DYNAMIC SPECTRUM MANAGEMENT: WHEN IS FDMA SUM-RATE OPTIMAL?

Shunsuke Hayashi[*]

Graduate School of Informatics
Kyoto University
Kyoto 606-8501, Japan

Zhi-Quan Luo[†]

Dept. of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455

ABSTRACT

Consider a multiuser communication system in a frequency selective environment whereby users share a common spectrum and can interfere with each other. Assuming Gaussian signaling and treating interference as noise, we study optimal spectrum sharing strategies for the maximization of weighted sum-rate. In this work, we show that, if the normalized crosstalk gains are larger than a given threshold (roughly equal to $\frac{1}{2}$), then the optimal spectrum sharing strategy is Frequency Division Multiple Access (FDMA). We also propose several simple distributed spectrum allocation algorithms that can approximately maximize weighted sum-rates. Numerical simulation of DSL applications shows that these algorithms are efficient and can achieve substantially larger weighted sum-rates than those obtained by the existing Iterative Waterfilling algorithm.

Index Terms— Weighted sum-rate maximization, FDMA, distributed optimization, dynamic spectrum management

1. INTRODUCTION

With the proliferation of radio devices and communication services, multiple systems sharing a common spectrum must coexist. Examples of this type include both wirelined applications, like unbundled DSL services from different service providers, and wireless applications such as cognitive radio. Conventional spectrum management is via static FDMA whereby each user has a pre-assigned band. As an orthogonal transmission strategy, FDMA eliminates multiuser interference. However, it may also lead to low system utilization.

Recently there has been increasing interest in dynamic spectrum management whereby users are allowed to access the entire spectrum simultaneously. In such a system, each user's performance depends on not only the power allocation (across spectrum) of his own, but also those of other users in the system. Thus, proper spectrum management is needed to ensure a social optimum (e.g., sum-rate maximization) is reached. A popular dynamic spectrum access strategy is the iterative water-filling algorithm (IWFA) [1, 2], whereby each user greedily maximizes his own data rate by using the well-known water-filling strategy, while assuming all other users' power spectrum are fixed. When crosstalk interference is small, IWFA is known to generate a sequence of power allocations converging to the unique Nash equilibrium of an non-cooperative game. However, when the crosstalk interference is strong, the solution offered by IWFA can be far from social optimum.

In this paper, we consider the maximization of the weighted sum of achievable rates of all users in the system. Suppose the common spectrum is divided into N frequency slots (or tones). We show that, if the normalized crosstalk gains for all users across all tones are greater than $1/2$ (roughly), then the optimal spectrum sharing strategy is FDMA[1]. This result generalizes our early work [3] which considered only equal-weight sum-rate maximization problem. We also present some simple algorithms that can approximately maximize weighted sum-rates. Similar studies on the optimality of FDMA were presented in [4,5], albeit only for the two user case.

2. PRELIMINARIES

Suppose there are K users sharing a common spectrum which is divided into N frequency tones numbered by $\{1, 2, ..., N\}$. For notational simplicity, we assume that each user acts both as a transmitter and as a receiver[2], and we number the transmitters and receivers by the same index set $\{1, 2, ..., K\}$. In this way, a physical user may act as transmitter k and receiver l, with $l \neq k$. Let x_k^n denote the transmitted complex Gaussian signal from transmitter k at tone n, and let $S_k^n := E|x_k^n|^2$ denote its power. The received signal y_k^n is given by

$$y_k^n = \sum_{l=1}^{K} h_{lk}^n x_l^n + z_k^n, \quad n \in \mathcal{N},\ k \in \mathcal{K},$$

where $\mathcal{N} := \{1, \ldots, N\}, \mathcal{K} := \{1, \ldots, K\}, z_k^n \sim CN(0, N_0)$ denotes the complex Gaussian channel noise with zero mean and variance N_0, and the complex scalars $\{h_{lk}^n\}$ represent

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[1] We also note that, if the crosstalk coefficients are small, then the FDMA solution is quite far from the true global optimum.

[2] There is no loss of generality with this assumption since we can always create a virtual channel with zero channel gain gains between pair of users who do not wish to communicate.

channel gain gains. In practice, h_{lk}^n can be determined by the distance between transmitter l and receiver k. Treating interference as white noise, we can write transmitter k's achievable data rate R_k^n at tone n [6] as

$$\begin{aligned} R_k^n(S_1^n, \ldots, S_K^n) &:= \log\left(1 + \frac{|h_{kk}^n|^2 S_k^n}{N_0 + \sum_{l \neq k} |h_{lk}^n|^2 S_l^n}\right) \\ &= \log\left(1 + \frac{S_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n S_l^n}\right), \end{aligned}$$

where $\sigma_k^n = N_0/|h_{kk}^n|^2$ denotes the normalized background noise power, and $\alpha_{lk}^n = |h_{lk}^n|^2/|h_{kk}^n|^2$ is the normalized crosstalk gain from transmitter l to receiver k at tone n. Due to normalization, we have $\alpha_{kk}^n = 1$ for all k.

Assume that transmitter k's power is bounded:

$$\sum_{n=1}^{N} S_k^n \leq P_k, \quad \text{for } k \in \mathcal{K}.$$

For a given power allocation $\{S_k^n\}$, transmitter k's total achievable data rate is given by $\sum_{n=1}^{N} R_k^n$. Hence, the weighted sum-rate maximization problem can be written as follows:

$$\begin{aligned} \underset{\{S_1^n, \ldots, S_K^n\}_{n=1}^N}{\text{maximize}} \quad & \sum_{k=1}^{K} w_k \sum_{n=1}^{N} R_k^n(S_1^n, \ldots, S_K^n) \\ \text{subject to} \quad & \sum_{n=1}^{N} S_k^n \leq P_k, \quad S_k^n \geq 0 \quad n \in \mathcal{N},\ k \in \mathcal{K}, \end{aligned} \quad (1)$$

where $w_k > 0$ is the positive parameter characterizing the weight to user k's achievable rate. If $w_1 = \cdots = w_K$, then the above problem is the standard (equal-weight) sum-rate maximization problem.

When interference is absent (or small), signal spreading across spectrum is optimal. In other words, if the crosstalk gains are sufficiently small, then all frequency tones should be utilized by all users. On the other hand, if the crosstalk gains are large, then the communication system becomes interference limited, and spectrum sharing is no longer optimal. Intuitively, FDMA should yield a larger sum-rate in this case. Mathematically, the set of FDMA solutions is defined as:

$$\mathcal{S} = \{\mathbf{S} \geq 0 \mid S_k^n S_l^n = 0,\ \forall\, k \neq l,\ \forall\, n\} \quad (2)$$

In the ensuing sections, we let $\mathbf{S}^n$ and $\mathbf{S}$ denote the power vectors at tone n, and in the whole system, respectively, i.e., $\mathbf{S}^n := (S_1^n, \ldots, S_K^n) \in \Re^K$, and $\mathbf{S} := (S_1^1, \ldots, S_K^N) \in \Re^{NK}$. We denote the power budget vector by $\mathbf{P}$, i.e., $\mathbf{P} := (P_1, \ldots, P_K) \in \Re^K$.

3. WHEN IS FDMA OPTIMAL?

In this section we show that FDMA type power allocation maximizes the weighted sum-rate if the crosstalk gains are larger than a certain explicit threshold.

Let us first introduce a mild assumption on a feasible power allocation vector $\mathbf{S}$.

Assumption 1 *Let $\mathcal{T}_k := \{n \in \mathcal{N} \mid S_k^n > 0\}$. Then, there holds* (a) $\min_{k \in \mathcal{K}} |\mathcal{T}_k| \geq C$ *for some integer $C \geq 2$, and* (b) $\sum_{n=1}^{N} \mathbf{S}^n = \mathbf{P}$.

Under this assumption, we obtain the following sufficient conditions under which every global maximum of (1) is FDMA.

Theorem 1 *Let $w_1, \ldots, w_K$ be arbitrary positive reals. Then, any global maximum of problem* (1) *satisfying Assumption 1 for some $C \geq 2$ must be FDMA, provided that*

$$\alpha_{lk}^n > \tfrac{1}{2},\ \alpha_{lk}^n \alpha_{kl}^n > \tfrac{1}{4}(1 + (C-1)^{-1})^2 \quad (3)$$

for all $n \in \mathcal{N}$ and $(k, l) \in \mathcal{K} \times \mathcal{K}$ with $k \neq l$.

When C is sufficiently large, we have $1 + (C-1)^{-1} \approx 1$. In this case, the condition $\alpha_{lk}^n \alpha_{kl}^n > \frac{1}{4}(1 + (C-1)^{-1})^2$ is essentially implied by the condition $\alpha_{lk}^n > \frac{1}{2}$. In a practical system with large number of tones, Assumption 1 often holds with a sufficiently large C. For the two-user case, we can strengthen the above sufficient conditions by exploiting the quasi-convexity of function $\sum_{k=1}^{K} w_k R_k^n$.

Theorem 2 *Let $K = 2$ and w_1 and w_2 be arbitrary positive reals. Then, any global maximum of problem* (1) *satisfying Assumption 1 for some $C \geq 2$ is FDMA, if the following inequalities hold for all $n \in \mathcal{N}$:*

$$\text{(i)}\ \ 2w_1 \alpha_{12}^n \left(\frac{\sigma_1^n}{\sigma_2^n}\right) + 2w_2 \alpha_{21}^n \left(\frac{\sigma_2^n}{\sigma_1^n}\right) > w_1 + w_2,$$

$$\text{(ii)}\ \ 2w_1 \alpha_{12}^n \left(\frac{\sigma_1^n}{\sigma_2^n}\right) > w_1 - w_2,\ \ 2w_2 \alpha_{21}^n \left(\frac{\sigma_2^n}{\sigma_1^n}\right) > w_2 - w_1$$

$$\text{(iii)}\ \ \alpha_{12}^n \alpha_{21}^n > \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^2$$

Note that at least one inequality in (ii) holds automatically. Moreover, if $w_1 = w_2$, then inequalities (ii) always hold and (i) is implied by (iii)[3].

In [3], it was shown that there exists a FDMA type local maxima for equal-weight sum-rate maximization problem, even when the crosstalk gains are small (but positive), so long as users' power budgets are sufficiently large. It is not clear if a similar result can be derived in the weighted sum-rate case.

4. FINDING AN OPTIMAL FDMA BANDWIDTH ALLOCATION

In this section, we focus our attention on how to design an optimal FDMA scheme for a multiuser communication system. For notational convenience, we restrict our attention to the equal-weight sum-rate case, i.e., $w_1 = \cdots = w_K = 1$. The algorithms can be extended to the weighted sum-rate maximization problems in a straightforward manner. Recall (2) that the set of FDMA solutions is denoted by

$$\mathcal{S} = \{\mathbf{S} \geq 0 \mid S_k^n S_l^n = 0,\ \forall\, k \neq l,\ \forall\, n\}.$$

[3] From the relation between arithmetic and geometric means, it can be easily seen that, if $w_1 = w_2$ and $\alpha_{12}^n \alpha_{21}^n > 1/4$, then (i) holds.

Then, the optimal FDMA frequency allocation problem can be described as follows:

$$\begin{aligned} &\underset{\mathbf{S}}{\text{maximize}} \quad \sum_{k=1}^{K}\sum_{n=1}^{N}\log\left(1+\frac{S_k^n}{\sigma_k^n}\right) \\ &\text{subject to} \quad \mathbf{S}\in\mathcal{S},\ \sum_{n=1}^{N} S_k^n \le P_k.\ (\forall\, k\in\mathcal{K}) \end{aligned} \tag{4}$$

Notice that, due to the FDMA condition, the interference term $\sum_{l\neq k}\alpha_{lk}^n S_l^n$ is absent from the objective function. This makes the objective function concave. However, problem (4) remains a nonconvex problem due to the constraint $\mathbf{S}\in\mathcal{S}$.

In what follows, we show some simple algorithms for computing an approximately optimal FDMA bandwidth allocations. The first one is based on dual decomposition, while the second is based on the idea of greedy local search.

A dual decomposition method

Define the bounded set $\tilde{\mathcal{S}}\subset\Re^{NK}$ by $\tilde{\mathcal{S}} := \{\mathbf{S}\in\mathcal{S}\,|\,0\le S_k^n\le P_k\ \ \forall\, k,n\}$. Then, we can easily see that the constraint region of (4) is unchanged if $\mathcal{S}$ is replaced by $\tilde{\mathcal{S}}$. Hence, by using multipliers $\{\lambda_k\}$, the dual function can be written as

$$d(\boldsymbol{\lambda}) = \sum_{k=1}^{K}\lambda_k P_k + \sum_{n=1}^{N}\max_{k=1,\dots,K} M_k^n(\lambda_k),$$

where M_k^n is defined by $M_k^n(\lambda_k) = \log(1+\overline{S}_k^n/\sigma_k^n) - \lambda_k\overline{S}_k^n$ with the optimal power level

$$\overline{S}_k^n = \begin{cases} \mathcal{P}_k(\lambda_k^{-1}-\sigma_k^n) & \text{if } \lambda_k>0 \\ P_k & \text{if } \lambda_k\le 0. \end{cases}$$

Here $\mathcal{P}_k(\cdot)$ denotes the projection of a real number to the interval $[0,P_k]$. Then a subgradient of $d(\boldsymbol{\lambda})$ is given by

$$\nabla d(\boldsymbol{\lambda}) = \left(P_1 - \sum_{n\in\mathcal{N}_1(\boldsymbol{\lambda})}\overline{S}_1^n,\dots,\ P_K - \sum_{n\in\mathcal{N}_K(\boldsymbol{\lambda})}\overline{S}_K^n\right)^T$$

with the tone set $\mathcal{N}_k(\boldsymbol{\lambda})$ defined by

$$\mathcal{N}_k(\boldsymbol{\lambda}) := \left\{ n\in\mathcal{N} \,\middle|\, M_k^n(\lambda_k) = \max_{k'=1,\dots,K} M_{k'}^n(\lambda_{k'}) \right\}.$$

Notice that the components of subgradient $\nabla d(\boldsymbol{\lambda})$ correspond to each user's unused power (or deficit power if negative).

The dual minimization problem is given by

$$\text{minimize}\ \ d(\boldsymbol{\lambda}) \quad \text{subject to}\ \ \boldsymbol{\lambda}\ge 0.$$

In the standard dual descent method for this problem, the dual variable is updated as $\boldsymbol{\lambda}^{(\nu+1)} := [\boldsymbol{\lambda}^{(\nu)} - \alpha^{(\nu)}\nabla d(\boldsymbol{\lambda}^{(\nu)})]_+$, where $[\cdot]_+$ denotes the projection onto the nonnegative orthant. In actual, there are many possible rules to select stepsizes $\{\alpha^{(\nu)}\}$, the choice of which can have significant impact on the convergence speed and the implementation complexity of the algorithm. In the later numerical experiments, we adopt the following stepsize rule:

Stepsize Rule $\alpha^{(\nu)} := \theta^{(\nu)}(d(\boldsymbol{\lambda}^{(\nu)}) - L^*)/\|\nabla d(\boldsymbol{\lambda}^{(\nu)})\|^2$, where L^* is a known lower bound of the dual function d, and $\theta^{(\nu)}$ is calculated according to the following rule: (i) $\theta^{(0)}=2$, (ii) $\theta^{(\nu+1)}=\theta^{(\nu)}/2$ if $d(\boldsymbol{\lambda}^{(\nu)})\ge d(\boldsymbol{\lambda}^{(\nu-10)})$ for $\nu\ge 10$, and (iii) $\theta^{(\nu+1)}=\theta^{(\nu)}$ if $d(\boldsymbol{\lambda}^{(\nu)}) < d(\boldsymbol{\lambda}^{(\nu-10)})$ or $\nu\le 9$.

Local search algorithm

We next present a efficient combinatorial local search algorithm which has an overall complexity of $O(NK\log N)$. In the algorithm, we sequentially allocate each tone to a certain user at each iteration. Hence, the algorithm terminates in N iterations, and the obtained solution is always FDMA. The outline of the algorithm is as follows.

Step 0 Set the unallocated tone set $\mathcal{U}^{(0)} := \mathcal{N}$. Set $\nu := 0$.

Step 1 Consider all the possible combinations of $(n,k)\in\mathcal{U}^{(\nu)}\times\mathcal{K}$. Calculate the rate increment for each (n,k).

Step 2 Find the tone $\overline{n}$ and the user $\overline{k}$ which yield the largest rate increment. Allocate tone $\overline{n}$ to user $\overline{k}$.

Step 3 Let $\mathcal{U}^{(\nu+1)} := \mathcal{U}^{(\nu)}\setminus\{\overline{n}\}$. If $\mathcal{U}^{(\nu+1)}=\emptyset$, then terminate. Otherwise, return to Step 1 with setting $\nu := \nu+1$.

In most cases, the computational complexity for the rate increment in Step 2 is $O(1)$. Therefore, a direct implementation of the above procedure will result in $O(N^2K)$. However, we note that, by sorting the noise parameters $\{\sigma_k^n\}$ appropriately, we can reduce its complexity to $O(NK\log N)$. Because of the limitation of space, we omit the details of the algorithm here.

5. NUMERICAL RESULTS ON DSL SCENARIOS

We consider two DSL scenarios: upstream and downstream. We set the background noise level $N_0=-140$ dB, and the capacity gap $\Gamma=12$ dB. Then, the normalized crosstalk gains and noise powers are chosen as $\alpha_{lk}^n := \Gamma|h_{lk}^n|^2/|h_{kk}^n|^2$ and $\sigma_k^n := \Gamma N_0/|h_{kk}^n|^2$.

Downstream scenario: we consider a ADSL downstream scenario with two users, in which users 1 and 2 has a loop length of 10k feet and 13k feet, respectively, with a crosstalk distance of 5k feet. The loop topology is shown in Figure 1. We set the number of tones $N=256$ and the power budget $(20.4+\beta)$ dB, where β is chosen from -20 dB to 20 dB. The obtained sum-rates by three algorithms are plotted in Figure 2. As the figure shows, the sum-rates obtained by the dual decomposition method are slightly (0.2%–0.8%) better than those by the local search algorithm, and both increase linearly with the power budget level β. However, the sum-rate obtained by IWFA peaked as β is increased.

Upstream scenario: we consider a VDSL upstream scenario with eight users, in which 4 users have loops length of 4k feet and other 4 users have those of 2k feet. The loop topology is shown in Figure 3. We set the number of tones $N = 1024$, and the power budget $(20.4 + \beta)$ dB where β is chosen from -20 dB to 20 dB. Figure 4 shows the obtained sum-rates for each β, from which we can see a trend similar to Figure 2. Figure 5 shows the CPU time for each method to terminate. Compared to the dual decomposition method and IWFA, the local search algorithm terminates in very short time (approximately 0.15 seconds). In most cases the local search algorithm is faster than the dual decomposition method and IWFA. This speed advantage becomes more pronounced when N and K are large.

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6. REFERENCES

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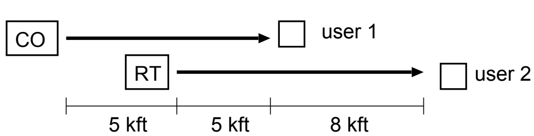

Fig. 1. ADSL downstream scenario

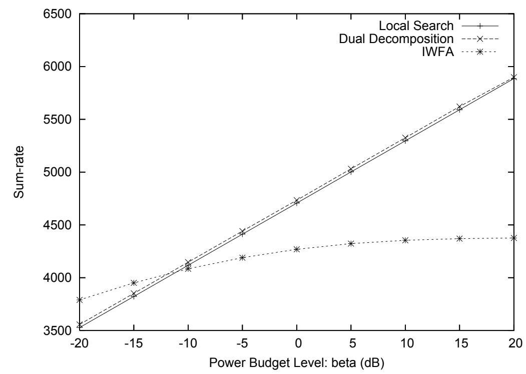

Fig. 2. Sum-rate v.s. power levels

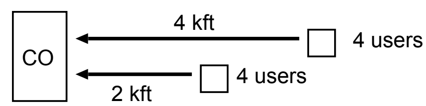

Fig. 3. VDSL upstream scenario

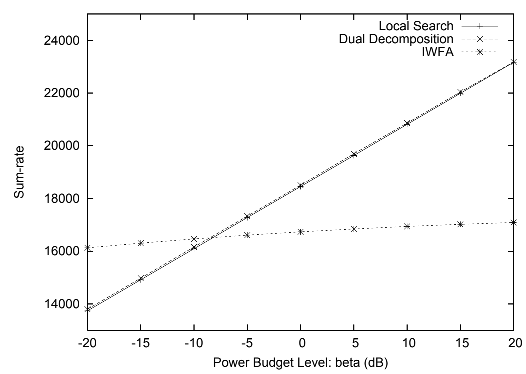

Fig. 4. Sum-rate v.s. power levels

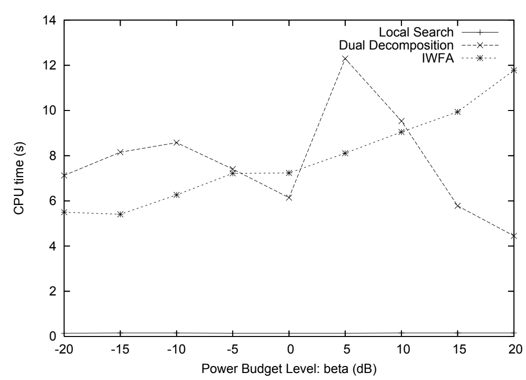

Fig. 5. CPU time v.s. power levels