# ASYMPTOTIC ML DETECTION FOR THE PHOTON COUNTING CHANNEL

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**ABSTRACT**

Recent technological advances in single photon detectors have paved the way for laser-based interplanetary communications, targeting data rates higher than those achievable in the radio part of the spectrum. The development of photon counting receiver algorithms specific to this channel is required. In particular, Maximum Likelihood (ML) Detection must incorporate the typical Bose-Einstein distribution of background noise photons. We overcome the unwieldy expression of the ML detection metric by deriving an asymptotically tight approximation, converging in variance, which is linear in the data and whose coefficients can be explicitly calculated.

***Index Terms***— photon counting channel, quantum communication, laser, photon, maximum likelihood

## 1. INTRODUCTION

Optical channels can be classified as: (i) optical (high) intensity channels [1], where the receiver is sensitive to the amplitude of the optical field; and, in the low-intensity regime, (ii) photon counting channels, where the receiver records the times-of-arrival (TOAs) of single photons. The recent development of a nano-technology superconducting single-photon detector (SSPD) [2] with a 2 GHz counting rate, in contrast to the few KHz range of to-date APD detectors, is paving the way for photon-based high-rate communications. SSPDs are constituted by a nano-wire, cooled down to exhibit superconductivity. When a photon is absorbed, its energy heats the nano-wire so that superconductivity is lost and a photon is detected. Interplanetary communications, traditionally dominated by radio technology, constitute a possible application field. In Mars exploration, the extremely low link budget resulting from the combination of power-constrained spacecraft and solar system scale distances, requires large radio telescopes to achieve reasonable data rates. In the last few years, space agencies have conducted studies on the feasibility of optical space communications as a promising candidate for high-rate interplanetary communications. We address here the derivation of a detection metric for photon counting receivers.

## 2. SIGNAL MODEL AND ASYMPTOTIC ML

In the photon counting channel, the signal is modeled by the TOAs of single photons, which conform to an inhomogeneous Poisson distribution, with each event a photon arrival. The simpler *stationary* (homogeneous) Poisson point process is characterized by its mean event density (ideally, each event has zero duration) as $\lambda = \lim_{T_o\to\infty}[n(T_o)/T_o]$ with $T_o$ the observation time and $n(T_o)$ the number of events in $T_o$. The detector evaluates the absence/presence of photons in a time bin of duration $T_b$, such that if more than one photon arrives within the interval $[qT_b,(q+1)T_b)$, it is detected as a single photon (with $\Pr(k) = ((\lambda T_b)^k/k!)\cdot\exp[-\lambda T_b]$ the probability that $k$ Poisson points occur in an interval of duration $T_b$). For single-photon detectors, we consider the following complementary events: (*0-event*) no photon has arrived in a time-bin of duration $T_b$, which occurs with probability $P_0 = e^{-\lambda T_b}$; (*1-event*) more than one photon has arrived in the given time-bin, which occurs with probability $P_1 = 1 - e^{-\lambda T_b}$. In nature, $\lambda$ is time-dependent and the received signal must be modeled as an inhomogeneous Poisson process: let $\mathrm{n}(t,t+T_b)$ be the number of photons detected in $[t,t+T_b)$. Then, the corresponding probability generating function is given by,

$$\begin{aligned} G_{t,t+T_b}(z) &= \sum_{k=0}^{+\infty}\Pr(\mathrm{n}(t,t+T_b)=k)\cdot z^k \\ &= \exp\left((z-1)\int_t^{t+T_b}\lambda(\tau)d\tau\right) \end{aligned} \quad (1)$$

We now link this to the usual formulation: let the received (noisy) pass-band signal at central frequency $\nu_c$ be expressed as,

$$s(t) = \mathrm{Re}\left[b_s(t)\cdot e^{j2\pi\nu_c t}\right] \quad (2)$$

with $b_s t) = I_s(t) + jQ_s(t)$ the complex equivalent baseband signal, such that the instantaneous signal power is $|b_s(t)|^2$. For $B_s << \nu_c$ the effective signal bandwidth, the average number of photons detected in an interval $[t,t+T)$ is determined by the expression,

$$\overline{\mathrm{n}}(t,t+T) = \eta\cdot\frac{1}{h\nu_c}\int_t^{t+T}|b_s(\tau)|^2 d\tau = \int_t^{t+T}\lambda(\tau)\mathrm{d}\tau \quad (3)$$

with $h$ Planck's constant, $\eta$ the efficiency of the detector and the integral term the signal energy within the specified interval. Therefore, for $T\to 0$, we may establish the useful relationship,

$$\lambda(t) = \frac{\eta}{h\nu_c}|b_s(t)|^2 \quad (4)$$

The total equivalent baseband signal is the addition of a noise term $b_n(t)$ plus the useful signal $b_0(t)$. The laser is a monochromatic source that modulates the amplitude of $b_0(t)$. Therefore, we define $b_0(t)$ to be real (random phases are translated to the noise, an approach equivalent to [3], but simplified) so that,

$$|b_s(t)|^2 = (I_n(t)+b_0(t))^2 + Q_n^2(t) \quad (5)$$

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and $I_n(t)$, $Q_n(t)$ the thermal Gaussian in-phase and in-quadrature noise components, such that $I_n(t) + jQ_n(t) = \sqrt{h\nu_c\lambda_n(t)/\eta}\exp[j\theta_n(t)]$, with $\lambda_n(t)$ and $\theta_n(t)$ independent exponentially and uniformly distributed random processes, respectively, with $\theta_n(t)$ the relative phase difference between signal and noise. The total mean arrival density $\lambda(t)$ takes the form,

$$\begin{aligned}\lambda(t) &= \frac{\eta}{h\nu_c}\left(|b_n(t)|^2 + |b_0(t)|^2 + 2\mathrm{Re}[b_n^*(t)b_0(t)]\right)\\ &= \lambda_n(t) + \lambda_0(t) + 2\sqrt{\lambda_n(t)\lambda_0(t)}\cos\theta_n(t) \qquad (6)\end{aligned}$$

We assume an SSPD of timing resolution $T_b$. Hence, for given realizations $b_n(t)$ and $b_0(t)$, the 0- and 1-event probabilities in the $q$-th bin are calculated from (1) by setting $z = 0$,

$$P_0[q|b_n, b_0] = \exp\left[-\int_{qT_b}^{(q+1)T_b}\lambda(\tau)d\tau\right] \qquad (7)$$

$$P_1[q|b_n, b_0] = 1 - P_0[q|b_n, b_0] \qquad (8)$$

so that the true 0- and 1-event probabilities for a given $b_0(t)$ are expressed in terms of the expectation with respect to $b_n(t)$ as,

$$P_0[q|b_0] = \mathbb{E}_{b_n} P_0[q|b_n, b_0] \quad , \quad P_1[q|b_0] = 1 - P_0[q|b_0] \qquad (9)$$

We denote the sequence of detector outputs $x_q$ ($x_q = 0$ indicates the absence of detected photons in $t \in [qT_b, (q+1)T_b)$). Let $\mathbf{x} = [x_1, \cdots, x_{N_b}]^{\mathrm{T}}$ and let $N_b$ be the number of bins. For statistically independent detector outputs (the detection of one photon in one time bin does not affect the detection in other time bins), the discrete probability mass function of $\mathbf{x}$ is expressed as,

$$p(\mathbf{x}|b_n, b_0) = \prod_{q=1}^{N_b} P_0[q|b_n, b_0]^{1-x_q}(1 - P_0[q|b_n, b_0])^{x_q}$$

The final probability density is $p(\mathbf{x}|b_0) = \mathbb{E}_{b_n} p(\mathbf{x}|b_n, b_0) =$

$$\mathbb{E}_{b_n} \exp\left[\sum_{q=1}^{N_b}((1-x_q)\ln P_0[q|b_n, b_0] + x_q\ln(1 - P_0[q|b_n, b_0]))\right]$$

We examine the large sample behaviour of the argument of $\exp[\cdot]$,

$$J_{b_n} = \sum_{q=1}^{N_b}((1-x_q)a_n(q) + x_q b_n(q)) \qquad (10)$$

$$a_n(q) = \ln P_0[q|b_n, b_0] \quad , \quad b_n(q) = \ln(1 - P_0[q|b_n, b_0]) \qquad (11)$$

where we will establish the following stochastic convergence in variance: $J_{b_n} \to \mathbb{E}_{b_n}[J_{b_n}] + w_n$, with the variance of $w_n$ tending to zero for $N_b \to \infty$. In this way, asymptotically,

$$p(\mathbf{x}|b_0) = \mathbb{E}_{b_n}[e^{J_{b_n}}] \to \mathbb{E}_{b_n}[e^{\mathbb{E}_n[J_{b_n}]+w_n}] \to e^{\mathbb{E}_{b_n}[J_{b_n}]}$$

### 2.1. Stochastic Convergence Analysis

We provide only an outline for the convergence in variance of $J_{b_n}$, where $\sigma^2 = \mathbb{E}_{b_n}(J_{b_n} - \mathbb{E}_{b_n}[J_{b_n}])^2$ is compared with $\mathbb{E}_{b_n}[J_{b_n}^2]$ for $N_b \to \infty$. We define the zero-mean variables $\overline{a}_n(q) = a_n(q) - \mathbb{E}_{b_n}[a_n(q)]$ and $\overline{b}_n(q) = b_n(q) - \mathbb{E}_{b_n}[b_n(q)]$, so that,

$$\begin{aligned}\sigma^2 &= \mathbb{E}_{b_n}\sum_{q=1}^{N_b}\left((1-x_q)^2\overline{a}_n^2(q) + x_q^2\overline{b}_n^2(q)\right) + \\ &+ 2\mathbb{E}_{b_n}\sum_{q,q'=1}^{N_b}(1-x_q)x_{q'}\overline{a}_n(q)\overline{b}_n(q') \qquad (12)\end{aligned}$$

where we assume that $(\overline{a}_n(q), \overline{a}_n(q'))$, $(\overline{b}_n(q), \overline{b}_n(q'))$ and $(\overline{a}_n(q), \overline{b}_n(q'))$ are pairs of independent random variables for $q \neq q'$ as they depend on independent values of $b_n(t)$. Otherwise, the stochastic convergence result is also true but the proof is omitted for extension reasons. Therefore, $c_{q,q'} = \mathbb{E}_{b_n}[\overline{a}_n(q)\overline{b}_n(q')] = c_0 \cdot \delta_{q,q'}$ so that the cross term becomes $\sum_{q=1}^{N_b}(1-x_q)x_q c_0 = 0$ and,

$$\begin{aligned}\sigma^2 &= \mathbb{E}_{b_n}[\overline{a}_n^2]\sum_{q=1}^{N_b}(1-x_q) + \mathbb{E}_{b_n}[\overline{b}_n^2]\sum_{q=1}^{N_b}x_q \qquad (13)\\ &\to N_b\pi_0\mathbb{E}_{b_n}[\overline{a}_n^2] + N_b\pi_1\mathbb{E}_{b_n}[\overline{b}_n^2]\end{aligned}$$

with $\pi_0$ and $\pi_1$ the 0- and 1-event probabilities of $x_q$, independent of $q$. Hence, $\sigma^2$ is linear in $N_b$. As $J_{b_n}^2$ can be shown to grow in the order of $N_b^2$ (it is not zero-mean), the proposition is proved.

### 2.2. Asymptotic Value of $\mathbb{E}_{b_n}[J_{b_n}]$ and Signal Modulation

We evaluate $\mathbb{E}_{b_n}[a_n(q)]$ and $\mathbb{E}_{b_n}[b_n(q)]$ in $\mathbb{E}_{b_n}[J_{b_n}]$, which requires a definition of the signal format. We assume that the laser is either on or off over each time bin, so that $b_0(t)$ is defined by,

$$b_0(t) = B_0\sum_{k=-\infty}^{+\infty}\alpha_k(\mathbf{b})\phi(t - kT_c) \qquad (14)$$

with $\phi(t)$ a rectangular pulse, $\alpha_k(\mathbf{b}) \in \{0, 1\}$ the chip values, $\mathbf{b}$ the vector of source bits, $T_c$ the chip period and $N_{sc} = T_c/T_b$ the number of samples per chip, so that within a given chip, we have: $b_0(qT_b) = B_0\alpha_{\lfloor q/N_{sc}\rfloor}$. If we set $a_q = \alpha_{\lfloor q/N_{sc}\rfloor}$, we can define the following metrics,

$$\Lambda_{i,l} = \mathbb{E}_{b_n}[\ln P_i[q|b_n, b_0 = B_0 \cdot l]] \qquad (15)$$

which are independent of $q$, due to the stationarity of noise. Therefore, as $a_q \in \{0, 1\}$, we may define,

$$\begin{aligned}\Lambda_i[q] &= \mathbb{E}_{b_n}[\ln P_i[q|b_n, b_0 = a_q B_0]] \qquad (16)\\ &= (1-a_q)\Lambda_{i,0} + a_q\Lambda_{i,1} \qquad (17)\end{aligned}$$

which, for the signal part, does depend on $q$. Hence,

$$\begin{aligned}\mathbb{E}_{b_n}[J_{b_n}] &= \sum_{q=1}^{N_b}\big((1-x_q)((1-a_q)\Lambda_{0,0} + a_q\Lambda_{0,1}) + \\ &+ x_q((1-a_q)\Lambda_{1,0} + a_q\Lambda_{1,1})\big) \qquad (18)\end{aligned}$$

whence we may easily establish the Log-Likelihood Ratio (LLR) between the hypotheses: $\mathrm{H}_1 : \{a_q \neq 0\}_{1\leq q\leq N_b}$ (signal present) and $\mathrm{H}_0 : \{a_q = 0\}_{1\leq q\leq N_b}$ (signal absent), as,

$$\begin{aligned}J_{\mathrm{H}_1/\mathrm{H}_0} &= \mathbb{E}_{b_n}[J_{b_n}]|_{\mathrm{H}_1} - \mathbb{E}_{b_n}[J_{b_n}]|_{\mathrm{H}_0} = J_{\mathrm{H}_1} - J_{\mathrm{H}_0} \qquad (19)\\ &= \sum_{q=1}^{N_b}a_q\left((1-x_q)(\Lambda_{0,1} - \Lambda_{0,0}) + x_q(\Lambda_{1,1} - \Lambda_{1,0})\right)\end{aligned}$$

In more general terms, and defining $\Delta_i = \Lambda_{i,1} - \Lambda_{i,0}$, the LLR between two different sequences $a_q$ and $a_q'$ is given by,

$$J_{\mathrm{H}/\mathrm{H}'} = \sum_{q=1}^{N_b}(a_q - a_q')\left((1-x_q)\Delta_0 + x_q\Delta_1\right) \qquad (20)$$

The following two sections address the evaluation of $\Delta_i$ in terms of: (i) the mean arrival densities of noise and of the useful signal; and (ii) the 0-event probabilities under active/inactive signal $b_0(t)$.

### 2.3. Evaluation of $\Lambda_{0,0}$

In computing $\Lambda_{i,l}$, we need to consider the 0-event probabilities under active ($a_q = 1$)/inactive ($a_q = 0$) signal. We assume a typical approximation where $\lambda_n(t) = \lambda_n, \theta_n(t) = \theta$ and $\lambda_0(t) = \lambda_0$ in (6) are constant (but random) over each time bin. Integrating (6) we get,

$$\int_{qT_b}^{(q+1)T_b} \lambda(\tau)\mathrm{d}\tau \simeq n + n_0 + 2\sqrt{n_0 n}\cos\theta \tag{21}$$

where $n = \lambda_n T_b$ and $n_0 = \lambda_0 T_b$ constitute the average photon count of noise and signal within each time bin, respectively, and $\theta$ is independent uniformly distributed over $[0, 2\pi)$. We saw that $\lambda_n(t) = (\eta/h\nu)(I_n^2(t) + Q_n^2(t))$, with $I_n(t)$ and $Q_n(t)$ independent Gaussian distributed. Therefore, $\lambda_n$ is $\chi^2$-distributed with two degreees of freedom (exponentially distributed), and so is $n$. Hence, $n = v_1^2 + v_2^2$, with $v_1$ and $v_2$ independent Gaussian random variables of variance $\sigma^2$ and $\overline{n} = \mathbb{E}_n[n] = 2\sigma^2$, with the pdf of $n$ given by,

$$p_N(n) \quad = \quad \frac{1}{\overline{n}} \cdot e^{-n/\overline{n}} \cdot u(n) \tag{22}$$

From (7) and (15), we have $\Lambda_{0,0} = -\overline{n}$ (average number of photons in $T_b$ in the absence of the useful signal), where $\overline{n}$ is to be derived from the 0-event probabilities when the signal is inactive: $a_q = 0 \Rightarrow n_0 = 0$. Hence, the probability of not detecting a photon becomes,

$$P_0 = \mathbb{E}_n[e^{-n}] = \int_0^\infty \frac{1}{\overline{n}} \cdot e^{-n\cdot(1/\overline{n}+1)}\mathrm{d}n = (\overline{n}+1)^{-1} \tag{23}$$

Therefore, $\Lambda_{0,0} = -\overline{n} = 1 - P_0^{-1}$ and $\overline{n} = P_0^{-1} - 1$. Note that if a training sequence is used, this expression also serves as an estimator of $\overline{n}$ in terms of the measured $P_0$. Otherwise, the formulation of the ML estimator of $\overline{n}$ and $n_0$ leads to a non-linear equation system.

### 2.4. Evaluation of $\Lambda_{1,0}$ and $\Lambda_{i,1}$

$\Lambda_{1,0}$ is calculated as $\Lambda_{1,1}|_{n_0=0}$. We will first compute $P_{0,1}$, the 0-event probability under an active signal: $a_q = 1 \Rightarrow n_0 = \lambda_0 T_b$. Hence, $P_{0,1} = \mathbb{E}_{n,\theta}[\exp(-(n + n_0 + 2\sqrt{n_0 n}\cos\theta))]$ becomes,

$$\begin{aligned} P_{0,1} &= \mathbb{E}_\theta \int_0^\infty \frac{1}{\overline{n}} e^{-n/\overline{n}} e^{-(n+2\sqrt{n_0 n}\cos\theta + n_0)}\mathrm{d}n \qquad (24) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2} e^{-\frac{v_1^2+v_2^2}{2\sigma^2}} e^{-((v_1+v_0)^2+v_2^2)}\mathrm{d}v_1\mathrm{d}v_2 \end{aligned}$$

with $n_0 = v_0^2$ and using $n = v_1^2 + v_2^2$ and $\mathrm{tg}\theta = v_2/v_1$ as mentioned before (22). Defining $1/2\sigma_e^2 = 1 + 1/2\sigma^2$,

$$P_{0,1} = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{+\infty} e^{-v_1^2/2\sigma^2 - (v_1+v_0)^2} \int_{-\infty}^{+\infty} e^{-v_2^2/2\sigma_e^2}\mathrm{d}v_2\mathrm{d}v_1$$

Hence,

$$P_{0,1} = \frac{\sigma_e}{\sqrt{2\pi}\sigma^2}\int_{-\infty}^{+\infty} e^{-v_1^2/2\sigma^2 - (v_1+v_0)^2}\mathrm{d}v_1 \tag{25}$$

$$\begin{aligned} &= \frac{\sigma_e}{\sqrt{2\pi}\sigma^2} e^{-(1-2\sigma_e^2)v_0^2} \int_{-\infty}^{+\infty} e^{-(v_1+2\sigma_e^2 v_0)^2/2\sigma_e^2}\mathrm{d}v_1 \\ &= \frac{\sigma_e^2}{\sigma^2} e^{-(1-2\sigma_e^2)v_0^2} \qquad (26) \end{aligned}$$

But it follows that, as $\overline{n} = 2\sigma^2$, we get $\sigma_e^2 = \frac{1}{2}\frac{\overline{n}}{\overline{n}+1}$, with $1 - 2\sigma_e^2 = (\overline{n}+1)^{-1}$. Hence,

$$P_{0,1} \quad = \quad (\overline{n}+1)^{-1} e^{-n_0/(\overline{n}+1)} \tag{27}$$

which, combined with (23), yields $n_0 = P_0^{-1}\ln(P_0/P_{0,1})$. From (7), (15) and (21), we have $\Lambda_{0,1} = -\overline{n} - n_0$, and hence,

$$\Lambda_{0,1} = 1 - \frac{1}{P_0} + \frac{1}{P_0}\ln\frac{P_{0,1}}{P_0} \Rightarrow \Delta_0 = -n_0 = \frac{1}{P_0}\ln\frac{P_{0,1}}{P_0} \tag{28}$$

The computation of $\Lambda_{1,1}$ is, by far, the most difficult. We have,

$$\begin{aligned} \Lambda_{1,1} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2} e^{-(v_1^2+v_2^2)/2\sigma^2} \cdot \\ &\quad \cdot \ln\left(1 - e^{-((v_1+v_0)^2+v_2^2)}\right)\mathrm{d}v_1\mathrm{d}v_2 \qquad (29) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma^2} e^{-((v_1-v_0)^2+v_2^2)/2\sigma^2} \cdot \\ &\quad \cdot \ln\left(1 - e^{-(v_1^2+v_2^2)}\right)\mathrm{d}v_1\mathrm{d}v_2 \qquad (30) \end{aligned}$$

Now: $n = v_1^2 + v_2^2, \theta = \mathrm{arctg}(v_2/v_1)$. Therefore, $v_1 = \sqrt{n}\cos\theta$ and $\mathrm{d}v_1\mathrm{d}v_2 = \sqrt{n}(\mathrm{d}\sqrt{n})\mathrm{d}\theta = \frac{1}{2}\mathrm{d}n\mathrm{d}\theta$.

$$\begin{aligned} \Lambda_{1,1} &= \int_{-\pi}^{+\pi}\int_0^{+\infty} \frac{1}{\pi\overline{n}} e^{-(n-2\sqrt{n_0 n}\cos\theta + n_0)/\overline{n}} \cdot \\ &\quad \cdot \ln\left(1 - e^{-n}\right)\frac{1}{2}\mathrm{d}n\mathrm{d}\theta \\ &= \int_0^{+\infty} \frac{1}{\overline{n}} e^{-(n+n_0)/\overline{n}} \cdot \mathrm{I}_0\left(2\frac{\sqrt{n_0 n}}{\overline{n}}\right)\ln(1-e^{-n})\mathrm{d}n \\ &= e^{-n_0/\overline{n}}\mathbb{E}_n\left[\mathrm{I}_0\left(2\frac{\sqrt{n_0 n}}{\overline{n}}\right)\ln(1-e^{-n})\right] \qquad (31) \end{aligned}$$

We set $n = r^2$. Hence, $\mathrm{d}n = 2r\mathrm{d}r$ and,

$$\Lambda_{1,1} = e^{-n_0/\overline{n}}\int_0^\infty \frac{1}{\overline{n}} e^{-r^2/\overline{n}}\mathrm{I}_0\left(2r\frac{\sqrt{n_0}}{\overline{n}}\right)\ln\left(1 - e^{-r^2}\right)2r\mathrm{d}r$$

which is related with Laguerre polynomials $L_p(x)$ as $L_p(-x) = (2^p p!)^{-1} e^{-x}\int_0^\infty t^{2p+1} e^{-t^2/2}\mathrm{I}_0(t\sqrt{2x})\mathrm{d}t$. From $L_0(x) = 1$, we have $e^x = \int_0^\infty t e^{-t^2/2}\mathrm{I}_0(t\sqrt{2x})\mathrm{d}t$, which can be used in the computation of $\Lambda_{1,1}$ if we expand $\ln(1 - e^{-r^2})$ in powers of $e^{-r^2}$,

$$\Lambda_{1,1} \quad = \quad e^{-n_0/\overline{n}}\sum_{k=1}^\infty -\frac{1}{k}\int_0^\infty \frac{2r}{\overline{n}} e^{-r^2(k+1/\overline{n})}\mathrm{I}_0\left(2r\frac{\sqrt{n_0}}{\overline{n}}\right)\mathrm{d}r$$

Changing variables: $r^2(k + 1/\overline{n}) = t^2/2$, we get,

$$\begin{aligned} \Lambda_{1,1} &= \sum_{k=1}^\infty -\frac{1}{k}\frac{e^{-n_0/\overline{n}}}{\overline{n}(k+1/\overline{n})}\int_0^\infty t e^{-t^2/2}\mathrm{I}_0\left(t\sqrt{\frac{2n_0/\overline{n}}{\overline{n}k+1}}\right)\mathrm{d}t \\ &= -\sum_{k=1}^\infty \frac{1}{k(1+\overline{n}k)}\exp\left[-\frac{\overline{n}k}{1+\overline{n}k}\frac{n_0}{\overline{n}}\right] \qquad (32) \end{aligned}$$

which yields,

$$\Delta_1 \quad = \quad \sum_{k=1}^\infty \frac{1}{k(1+\overline{n}k)}\left(1 - \exp\left[-\frac{\overline{n}k}{1+\overline{n}k}\frac{n_0}{\overline{n}}\right]\right) \tag{33}$$

Hence, combining (23),(28) and (33), we have established that $P_0$ and $P_{0,1}$ can be used to determine both $(\overline{n}, n_0)$ and $(\Delta_0, \Delta_1)$.

## 3. DETECTION PROBABILITIES AND RESULTS

In this section, we compute and experimentally verify the detection probability of the signal versus noise hypothesis. In the large sample regime, $\mathbb{E}_{b_n}[J_{b_n}]$ becomes asymptotically Gaussian, so that only the first- and second-order statistics of $x_q$ are needed for establishing the detection probabilities. We define $x_q = \mathbb{E}_{b_n}[x_q] + v_q$, with $v_q \in \{-\mathbb{E}_{b_n}[x_q], 1 - \mathbb{E}_{b_n}[x_q]\}$ an equivalent uncorrelated zero-mean noise sequence. The power of $v_q$ becomes,

$$\begin{aligned}\sigma_v^2 &= \mathbb{E}_{b_n}\left[(x_q - \mathbb{E}_n[x_q])^2\right] = \mathbb{E}_{b_n}[x_q^2] - \mathbb{E}^2_{b_n}[x_q] \quad (34)\\ &= \mathbb{E}_{b_n}[x_q] - \mathbb{E}^2_{b_n}[x_q] = \mathbb{E}_{b_n}[x_q]\cdot(1 - \mathbb{E}_{b_n}[x_q])\end{aligned}$$

Defining $\mu_i = \mathbb{E}_{b_n}[x_q = 1|a_q = i]$ for $i \in \{0, 1\}$, we have,

$$\mathbb{E}_{b_n}[x_q] = \Pr[x_q = 1|a_q] = (1 - a_q)\mu_0 + a_q\mu_1 \quad (35)$$

where from (23) and (27), we establish that,

$$\mu_i = 1 - (\overline{n} + 1)^{-1}e^{-i\cdot n_0/(\overline{n}+1)} \quad (36)$$

From (34) and (35), and using $a_q \in \{0, 1\}$, the covariance becomes,

$$\begin{aligned}\sigma_v^2 &= (1 - a_q)\mu_0 + a_q\mu_1 + \quad (37)\\ &- (1 - a_q)^2\mu_0^2 - 2a_q(1 - a_q)\mu_0\mu_1 - a_q^2\mu_1^2\\ &= (1 - a_q)\mu_0(1 - \mu_0) + a_q\mu_1(1 - \mu_1)\\ &= (1 - a_q)\sigma_0^2 + a_q\sigma_1^2\end{aligned}$$

with $\sigma_i^2 = \mu_i(1 - \mu_i)$. Setting $\zeta = (\Delta_1 - \Delta_0)/\Delta_0 < 0$ in (20), and using (35) for $x_q = \mathbb{E}_{b_n}[x_q] + v_q$, we have,

$$J_{\mathrm{H/H'}} = \Delta_0 \sum_{q=1}^{N_b}(a_q - a_q')\cdot(1 + x_q\zeta) \quad (38)$$

$$= \Delta_0 \sum_{q=1}^{N_b}(a_q - a_q')\cdot(1 + ((1 - a_q)\mu_0 + a_q\mu_1 + v_q)\zeta)$$

To test the signal versus noise hypothesis, we set $a_q' = 0$. Hence, for $N_a = \sum_{q=1}^{N_b} a_q$ the number of active signal bins in the sequence,

$$J_{\mathrm{H_1/H_0}} = \Delta_0\left(N_a(1 + \mu_1\zeta) + \zeta\sum_{q=1}^{N_b} a_q v_q\right) \quad (39)$$

where $J_{\mathrm{H_1/H_0}} = \Delta_0 N_a(1 + \mu_1\zeta) + \varrho$ , with $\varrho$ an asymptotically (for large $N_a, N_b$) zero-mean Gaussian noise term, of power $\sigma_\varrho^2 = \Delta_0^2\zeta^2 N_a\sigma_1^2$. The probability of not detection is calculated in terms of the complementary error function $\mathrm{erfc}(x) = \sqrt{4/\pi}\int_x^\infty e^{-t^2}\mathrm{d}t$,

$$P_e = \frac{1}{2}\mathrm{erfc}\left(\frac{1 + \mu_1\zeta}{\zeta\sigma_1}\sqrt{\frac{N_a}{2}}\right) \quad (40)$$

It is important to note that this expression is only valid in those ranges where the cumulative probability distribution of $\varrho$ can be approximated to that of a Gaussian (which excludes the extreme values of $\varrho$). In fact, $\zeta$ cannot not be an arbitrary value. Generally, when testing $a_q$ versus $a_q'$ using (38), it can be shown after lengthy calculations that the probability of error is $P_e = \Pr[\hat{a}_q = a_q'|a_q] =$

$$\frac{1}{2}\mathrm{erfc}\left(\frac{N_a - N_{a'} + ((N_a - N_{aa'})\mu_1 - (N_{a'} - N_{aa'})\mu_0)\zeta}{\zeta\sqrt{2((N_a - N_{aa'})\sigma_1^2 + (N_{a'} - N_{aa'})\sigma_0^2))}}\right)$$

where $N_{a'} = \sum_{q=1}^{N_b} a_q'$ and $N_{aa'} = \sum_{q=1}^{N_b} a_q a_q'$ the number of common active signal bins. When each sequence has the same number of active bins: $N_a = N_{a'}$, it reduces to the simpler expression,

$$P_e = \frac{1}{2}\mathrm{erfc}\left(\frac{\mu_1 - \mu_0}{\sqrt{\sigma_0^2 + \sigma_1^2}}\sqrt{\frac{N_a - N_{aa'}}{2}}\right) \quad (41)$$

with associated metric $J_{\mathrm{H/H'}} = \Delta_0\zeta\sum_{q=1}^{N_b}(a_q - a_q')x_q$ such that knowledge of $\zeta$ becomes irrelevant. In conclusion, for a sequence set whose members share the same Hamming weight, the previous LLR is equivalent to applying the sequence matched filter to the detector outputs $x_q$ for all members in the set, and this is asymptotically ML. When the dependence on $\overline{n}$ and $n_0$ is made explicit in (40-41), the error rates do not solely depend on the signal to noise ratio $n_0/\overline{n}$ as is usual in radio communications, but the relationship is more complex. Equation (41) sets the basis for computing the achievable rates (i.e., the $\log_2$-size of the set of sequences $a_q$) under a given $P_e$. Finally, [4] is and interesting reference for photon detector arrays.

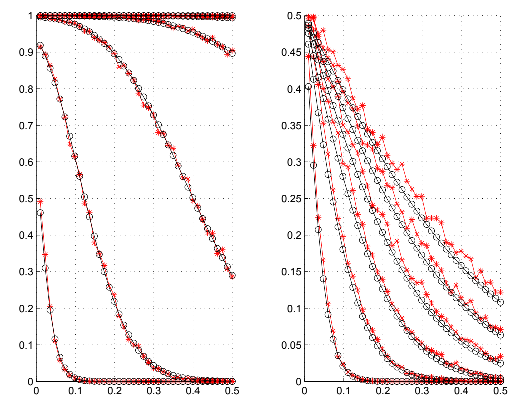


**Fig. 1**. [Left] Probability of detection error (40) and [Right] probability of error (41) versus $n_0$ for a length of 200 chips (3 bins/chip) for several $\overline{n}$: $0.1 \le \overline{n} \le 1.6$ in steps of 0.3. The theoretical prediction (noughts) closely agrees with the experimental probabilities (crosses) measured from sets of 4000 runs. Note that for 200 chips, the optimum detector is far too complex for implementation.

## 4. REFERENCES

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