but not on the degree N. However, this approximation $\mathcal{S}_N^{(1)} f$ is not optimal with with respect to the minimal approximation error $\|f - g\|_\infty$, in general.

A second possible approximation method takes this g which minimizes the approximation error $\|f - g\|_\infty$. This optimal g is uniquely determined by its Fourier coefficients and has therefore also a representation of the form $g(e^{j\omega}) = (\mathcal{S}_N^{(2)} f)(e^{j\omega}) = \sum_{k=0}^{N} c_k(f, N)\, e^{j\omega k}$, but now the Fourier coefficients $c_k(f)$ depend non-linear on f and on N.

Here, we considers general approximation methods of the form (1) with arbitrary basis functions φ_k. Basically, we require that the approximation method (1) satisfies the following three natural properties:

(A) *Stability*: The approximation error $\|f - \mathcal{S}_N f\|_\infty$ should decrease as the degree N increases, and $\sup_{N \in \mathbb{N}} \|\mathcal{S}_N f\|_\infty$ ought to be bounded.

(B) *Causality*: The approximation $\mathcal{S}_N f$ should represent a causal transfer function. This is certainly achieved if all individual transfer functions φ_k are causal.

(C) *Linearity*: The calculation of the coefficients $c_k(f, N)$ should be sufficiently simple. Therefore, we assume that the coefficients have the following general representation

$$c_k(f) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{j\theta})\, \gamma_k(e^{j\theta})\, \mathrm{d}\theta \qquad (4)$$

with certain functions $\gamma_k \in L^1$. Obviously, the so defined coefficients $c_k(f)$ depend linear on f.

This paper considers approximation methods with these three properties on the space $\mathcal{C}(\mathbb{T})$ of all continuous functions. It will show that there exists no approximation method $\mathcal{S}_N : \mathcal{C}(\mathbb{T}) \to \mathcal{A}$ which satisfies all three of the above properties.

The approximation problem on the space $\mathcal{C}(\mathbb{T})$ arises in particular due to distortions of the given data f by certain errors (e.g. estimation or quantization errors). For example, assume that a causal and stable transfer function f is given which should be approximated in the filterbank (1). In many cases f is disturbed by a function Δ such that only $\widetilde{f} = f + \Delta$ is known. Then, the approximation error

$$\|\mathcal{S}_N \widetilde{f} - f\|_\infty \leq \|\mathcal{S}_N f - f\|_\infty + \|\mathcal{S}_N \Delta\|_\infty$$

has to be controlled. Even if the approximation error of the perfect f (first term on the right hand side) tends to zero as $N \to \infty$, the second term may become arbitrary large. Thus, in order to controll the overall approximation error, the approximation of the disturbance $\|\mathcal{S}_N \Delta\|_\infty$ should remain bounded for all possible disturbances Δ. In general, it can not be assumed that the disturbance Δ belongs to the same class as the transfer function f (causal and stable, in the present case). One reasonable (and already quite optimistic) model for the disturbance Δ, is that Δ is a continuous function. For these reasons, the approximation problem on the space $\mathcal{C}(\mathbb{T})$ is considered in this paper. It is clear that all the results hold also for all kinds of worst disturbances which may even be not continuous.

3. APPROXIMATION METHODS

3.1. Non-linear approximations

Let $f \in \mathcal{C}(\mathbb{T})$, we look for a causal and stable transfer function $g \in H^\infty$ which approximates f as close as possible. Moreover, we if demand that the approximative transfer function g is also continuous, we have to look for such an optimal $g \in \mathcal{A}$. It is known [7] that

$$E(f, \mathcal{A}) = \inf_{g \in \mathcal{A}} \|f - g\|_\infty = \inf_{g \in H^\infty} \|f - g\|_\infty = E(f, H^\infty)$$

in which $E(f, \mathcal{A})$ and $E(f, H^\infty)$ are called the *best approximation* of f in $\mathcal{A}$ and H^∞, respectively [8]. Thus, the best approximation is equal in $\mathcal{A}$ and H^∞ which means that the remaining approximation error coincide in both spaces. Since $A \subset H^\infty$ the optimal g, for which the best approximation is attained, belongs to H^∞, in general and there exists a unique function $g^{opt} \in H^\infty$ such that

$$E(f, \mathcal{A}) = E(f, H^\infty) = \|f - g^{opt}\|_\infty .$$

Thus, the optimal approximation g^{opt} of a continuous function f is non-continuous, in general. Only if the approximation problem is considered on a subspace $\mathcal{C}_\omega(\mathbb{T})$ of smooth functions in $\mathcal{C}(\mathbb{T})$, the optimal g^{opt} will always be continuous on $\mathbb{T}$. It can be shown that $g^{opt} \in \mathcal{A}$ if the modulus of continuity ω_f of f is a regular majorant (see e.g. [9]). In the following, such subspaces $\mathcal{C}_\omega(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$ are considered for which the best approximation g^{opt} belongs to $\mathcal{A}$.

How does g^{opt} depends on the given f and does there exists a linear mapping $\mathcal{M}_+^{opt} : f \mapsto g^{opt}$ which gives to every $f \in \mathcal{C}_\omega(\mathbb{T})$ the optimal approximation $g^{opt} \in \mathcal{A}$? The general interrelation between f and g^{opt} is quite complicated. However, it is known [7, Section IV] that to every $f \in \mathcal{C}(\mathbb{T}) \subset L^\infty(\mathbb{T})$ there exists an $F \in H_0^1$ such that $f - g^{opt} = E(f, \mathcal{A}) \frac{\overline{F}}{|F|}$. Consequently, the causal and stable transfer function, which approximates the given f best, can be written as

$$g^{opt} = f - E(f, \mathcal{A})\, e^{-j \arg(F)}$$

with a certain function $F \in H_0^1$. This last relation shows that 1) $\mathcal{M}_+^{opt} f = f$ whenever $f \in \mathcal{A}$ and 2) that the mapping $\mathcal{M}_+^{opt}$ is *non-linear*, in general.

Similarly, one can consider the approximation by FIR filters. Then the best approximation by a (causal) polynomial in $\mathcal{P}_N$ is given by

$$E_+(f, N) = \inf_{p \in \mathcal{P}_N} \|f - p\|_\infty = \|f - p_N^{opt}\|_\infty \, . \qquad (5)$$

The mapping $\mathcal{M}_{+,N}^{opt} : f \mapsto p_N^{opt}$ onto the optimal polynomial is unique but again $\mathcal{M}_{+,N}^{opt}$ is a non-linear operator, in general. We summarize this result by the following lemma.

Lemma 1: *There exists an approximation method* $\mathcal{M}_{+,N}^{opt} : \mathcal{C}(\mathbb{T}) \to \mathcal{P}_N$ *which maps every* $f \in \mathcal{C}(\mathbb{T})$ *onto a unique causal polynomial* $p_N^{opt} \in \mathcal{P}_N$ *which satisfies* (5)*. This method has the properties (A) and (B) but it is non-linear.*

3.2. Non-causal, linear approximations

Next we consider linear approximations of the form

$$(\mathcal{S}_N^{(w)} f)(e^{j\omega}) = \sum_{k=-N}^{N} w(\tfrac{k}{N})\, \hat{f}_k\, e^{jk\omega} \qquad (6)$$

in which $\hat{f}_k$ are the usual Fourier coefficients (3) of f, and $w(x)$ is a window function defined for $-1 \le x \le 1$ and with $w(x) = 0$ for all $|x| > 1$. Since the representation (6) contains non-zero negative Fourier coefficients, it is clear that the approximation $\mathcal{S}_N^{(w)} f$ is *non-causal*. It is clear that the coefficients $c_k(f) = w(k/N)\, \hat{f}_k$ depend linear on f. Therefore (6) has property (C). For the window function $w(x) = 1$, the usual Fourier series (2) is obtained. However, it is well known [8] that this series does not converge uniformly to f for all $f \in \mathcal{C}(\mathbb{T})$ which implies that $\lim_{N\to\infty} \|\mathcal{S}_N^{(w)} f\|_\infty = \infty$ for some $f \in \mathcal{C}(\mathbb{T})$. For this reason, other windows w are considered. Most important is the triangular window $w(x) = 1 - |x|$ and the trapezoid window $w(x) = 1$ for $0 \le |x| \le 1/2$ and $w(x) = 2(1 - |x|)$ for $1/2 < |x| \le 1$. For the triangular window, the series (6) is also known as *Fejér mean* of f and for the trapezoid window, (6) is called *de la Vallée-Poussin mean* of f. It is well known [8] that for these window functions hold that $\lim_{N\to\infty} \|f - \mathcal{S}_N^{(w)} f\|_\infty = 0$ for all $f \in \mathcal{C}(\mathbb{T})$, which shows that the approximation methods possess property (A). We summarize this in the following lemma.

Lemma 2: *There exist approximation methods* $\mathcal{S}_N^{(w)} : \mathcal{C}(\mathbb{T}) \to \mathcal{C}(\mathbb{T})$ *witch are stable (A) and linear (C) but which are non-causal.*

4. BEHAVIOR OF CAUSAL APPROXIMATIONS

This section investigates approximation methods of the form (1) with properties (B) and (C). Thus, we assume that the transfer functions $\{\varphi_k\}_{k=1}^\infty$ in (1) are causal and stable ($\varphi_k \in \mathcal{A}$ for all k) and that the coefficients $c_k(f)$ are of the form (4). In order that even an approximation is possible, we have to make sure that the system $\{\varphi_k\}_{k=1}^\infty$ spans the whole $\mathcal{A}$. Thus, as a minimal property of the method $\mathcal{S}_N$, we require that it converges at least for all polynomials of the form $p_m(z) = z^m$ for all integers $m \ge 0$. This means in the following, we always require that the method $\mathcal{S}_N$ satisfies the property

$$\lim_{N\to\infty} \|p_m - \mathcal{S}_N p_m\|_\infty = 0 \quad \text{for all } m \ge 0 \, .$$

The question arises, whether there exist such methods which have also property (A), i.e. which are also stable. The following theorem shows that no such method exist.

Theorem 3: *For every approximation method* $\mathcal{S}_N : \mathcal{C}(\mathbb{T}) \to \mathcal{A}$ *of the form* (1) *with* $\varphi_k \in \mathcal{A}$ *and with the coefficients* c_k *of the form* (4) *there exist functions* $f \in \mathcal{C}(\mathbb{T})$ *such that* $\sup_{N\in\mathbb{N}} \|\mathcal{S}_N f\|_\infty = \infty$ *and such that* $\limsup_{N\to\infty} \|f - \mathcal{S}_N f\|_\infty = \infty$.

This theorem is an extension of a result presented in [10]. Because of the limited space, a detailed proof can not be presented, but only a short outline of the main steps.

Sketch of proof: If the coefficients (4) are plugged into (1) an integral representation of the approximation operator (1) is obtained:

$$(\mathcal{S}_N f)(z) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{j\theta})\, K_N(e^{j\theta}, z)\, \mathrm{d}\theta$$

with the *reproducing kernel* $K_N(e^{j\theta}, z) = \sum_{k=1}^N \gamma_k(e^{j\theta})\, \varphi_k(z)$. It can be verified that the operator norm of $\mathcal{S}_N : \mathcal{C}(\mathbb{T}) \to \mathcal{A}$ is equal to

$$\|\mathcal{S}_N\|_{\mathcal{C}(\mathbb{T})\to\mathcal{A}} = \sup_{|z|<1} \underbrace{\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |K_N(e^{j\theta}, z)|\, \mathrm{d}\theta \right)}_{=: L_N(z)}$$

where the right hand side is the so called *Lebesgue constant* of $\mathcal{S}_N$, which was also investigated in [5] for one special system $\{\varphi_k\}_{k=1}^\infty$ of rational functions. Here, φ_k are arbitrary functions in $\mathcal{A}$.

Now, it can be shown (a detailed proof will be published elsewhere) that

$$\liminf_{N\to\infty} \tfrac{1}{2\pi} \int_{-\pi}^{\pi} |L_N(\rho e^{j\tau})|\, \mathrm{d}\tau \ge \tfrac{1}{\pi\,\rho} \log \tfrac{1}{1-\rho} \qquad (7)$$

for all $0 < \rho < 1$, and for all possible systems $\{\varphi_k\}_{k=1}^\infty$ with $\varphi_k \in \mathcal{A}$ for all k. It follows from (7) that

$$\liminf_{N\to\infty} \sup_{|z|<1} L_N(z) = \infty$$

which certainly implies that $\liminf_{N\to\infty} \|\mathcal{S}_N\|_{\mathcal{C}(\mathbb{T})\to\mathcal{A}} = \infty$. Together with the theorem of Banach-Steinhaus, this proves the statement of the theorem. □

It follows in particular, that to every such approximation operator $\mathcal{S}_N$, characterized in Theorem 3, there exist continuous functions $f \in \mathcal{C}(\mathbb{T})$ such that the approximation error $\|f - \mathcal{S}_N f\|_\infty$ increases as the approximation degree N is increased. In conclusion, the Theorem 3 shows that there exists no stable, causal and linear approximation method on $\mathcal{C}(\mathbb{T})$.

5. DISCUSSIONS AND CONCLUSIONS

It was shown in Section 3.2 that the windowed Fourier series (6) is an approximation method with properties (A) and (C), i.e. stable and linear. What happens if the non-causal part is truncated? Clearly this will give an approximation method with property (B) (causality), but unfortunately and in accordance with Theorem 3, the operators will

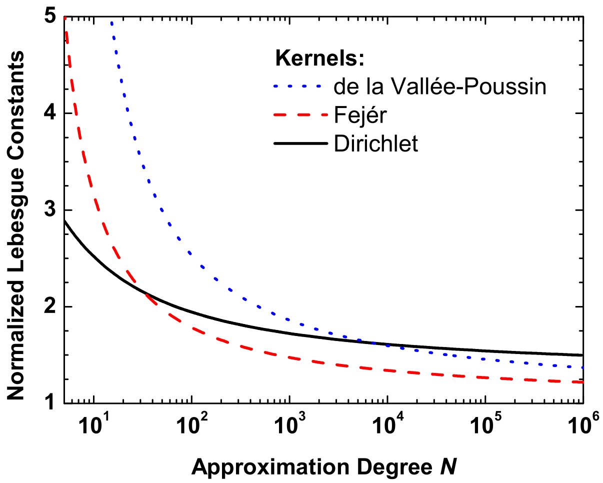

Fig. 1. The Lebesgue constants for different windowed Fourier series versus the approximation degree N. All curves are normalized by $\frac{1}{\pi}\log N$.

lose property (A), i.e. the stability, due to this truncation. To see this, consider the causal version of the windowed Fourier series (6)

$$(\mathcal{S}_N^{(w)} f)(z) = \sum_{k=0}^{N-1} w(\tfrac{k}{N})\, \hat{f}_k\, z^k\,, \quad (z \in \overline{\mathbb{D}})$$

with the same window functions $w(x)$ as in Section 3.2. The kernels of these approximation operators $\mathcal{S}_N^{(w)}$ become then $K_N(\zeta, z) = \sum_{k=0}^{N-1} w(k/N)\, \overline{\zeta}^k z^k$. As in the proof of Theorem 3 we use the corresponding Lebesgue constants to investigate the operator norms. For these Lebesgue constants hold that $\sup_{|z|<1} \|K_N(\cdot, z)\|_1 = \sup_{|z|<1} \|\overline{K_N(\cdot, z)}\|_1$ and since $\overline{K_N(\zeta, z)}$ is an analytic function for every fixed z and for all $\zeta \in \mathbb{D}$, we can apply *Hardy's inequality* (see e.g. [7]) and obtain

$$\|K_N(\cdot, z)\|_1 \geq \tfrac{1}{\pi} \sum_{k=0}^{N-1} \tfrac{w(k/N)}{k+1}\, \overline{z}^k\,.$$

This finally shows that the Lebesgue constants, and therewith the operator norms $\|\mathcal{S}_N^{(w)}\|_{\mathcal{C}(\mathbb{T})\to\mathcal{A}}$, diverge as N tends to infinity. For the Fejér mean, for instance, one obtains

$$\sup_{|z|<1} \|K_N(\cdot, z)\|_1 \geq \tfrac{1}{\pi}[\log(N+1) - 2]$$

and similar results hold for all usable window functions w. For the Lebesgue constants hold always a lower bound of the form $\sup_{|z|<1} \|K_N(\cdot, z)\|_1 \geq \frac{1}{\pi}[\log(N+1) - c(w)]$ with a certain constant $c(w)$ which depends on the actual window w.

Fig. 1 demonstrates that the lower bounds of the Lebesgue constants, obtained form Hardy's inequality, are quite tight. It shows the Lebesgue constants for the constant window $w(x) = 1$ (*Dirichlet kernel*) and for the Fejér and de la Vallée-Poussin means. All Lebesgue constants are normalized by $\frac{1}{\pi}\log N$ which is equal (up to a constant) to the lower bound obtained from Hardy's inequality. For $N \to \infty$, all graphs seems to converge to 1 which shows the tightness of the lower bounds.

It was shown in [11] that there exists a basis $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{A}$ such that every causal and stable transfer function $f \in \mathcal{A}$ can be expanded in this basis: $\lim_{N\to\infty} \|f - \sum_{k=1}^{N} c_k(f)\varphi_k\|_\infty = 0$ for all $f \in \mathcal{A}$. However, as a consequence of Theorem 3, for every such basis $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{A}$ there always exists an $f \in \mathcal{C}(\mathbb{T})$ such that $\limsup_{N\to\infty} \|f - \sum_{k=1}^{N} c_k(f)\varphi_k\|_\infty = \infty$. Thus, such a basis can not be used to approximate continuous functions. Also a windowing of the generalized Fourier coefficients will not improve this behavior, as it was shown on the example of the classical Fourier series.

Taking up the discussion from the introduction: In practical applications, the transfer functions $f \in \mathcal{A}$ are disturbed by a certain error $\Delta \in \mathcal{C}(\mathbb{T})$. Using a linear and causal approximation method (1), it will never be possible to control the approximation error induced by Δ. There are three possibilities to overcome this problem. Either a non-linear approximation method is used (cf. Section 3.1) which will yield quite complex approximation algorithms. Or, a non-causal approximation method is applied (cf. Section 3.2), which may physical non-realizable is certain applications. As a third possibility, the class of admissible disturbances may be reduced. Because it can be shown (see [9] for an first step in this direction) that for all smooth functions, namely for all $f \in \mathcal{C}_\omega(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$ with a regular majorant ω, stable, causal, and linear approximation methods exist.

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