EXACT IDENTIFICATION OF CONTINUOUS-TIME SYSTEMS FROM SAMPLED DATA

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ABSTRACT

Both direct and indirect methods exist for continuous-time system identification. A direct method estimates continuous-time input and output signals from their samples and then use them to obtain a continuous-time model, whereas an indirect method estimates a discrete-time model first. Both methods rely on fast sampling to ensure good accuracy. In this paper, we propose a more direct method where a continuous-time model is directly fitted to the available samples. This method produces an exact model asymptotically, modulo some aliasing ambiguity, even when the sampling rate is relatively low.

keywords: System identification, continuous-time system identification, parameter estimation.

1. INTRODUCTION

Dynamical systems encountered in the physical world are usually of a continuous-time nature. However, when measuring its input and output signals, we obtain discrete-time signals formed by samples of possibly noisy and prefiltered versions of these continuous-time signals. Continuous-time system identification involves identifying a continuous-time model using these samples [1]. Compared to its discrete-time counterpart where abundant techniques are available (see, e.g., [2] [3]), relatively less research has been done in this area. Available methods for continuous-time system identification are summarized below.

Indirect Methods: In this approach, the sampled signals are used to identify a discrete-time version of the system first. This is done by using any standard discrete-time system identification technique. The discrete-time model so obtained is then converted to a continuous-time model by using any standard conversion method (e.g., bilinear (Tustin) approximation, matched pole-zero method, etc. [4]). The advantage of this approach is that the actual identification is done using discrete-time system identification techniques which are well developed and their behaviors well understood. However, a drawback of this approach is that the accuracy of the standard model conversion methods depends on the sampling frequency. Hence, it can not be too small.

Direct Methods: In this case, the parameters of a continuous-time model is tuned to minimize an objective function constructed using the sampled signals. Generally speaking, the construction of the objective function requires (explicitly or implicitly) the estimation of the continuous-time signals from their samples. Therefore, for these methods to be accurate, the sampling frequency needs to be high (this requirement can be replaced by other assumptions on the input and output signals as, for example, is done in [5]). Based on the definition of the cost function, the direct methods can be further divided into time-domain and frequency-domain methods.

Time-Domain Methods: These methods aim at transforming the system's differential equation into an algebraic (linear) equation. In theory, this can be achieved by introducing some kind of pre-processing of the continuous-time signals before the sampling operation (e.g., state-variable filtering and integration [6]). In practice, however, the pre-processing is implemented in the discrete-time domain, after the sampling operation. For this to be done, the continuous-time signals are approximated from the available samples by using a polynomial approximation. A main advantage of this approach is that the continuous-time parameters can be solved via a least-squares solution.

Frequency-Domain Methods: Here, the available samples are used to estimate the spectrum of the input and the output signals. In order to do so, some assumptions on the continuous-time signals (e.g., band-limitedness [5]) are needed. A disadvantage of this approach is that the continuous-time parameters are obtained by solving a non-linear optimization problem.

In this paper, we propose a novel continuous-time system identification method. As in other direct methods, our method aims to tune the continuous-time parameters to minimize an objective function constructed from the available samples. However, the key difference to the existing direct methods is that we do not need to estimate the continuous-time signals. Instead, we identify a continuous-time model using the correlation functions of the continuous-time signals which can be estimated using the sampled data. It turns out that the proposed method produces, in theory, the exact system model asymptotically, modulo some aliasing ambiguity, as the number of samples tends to infinity, for any sampling frequency. However, numerical problems associated with the finite quantization of samples introduce a lower bound on the sampling frequency. Nevertheless, numerical simulations show that this lower bound can be as small as half of the -3dB

bandwidth of the system in practice.

The aliasing ambiguity mentioned above refers to the well-known fact that a continuous-time model produced using input-output data sampled at a given frequency always has poles ambiguous in the sense that their imaginary parts can be shifted by any integer multiple of the sampling frequency without changing the dynamic response of the system at the sampling points. This ambiguity is common to all identification methods and can only be avoided by *a priori* knowledge of the region of the poles.

The proofs of results are not included in the paper.

2. PROPOSED METHOD

The setting for continuous-time system identification is depicted in Fig. 1. The input signal $u(t)$ and the output noise $v(t)$ are random processes. The discrete-time signals $\mathbf{u}(k)$ and $\mathbf{y}(k)$ are generated by first filtering the input signal and the noisy output signal $y(t)$ using the anti-alias filters $H(p)$ and $L(p)$, respectively, and then sampling them with a sampling period τ. It is assumed that $G(p)$ is modeled by

$$G^\theta(p) = \frac{a_0 + a_1 p + \cdots + a_n p^n}{b_0 + b_1 p + \cdots + b_{n-1} p^{n-1} + p^n}, \qquad (1)$$

where $\theta = [a_0, \cdots, a_n, b_0, \cdots, b_{n-1}]$ is the vector of parameters. Our aim is to identify θ based on the sampled signals $\mathbf{u}(k)$ and $\mathbf{y}(k)$. We give an intuitive explanation of the proposed method below.

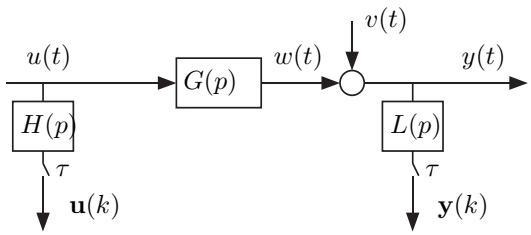

Fig. 1. Proposed identification method

Let $\mathbf{r_u}$ be the auto-correlation of $\mathbf{u}$ and $\mathbf{r_{y,u}}$ be the cross-correlation between $\mathbf{y}$ and $\mathbf{u}$. It turns out that

$$\mathbf{r_u} = \sigma_\tau\{h * h^* * r_u\}, \qquad (2)$$

$$\mathbf{r_{y,u}} = \sigma_\tau\{l * h^* * g * r_u\}, \qquad (3)$$

where h, g and l denote the impulse responses of $H(p)$, $G(p)$ and $L(p)$, respectively, r_u is the auto-correlation of u, $*$ denotes convolution, and $\sigma_\tau(\cdot)$ is the sampling operation. Now, suppose that r_u is known. Then, equation (3) states a relationship between the g and $\mathbf{r_{y,u}}$. Since $\mathbf{r_{y,u}}$ can be estimated from the available samples, (3) can be used to solve θ. This is a nonlinear optimization problem and the solution is in general non-unique. However, we can show (although not done in this conference version) that the non-uniqueness is only caused by the aliasing problem due to sampling, and therefore the solution is indeed unique modulo the aliasing ambiguity as mentioned in Introduction. A more detailed presentation is given in Section 2.1.

The method outlined above relies on the knowledge of r_u. Depending on the application, this information can be known *a priori*, or otherwise estimated from the available samples. In Section 2.3 we use an argument similar to the one given above to show that r_u can be estimated accurately from $\mathbf{r_u}$ using (2), modulo some aliasing ambiguity.

2.1. Identification with known input auto-correlation

In this section we assume that the auto-correlation function r_u of the input signal u is known, and we use (3) to identify θ. We denote the impulse response of $G^\theta(p)$ by g^θ and the parametric version of $\mathbf{r_{y,u}}$ by $\mathbf{r}^\theta_{\mathbf{y,u}}$, i.e.,

$$\mathbf{r}^\theta_{\mathbf{y,u}} = \sigma_\tau\{l * h^* * r_u * g^\theta)\},$$

where the overbar in $\overline{H}$ denotes complex conjugation. Also, we define $\underline{\mathbf{r}^\theta_{\mathbf{y,u}}}$ to be a vector of m samples of $\mathbf{r}^\theta_{\mathbf{y,u}}$ by

$$\underline{\mathbf{r}^\theta_{\mathbf{y,u}}} = [\mathbf{r}^\theta_{\mathbf{y,u}}(m_1), \cdots, \mathbf{r}^\theta_{\mathbf{y,u}}(m_2)]$$

with $m = m_2 - m_1 + 1$. On the other hand, we define an estimate $\mathbf{r}^{(N)}_{\mathbf{y,u}}$ of $\mathbf{r_{y,u}}$ by

$$\mathbf{r}^{(N)}_{\mathbf{y,u}}(k) = \frac{1}{N}\sum_{l=1}^{N} \mathbf{y}(l)\mathbf{u}(l+k)$$

and define the corresponding vector $\underline{\mathbf{r}^{(N)}_{\mathbf{y,u}}}$ for $\mathbf{r}^{(N)}_{\mathbf{y,u}}$ by

$$\underline{\mathbf{r}^{(N)}_{\mathbf{y,u}}} = [\mathbf{r}^{(N)}_{\mathbf{y,u}}(m_1), \cdots, \mathbf{r}^{(N)}_{\mathbf{y,u}}(m_2)]$$

Hence, we can define the optimal vector of parameters $\theta^{(N)}$ up to time N, as the one that makes $\underline{\mathbf{r}^\theta_{\mathbf{y,u}}}$ as close as possible to $\underline{\mathbf{r}^{(N)}_{\mathbf{y,u}}}$, i.e.,

$$\theta^{(N)} = \arg\min_\theta \left\| \underline{\mathbf{r}^\theta_{\mathbf{y,u}}} - \underline{\mathbf{r}^{(N)}_{\mathbf{y,u}}} \right\|_2^2. \qquad (4)$$

2.2. Parameter Optimization Algorithm

Solving equation (4) over all θ is a nonlinear least-squares problem. However, if we fix the denominator coefficient vector $b = [b_0, \cdots, b_{n-1}]$, then the optimization of the numerator coefficient vector $a = [a_0, \cdots, a_n]$ is a linear least-squares problem. We may denote by $a^{(N)}(b)$ the optimal numerator coefficients at sample N for a given denominator coefficient vector b. Then,

$$a^{(N)}(b) = \arg\min_a \left\| \underline{\mathbf{r}^{[a\ b]}_{\mathbf{y,u}}} - \underline{\mathbf{r}^{(N)}_{\mathbf{y,u}}} \right\|_2^2. \qquad (5)$$

Now, the optimal vector of denominator coefficient vector $b^{(N)}$ up to sample N can be computed by solving

$$b^{(N)} = \arg\min_{b} \left\| \underline{\mathbf{r}_{\mathbf{y},\mathbf{u}}^{[a^{(N)}(b),b]}} - \underline{\mathbf{r}_{\mathbf{y},\mathbf{u}}^{(N)}} \right\|_2^2 \quad (6)$$

and $\theta^{(N)}$ is given by

$$\theta^{(N)} = [a(b^{(N)}), b^{(N)}]. \quad (7)$$

The optimization problem in (6) is nonlinear but can be solved using the simplex method [7]. Numerical simulations show that the poles of the model always converge to those of the true system, provided that the initial guess of them are chosen such that they are closer to those of the true system than any of their aliased versions. On the other hand, if the initial guess is too far, the poles will converge to an aliased version.

2.3. Estimation of Input Auto-correlation

So far, we have assumed that the auto-correlation function r_u is known. However, this may not be a realistic assumption in certain applications. We now introduce a method to estimate r_u, which is analogous to the method in Section 2.1 for estimating g.

More precisely, we consider that the signal u is generated by filtering a continuous-time white random process by a linear time-invariant filter with a rational transfer function $F(p)$ parameterized by

$$F^{\rho}(p) = \frac{C(p)}{D(p)} = \frac{c_0 + c_1 p + \cdots + c_\nu p^\nu}{d_0 + d_1 p + \cdots + d_{\nu-1} p^{\nu-1} + p^\nu}, \quad (8)$$

where $\rho = [c,\ d] = [c_0, \cdots, c_\nu, d_0, \cdots, d_{\nu-1}]$. Consequently, we consider a parameterized model r_u^ρ of r_u. Its Fourier transform is given by

$$\Phi_u^\rho(\omega) = F^\rho(j\omega) F^\rho(-j\omega).$$

The parameter vector ρ can be used using a method similar to the one for θ, but (2) is used instead of (3). The only difference is in the parameter optimization algorithm. In this case, for the estimation of the numerator coefficients to be linear, Φ_u^ρ needs to be re-parameterized as

$$\Phi_u^\xi(\omega) = \frac{E(\omega)}{D(j\omega)D(-j\omega)},$$

where

$$E(\omega) = C(j\omega)C(-j\omega) = e_0 + e_1\omega^2 + \cdots + e_\nu \omega^{2\nu}$$

and $\xi = [e,\ d] = [e_0, \cdots, e_\nu, d_0, \cdots, d_{\nu-1}]$. It follows that the optimal e for a given d can be found by solving a linear least-squares problem, and the optimal d by using the simplex method, analogous to (5) and (6). Once ξ is estimated, $\Phi_u^\xi(\omega)$ and thus r_u are completely characterized. Once again, the estimate of Φ_u^ξ becomes exact asymptotically, modulo some aliasing ambiguity.

3. SIMULATIONS

It is known that direct methods are in general more accurate than indirect methods [8]. Therefore, we only compare the proposed methods with the direct methods. In particular, we consider the frequency domain method in [5] (Approach 1 in the reference). The following benchmark example from [1], [6] and [8] is used:

$$G(p) = \frac{6400p + 1600}{p^4 + 5p^3 + 408p^2 + 416p + 1600}$$

which has four poles at $-2 \pm j19.90$ and $-0.5 \pm j1.937$. The -3dB bandwidth of the system equals 26.85 rad/sec. The input signal u is generated using

$$F(p) = \frac{26.85}{p + 26.85}$$

which has the same bandwidth as $G(p)$. The frequency responses of $G(p)$ and $F(p)$ are shown in Fig. 2.

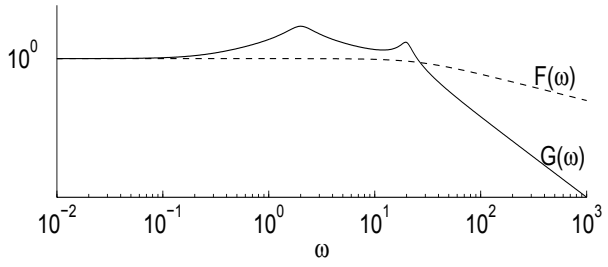

Fig. 2. Frequency responses of $G(p)$ and $F(p)$

In the simulations below, we compare the estimation errors of the different identification methods using

$$\varepsilon = \left\| G(p) - G^{\theta^{(N)}}(p) \right\|_2 / \left\| G(p) \right\|_2 .$$

As mentioned above, the initialization of the optimization algorithm in (6) is not critical. Therefore, we initialize the denominator of $G^\theta(p)$ to have four poles at $-5 \pm j5$. To resolve aliasing ambiguity, we correct the identified model by shifting the identified poles by an integer multiple of ω_s so that they are consistent with those of the true system.

In the first simulation we compare the performance of the proposed method in two scenarios: (S1) under the assumption that $F(p)$ is known, and (S2) including the estimation of $F(p)$. In Fig. 3 we compare the identification error obtained in the two scenarios, as a function of the sampling (radian) frequency $\omega_s = 2\pi/\tau$. It can be seen that the performance of the proposed method is similar in the two scenarios, with only a slightly bigger lost in performance in scenario (S2) for low sampling frequencies. The reason for this is that the time constant of F (i.e., 0.234 sec.) is significantly smaller than the smallest time constant in G (i.e., 3.14), and therefore the

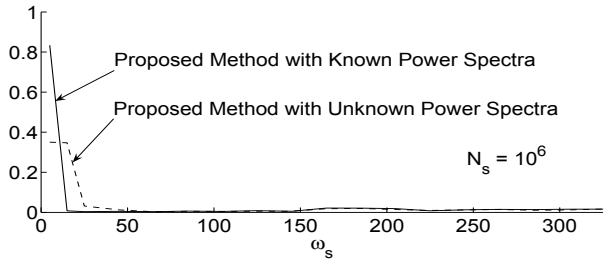

Fig. 3. Comparison of scenarios

minimum sampling frequency for identifying F is larger than that for G. In view of Fig. 3, we consider scenario (S1) for the rest of the simulations.

In the second simulation we compare the performance of the proposed method with that of the direct methods. Fig. 4 shows the identification errors as functions of ω_s for different values of N. We can see that the error of the proposed method does not depend on the sampling frequency up to a threshold frequency of about 5 rad/sec., under which numerical problems prevent the non-linear optimization algorithm to find the optimal solution. The estimation error depends on the number of available samples because this determines how well $\mathbf{r}_{\mathbf{y},\mathbf{u}}^{(N)}$ approaches $\mathbf{r}_{\mathbf{y},\mathbf{u}}$ in (4). In particular, when infinite number of samples are available, the error approaches zero at all sampling frequencies above the threshold. On the other hand, the performances of the direct methods improve as the sampling frequency increases but do not change noticeably with the number of samples. We conclude that the proposed method outperforms the direct methods when the sampling frequency is slow and the number of samples is large. In the opposite situation when the sampling frequency is high and the number of available samples is small, the direct methods can be shown to outperform the proposed method.

We have also tried the time domain method in [6] (also called state-variable filter method). But its performance is rather poorly for the range of sampling frequencies under our consideration.

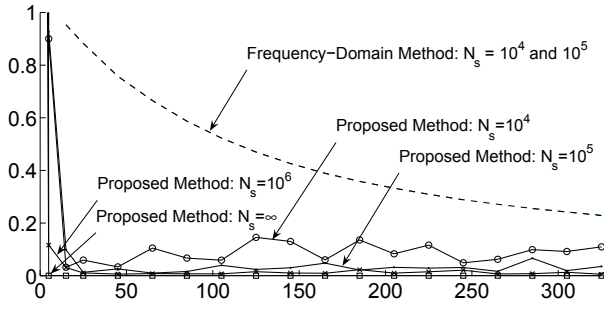

Fig. 4. Identification error vs. sampling frequency

3.1. Conclusion

We have proposed a novel identification method that directly fits a continuous-time model to the given sampled input and output signals. In this way, the proposed method is able to produce asymptotically, as the number of samples approaches infinity, the exact model of the system being identified, modulo some unavoidable aliasing ambiguity. This is valid in theory, for any sampling frequency. In practice, however, numerical problems introduce a lower bound on the sampling frequency. Nevertheless, simulation results show that this lower bound can be as small as half of the -3dB bandwidth of the system. In comparison with the available methods in the literature, the proposed method is a valid option when a slow sampling frequency must be used but a large number of samples available.

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