OPTIMAL QUANTIZATION SCHEMES FOR ORTHOGONAL RANDOM BEAMFORMING - A CROSS-LAYER APPROACH

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ABSTRACT

Orthogonal random beamforming (ORB) has recently attracted significant interest because of its ability to exploit multi-user diversity and spatial multiplexing gains by exclusively using partial channel state information (CSI). In this paper, we explore the impact of CSI quantization on the performance of ORB. By resorting to a dynamic programming formulation, we identify the *optimal* (i.e. pdf-matched) quantizer for which the sum-rate distortion is minimized. This is a cross-layer approach in the sense that the optimal quantizer (physical layer) actually depends on other layers' parameters such as the number of admitted users (link layer). Performance is assessed by means of computer simulations and compared with that of a uniform quantizer.

Index Terms: multi-user diversity, scheduling, quantization, random beamforming.

1. INTRODUCTION

The exploitation of multi-user diversity (MUD) [1] relies on the assumption that different users in a wireless multi-user system experience independent fading processes. In those circumstances, the cell throughput in the downlink of a Single-Input Single-Output (SISO) multi-user system can be maximized by scheduling in each time slot the user with the most favorable channel conditions [2]. To do so, only partial channel state information, namely SNRs, must be estimated and, in the case of FDD systems, be reported by the terminals to the Base Station (BS). In a context of Multiple-Input Multiple-Output (MIMO) Broadcast Channels, Dirty Paper Coding (DPC) is known to be the capacity-achieving strategy [3]. However, DPC is computationally intensive and requires full channel state information at the transmitter (CSIT). The computational complexity of Transmit Zero-Forcing (TxZF) [4] beamforming is far more affordable but, still, there is a need for full CSIT. Orthogonal random beamforming (ORB) schemes [5], instead, merely require partial CSIT, mostly SINR measurements for each transmit beam. Hence, ORB has emerged as a viable alternative to DPC and TxZF, in particular in the asymptotic case of a high number of users where the sum-rate exhibits the same growth rate as TxZF and DPC.

Since CSI (either partial or full) must be quantized before its transmission over a feedback channel, a number of authors have analyzed the impact of quantization on the exploitation of MUD. In [6], for instance, the authors found that in a SISO context most of the MUD gain can still be extracted when the measured SNRs are quantized with very few bits. In a MIMO context, we analyzed in [7] the impact of CSI quantization on the throughput and the sum-rate of ORB beamforming. The limiting case of one-bit quantizers deserves some attention as well. In [8], the authors designed a one-bit quantizer in such a way that sum-rate of the quantized ORB system exhibits the same growth rate as with *analog* CSI.

In this paper, we complement the work in [6] by (1) extending the study to encompass ORB (i.e. in a MIMO setting); (2) identifying the *optimal* set of quantization thresholds and, (3) comparing the performance of such optimal quantizer with that of a uniform one. We also go one step beyond [7] and move from a heuristic quantization law to the optimal one, being this defined by a set of pdf-matched quantization intervals. Since this problem cannot be solved analytically we must resort to a numerical solution. In particular, we adopt the dynamic programming formulation presented in [9] which, unlike other less sophisticated algorithms [10] possibly suffering from convergence to local minima, can always find the optimal quantizer.

2. SIGNAL MODEL

Consider the downlink of a wireless system with one Base Station (BS) equipped with M antennas, and K single-antenna terminals. In order to serve multiple users, we generate a pre-coding matrix $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_M]$[5], the columns of which, $\mathbf{w}_i \in \mathbb{C}^{M\times 1}$, $i = 1..M$, are isotropically-distributed random orthonormal vectors [11]. Each of those vectors is then used to transmit data to the users experiencing the highest SINRs. The received signal at the k-th terminal can be written as:

$$r_k = \mathbf{h}_k^T \mathbf{W}\mathbf{s} + n_k \tag{1}$$

where in the above expression the time index has been dropped for the ease of notation, $\mathbf{h}_k \in \mathbb{C}^{M\times 1}$ is the channel vector gain between the BS and the k-th terminal $\mathbf{h}_k \sim \mathcal{CN}(0, \mathbf{I}_M)$ (independent Rayleigh fading), $\mathbf{s} \in \mathbb{C}^{M\times 1}$ is the symbol vector, and $n_k \in \mathbb{C}$ denotes additive Gaussian noise (AWGN) with zero mean and variance σ^2. The active users in the system are assumed to undergo independent Rayleigh fading processes as well. Further, we consider block-fading, that is, the channel response remains constant during one time-slot and then it abruptly changes to a new independent realization.

Concerning CSI, we assume perfect knowledge at the terminals and the availability of a low-rate error- and delay-free feedback channel to convey partial CSI to the transmitter. Finally, the total transmit power, P_t, is constant and evenly distributed among the active beams, i.e., $\mathbb{E}\{\mathbf{s}^H\mathbf{s}\} = P_t$ and, hence, we can define $\rho = \frac{P_t}{\sigma^2}$ as the average SNR.

This work is partially supported by the IST projects NEWCOM (50725) and ACE2 (26957) and the Generalitat de Catalunya (SGR2005-00690)

3. SUM-RATE WITH ANALOG FEEDBACK

According to the signal model of the previous section, the received signal for user k when using beamformer i can be re-written from (1) as:

$$r_{k,i} = \mathbf{h}_k^T \mathbf{w}_i s_i + \sum_{\substack{j=1 \\ j\neq i}}^{M} \mathbf{h}_k^T \mathbf{w}_j s_j + n_k \tag{2}$$

where s_j stands for the symbol transmitted with beam j. The last two terms in the above expression account for the interference-plus-noise contribution and, hence, the corresponding SINR measured at the terminal reads:

$$\gamma_{k,i} = \frac{|\mathbf{h}_k^T \mathbf{w}_i|^2}{M/\rho + \sum_{\substack{j=1 \\ j\neq i}}^{M} |\mathbf{h}_k^T \mathbf{w}_j|^2} = \frac{z}{M/\rho + y} \tag{3}$$

Since we assume that all users experience i.i.d Rayleigh fading and the beamformers are orthonormal to each other, the two random variables z and y are independent chi-square distributed: $z \sim \chi_2^2$ and $y \sim \chi_{2M-2}^2$ [5]. Bearing this in mind, the cumulative density function (CDF) and the probability density function (pdf) of the *pre-scheduling* SINR (which, by symmetry, do not depend on subscripts k or i) can be expressed as:

$$F_{\mathsf{SINR}}(\gamma) = 1 - \frac{e^{-\frac{\gamma M}{\rho}}}{(1+\gamma)^{M-1}} \tag{4}$$

$$f_{\mathsf{SINR}}(\gamma) = \frac{e^{-\frac{\gamma M}{\rho}}}{(1+\gamma)^{M}} \left(\frac{M}{\rho}(1+\gamma) + M - 1 \right) \tag{5}$$

The scheduler in the BS operates in a slot-by-slot basis following a *max*-SINR (*greedy*) rule. That is, for beam i, the scheduler selects the active user k_i^* satisfying[1]:

$$k_i^* = \arg \max_{k=1..K} \{\gamma_{k,i}\} \qquad i = 1 \dots M$$

which experiences a *post-scheduling* SINR given by:

$$\gamma_i^* = \max_{k=1..K} \{\gamma_{k,i}\} \qquad i = 1 \dots M$$

Again, the statistics of γ_i^* do not depend on the beam index i and, hence, it will be dropped in the sequel. Since all users experience i.i.d Rayleigh fading, the CDF of the *post-scheduling* SINR can be readily expressed as [5]:

$$F_{\mathsf{SINR}^*}(\gamma) = \left(F_{\mathsf{SINR}}(\gamma)\right)^K = \left(1 - \frac{e^{-\frac{\gamma M}{\rho}}}{(1+\gamma)^{M-1}}\right)^K \tag{6}$$

and by differentiating the above expression the corresponding pdf results:

$$f_{\mathsf{SINR}^*}(\gamma) = K \frac{e^{-\frac{\gamma M}{\rho}}}{(1+\gamma)^{M}} \left(\frac{M}{\rho}(1+\gamma) + M - 1 \right) \times \left(1 - \frac{e^{-\frac{\gamma M}{\rho}}}{(1+\gamma)^{M-1}}\right)^{K-1} \tag{7}$$

[1] We implicitly assume that a different user is scheduled on each beam since the probability that one user achieves the highest SINR on more than one beam is negligible when $K >> M$[5].

Finally, one can readily express the sum-rate R in terms of the pdf above as:

$$\begin{aligned} R &\approx \mathbb{E}_{\gamma^*}\left[\sum_{i=1}^{M} \log_2\left(1 + \max_{1\leq k\leq K} \gamma_{k,i}\right)\right] \\ &= M \mathbb{E}_{\gamma^*}\left[\log_2\left(1 + \max_{1\leq k\leq K} \gamma_{k,i}\right)\right] \\ &= M \int_0^{\infty} \log_2(1+\gamma)\, f_{\mathsf{SINR}^*}(\gamma) d\gamma \end{aligned} \tag{8}$$

Throughout this section, we have unrealistically assumed that, in being transmitted over the feedback channel, the measured SINRs can be represented with infinite precision (i.e. analog representation). This shortcoming will be addressed in the next section where we present the optimal quantization strategy.

4. OPTIMAL QUANTIZATION STRATEGIES

In a realistic scenario, the BS is constrained to schedule users on the basis of a *quantized* version of the pre-scheduling SINRs, $Q(\gamma_{k,i})$. Let $\Gamma_d = \{\gamma_{d_0} < \gamma_{d_1} < \dots < \gamma_{d_{N_q}}\}$ be the input decision levels and let $\Gamma_q = \{\gamma_{q_0} < \gamma_{q_1} < \dots < \gamma_{q_{N_q-1}}\}$ be the output representative levels of an $N_q = 2^{L_q}$-level quantizer $\mathcal{Q}(\cdot)$ which is defined as:

$$Q(\gamma) = \gamma_{q_j} \qquad \text{if } \gamma_{d_j} \leq \gamma < \gamma_{d_{j+1}}. \tag{9}$$

Hence, the (quantized) *post-scheduling* SINR on beam i becomes[2]:

$$\max_{k=1..K} \{Q(\gamma_{k,i})\} \qquad i = 1 \dots M$$

or, equivalently, by exchanging the $\max$ and Q operators

$$Q\left(\max_{k=1..K} \{\gamma_{k,i}\}\right) = Q(\gamma_i^*) \qquad i = 1 \dots M$$

After all these manipulations, we conclude that the problem to solve is that of identifying an optimum (i.e. pdf-matched) set of decision and representative levels, such that the average distortion introduced by the N_q-level quantizer:

$$\begin{aligned} D_{N_q} &= \mathbb{E}_{\gamma^*}\left[e\left(\gamma^*, Q(\gamma^*)\right)\right] \\ &= \sum_{i=0}^{N-1} \int_{\gamma_{d_i}}^{\gamma_{d_{i+1}}} e\left(\gamma, \gamma_{q_i}\right) f_{\mathsf{SINR}^*}(\gamma) d\gamma \end{aligned} \tag{10}$$

is minimized, with $e(\cdot,\cdot)$ standing for an appropriate error weighting function. Therefore, we are facing a cross-layer design in the sense that the optimal quantizer in the *physical* layer is tightly coupled with system-level parameters (introduced through f_{SINR^*}), such as the number of admitted users (K) or the number of antennas in the base station. Clearly, the necessary conditions for an optimal quantizer $\{\Gamma_d^*, \Gamma_q^*\}$ are given by

$$\begin{aligned} \frac{\partial D_{N_q}}{\gamma_{d_i}} &= 0 \qquad i = 1, \dots, N_q - 1 \\ \frac{\partial D_{N_q}}{\gamma_{q_i}} &= 0 \qquad i = 0, \dots, N_q - 1 \end{aligned} \tag{11}$$

Unfortunately, the resulting equation set is in general non-linear and, thus, extremely difficult to solve. Alternatively, one can resort to a

[2] If more than one SINRs belong to the highest quantization interval, one of them is selected at random.

dynamic program formulation [9] and obtain a numerical solution. As expressed in (10), the average distortion D_{N_q} depends explicitly both on the Γ_d and Γ_q sets. However, as proved in [9], we can first determine the optimal Γ_d and then find the optimal Γ_q, in such a way that each γ_{q_i} is determined from γ_{d_i} and $\gamma_{d_{i+1}}$ (notice that a modification of any one γ_{q_i} concerns only one integral term in the sum (10)). In this context, we define two functions $D_1(\alpha,\beta)$ and $D_n(\alpha,\beta)$ as follows.

$D_1(\alpha,\beta)$ is defined as the minimum value of the distortion measure when just one output level is placed in (α,β), a subrange of $(\gamma_{d_0},\gamma_{d_{N_q}})$. That is,

$$D_1(\alpha,\beta) \triangleq \min_y \int_\alpha^\beta e(\gamma,y)\, f_{\mathsf{SINR}^*}(\gamma)d\gamma \tag{12}$$

Next, $D_n(\alpha,\beta)$ is defined as the minimum distortion when n levels are place in the range (α,β), where $n \geq 2$. That is,

$$D_n(\alpha,\beta) \triangleq \min_{\substack{\gamma_{d_1},\ldots,\gamma_{d_{n-1}} \\ \gamma_{q_0},\ldots,\gamma_{q_{n-1}} \\ (\alpha<\gamma_{d_1}\ldots<\gamma_{d_{n-1}}<\beta)}} \sum_{i=0}^{n-1} \int_{\gamma_{d_i}}^{\gamma_{d_{i+1}}} e(\gamma,\gamma_{q_i})\, f_{\mathsf{SINR}^*}(\gamma)d\gamma \tag{13}$$

This expression can be rewritten in terms of (12) as

$$D_n(\alpha,\beta) = \min_{\substack{\gamma_{d_1},\ldots,\gamma_{d_{n-1}} \\ (\alpha<\gamma_{d_1}\ldots<\gamma_{d_{n-1}}<\beta)}} \sum_{i=0}^{n-1} D_1(\gamma_{d_i},\gamma_{d_{i+1}}) \tag{14}$$

where $\gamma_{d_0} = \alpha$ and $\gamma_{d_n} = \beta$. Notice that γ_{d_i} and γ_{q_i} denote the interim search variables whereas their optimal counterparts will be labeled with the subscript *.

The search algorithm consists of the following five steps:

1. **Initialization**: Compute and store the values of $D_1(\alpha,\beta)$ for all discrete α and β in $(\gamma_{d_0},\gamma_{d_{N_q}})$. To do so, we assume N_q,γ_{d_0} and $\gamma_{d_{N_q}}$ to be set in advance and, also, that the range $(\gamma_{d_0},\gamma_{d_{N_q}})$ is divided into an appropriate number of segments.
2. **Insertion of decision levels**: For each n from two to N_q, compute both
$$D_n(\gamma_{d_0},\gamma) = \min_{\substack{\alpha \\ \gamma_{d_0}<\alpha<\gamma}} [D_{n-1}(\gamma_{d_0},\alpha) + D_1(\alpha,\gamma)]$$
and $\gamma_{d_n}(\gamma_{d_0},\gamma)$, which denotes the optimum value of variable α for which $D_n(\gamma_{d_0},\gamma)$ is minimized. Store all these values. This is done for all discrete γ in $(\gamma_{d_0},\gamma_{d_{N_q}})$ and, by doing so, one can identify the best point to insert an additional decision level within (γ_{d_0},γ) and, also, compute the associated distortion.
3. **Computation of the optimal decision levels**: For each n from N_q to two, set
$$\gamma^*_{d_{n-1}} = \gamma_{d_{n-1}}(\gamma^*_{d_0},\gamma^*_{d_n}) \tag{15}$$
with $\gamma^*_{d_{N_q}} = \gamma_{d_{N_q}}$ and $\gamma^*_{d_0} = \gamma_{d_0}$.
4. **Computation of the optimal representative levels**: For each n from zero to N_q-1, compute $\gamma^*_{q_n}$ so that $D_1(\gamma^*_{d_n},\gamma^*_{d_{n+1}})$ is the minimum value as given in (12).
5. **End of algorithm**.

Clearly, this algorithm attains the absolutely optimal quantizer without requiring differentiation of D with respect to either the decision or representative levels.

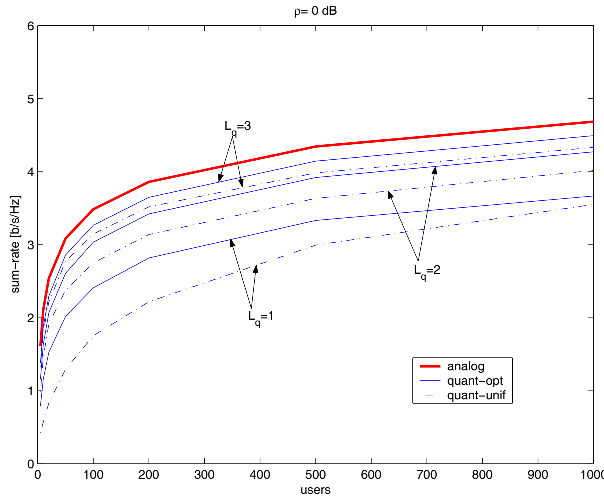

Fig. 1. Sum-rate vs. number of users with analog and quantized CSI. Noise-limited scenario (ρ=0 dB).

4.1. Specific considerations

In our discussions, the error weighting function $e(\gamma,\gamma_{q_i})$ has been left as a free parameter. Since the optimally-quantized SINRs should minimize the *sum-rate* distortion (8), we define the error function as

$$\begin{aligned} e(\gamma,\gamma_{q_i}) &= \log_2(1+\gamma) - \log_2(1+\gamma_{q_i}) \\ &= \log_2\left(\frac{1+\gamma}{1+\gamma_{q_i}}\right) \end{aligned} \tag{16}$$

Besides, some system-level restrictions apply to the optimal set of representative levels Γ_q. Since, ultimately, the quantized SINRs will be used to adjust the constellation size and the coding scheme at the transmitter, one should make sure that such quantized versions are never above the actual SINR value. Otherwise, the estimated data rate ($\log_2(1+\gamma_{q_i})$) would exceed that which can be reliably supported by channel ($\log_2(1+\gamma_q)$ and an outage would result. Hence, we impose the representative level for each quantization interval to be equal to its lower decision level:

$$\gamma_{q_i} = \gamma_{d_i}; \qquad i = 0\ldots N_q-1 \tag{17}$$

This results in an increase of the overall distorsion but, interestingly, it also considerably simplifies steps 1) and 4) of the algorithm.

5. COMPUTER SIMULATION RESULTS

We consider a system with $K = 1\ldots 1000$ active users and one BS with $M = 4$ antennas. In Figs.1-2, we assess the performance of the optimal quantizer and compare it with that of (1) a uniform quantizer, and (2) a system where SINRs have *analog* precision which constitutes an upper bound of performance. Results are given for a varying number of quantization bits/intervals and both in noise- and interference-limited scenarios ($\rho = 0, 20$ dB, respectively). In all cases, we force the quantization intervals to cover 99% of the dynamic range of the SINRs, that is, we set $\gamma_{d_0} = 0$ and $\gamma_{d_{N_q}}$ in the 99% percentile.

First, we can observe that in both scenarios the performance exhibited by the optimal quantizer with $L_q = 3$ bits is very close to that of the analog system: $\frac{4.49}{4.68} = 95\%$ for the noise-limited case

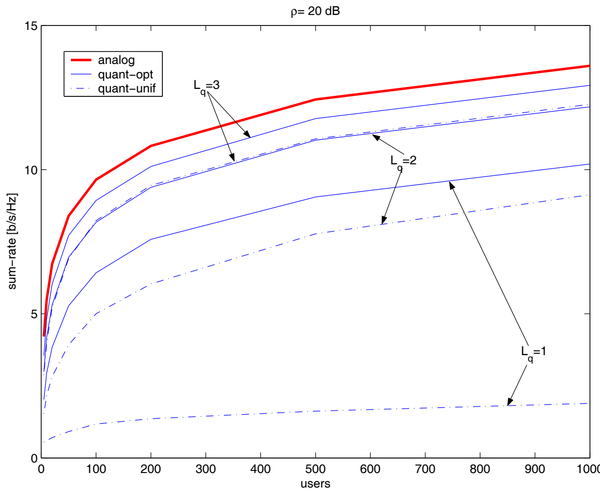

Fig. 2. Sum-rate vs. number of users with analog and quantized CSI. Interference-limited scenario (ρ=20 dB).

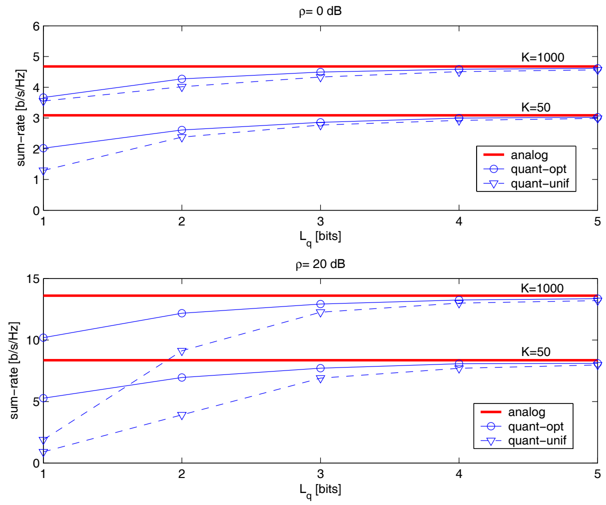

Fig. 3. Sum-rate vs. number of quantization bits. Top: ρ=0 dB. Bottom: ρ=20 dB.

and similar values in the interference-limited scenario ($K = 1000$). In absolute terms, though, the gap between both curves is wider in the second case because the dynamic range of the *post-scheduling* SINR is higher there. In other words, the pdf gets wider for an increasing ρ, (see (5)) and, accordingly, the quantization intervals are wider too. With $L_q = 1$ quantization bits the system still manages to retain up to $\frac{3.66}{4.68} = 78\%$ and $\frac{10.19}{13.60} = 75\%$ of the analog performance in noise- and interference-limited scenarios, respectively ($K = 1000$).

Besides, the pdf-matched quantizer clearly outperforms its uniform counterpart in all cases and scenarios. The performance loss that the uniform quantizer suffers from is in general more severe in the $L_q = 1$ bit case: up to $\frac{10-2}{10} = 80\%$ w.r.t. the optimal quantizer for $K = 1000$ (interference-limited scenario). The fact that performance relies on the value that the *single* decision level takes (recall that γ_{d_0} and $\gamma_{d_{N_q}}$ are set in advance) makes it very sensitive with respect to non-optimal designs. Still, the distance between the decision thresholds associated with the optimal and uniform quantizers strongly depends on parameters such as ρ or K (since so does the pdf of the *post-scheduling* SINR) and in some cases (e.g. $\rho = 0$, $K = 1000$) the gap between both curves is relatively narrow. However, only the optimal quantizer can guarantee close-to-analog performance in a general case.

When the number of quantization levels increases to $L_q = 2, 3$ bits, the loss of the uniform quantizer w.r.t the optimal one decreases. Indeed, when then number of intervals is higher some of the quantization intervals derived with the uniform quantizer partially overlap with the optimal ones, this making performance loss less severe.

Finally, in Fig.3 we analyze the impact of the number of quantization levels for a longer series of L_q values. Interestingly, from $L_q = 4$ bits on, the performance of the optimal quantizer is practically undistinguishable from that of the analog system, in particular for the noise-limited scenario.

6. CONCLUSIONS

In this paper, we have analyzed the impact of CSI quantization on the performance of orthogonal random beamforing. With as few as 3 or 4 bits, the optimal quantizer attains a sum-rate virtually identical to that of the analog system. In this case, the performance loss associated with the uniform quantizer is moderate. With one quantization bit, which is a case that recently has attracted lots of attention, the optimal quantizer still retains on the order of 75% of the analog sum-rate. In those conditions, however, the uniform quantizer often suffers from substantial performance losses.

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