OPTIMAL SPATIAL POWER ALLOCATION FOR RAYLEIGH FADING RELAY CHANNELS

Yonglan Zhu, Yan Xin, and Pooi-Yuen Kam

Department of Electrical and Computer Engineering
National University of Singapore, Singapore, 117576
Email: {zhuyonglan, elexy, elekampy}@nus.edu.sg

ABSTRACT

In this paper, we derive an optimal spatial power allocation strategy that maximizes the achievable rate of the decode-and-forward (DF) relay channels in a Rayleigh fading environment. Different from prior work, the derived strategy only requires the knowledge of the variances of the channels at the transmitters, which does not require frequent updates. Furthermore, we demonstrate that the optimal spatial power allocation strategy leads considerable performance improvements over the equal power allocation one.

Index Terms— Achievable rate, cooperative communications, Rayleigh fading, relay channel.

1. INTRODUCTION

Recently, considerable research efforts have been made to design cooperative relaying techniques to improve spectral efficiency and reliability of wireless networks. So far, the fundamental performance limit, capacity, of general relay channel remains unknown except for some special cases [1]. As an alternative, the information rates achieved by various relaying protocols such as decode-and-forward (DF), amplify-and-forward (AF) and compress-and-forward (CF) ha-ve been extensively investigated [1, 2, 3]. Among these information rates, till now, the highest information rate proved for the relay channels was obtained in [1] and achieved by DF relaying protocol. When the fading processes of the channels are ergodic, we can signal at the rate termed as the *achievable rate* with vanishing error if asymptotic optimal codebooks are used. The achievable rate is commonly served as an information-theoretic performance measure, and it depends on the statistical correlation of the signals transmitted from source and relay, and the spatial power allocation between these two nodes. It has been pointed out in [3, 4] that the transmit signals from source and relay, that maximize the achievable rate, are statistically independent for Rayleigh fading DF relay channels. A spatial power allocation strategy that maximizes the upper bound of the ergodic capacity was derived in [4] for low signal-to-noise ratio (SNR) regime. In [5], several power allocation strategies that maximize the lower or upper bounds of ergodic capacity were developed for various settings, but require the acquisition of the *instantaneous* channel state information (CSI) at the source and relay.

In this paper, we focus on deriving the optimal spatial power allocation strategy that maximizes the achievable rate of Rayleigh fading DF relay channels. One appealing feature of this strategy is that it only relies on the knowledge of the variances of the channels (*statistical* CSI) at the transmitters of the source and relay, which can be readily obtained and does not require frequent update. We first derive an analytical expression of the achievable rate, which depends on the correlation coefficient of the transmit signals from the source and relay, as well as the spatial power allocation. We show in

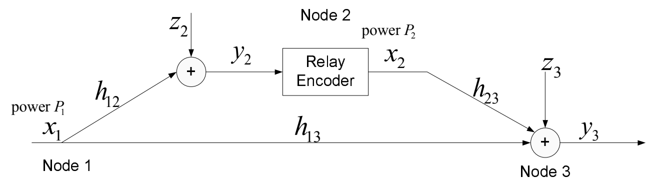

Fig. 1. The three-node (single-relay) channel.

mathematically rigorous manner that for any fixed power allocation, the achievable rate is a monotonically decreasing function of the correlation coefficient of the transmit signals from the source and relay, and thus the optimal transmit signals that maximize the achievable rate are always independent. Therefore, we solely concentrate on spatial power allocation strategy with independent transmit signaling and derive the optimal power allocation strategy through solving certain transcendental equations numerically. Our findings suggest that it is not always beneficial to use the relay, and the optimal power allocation generally depends on the variances of the channels and the total transmit power. Several examples demonstrate that the optimal spatial power allocation strategy considerably outperforms the equal power allocation one.

2. SYSTEM MODEL

Fig. 1 depicts a DF relay system that consists of a source (Node 1), a relay (Node 2), and a destination (Node 3). The transmit power at the source and relay are P_1 and P_2, respectively.

The source transmits $\sqrt{P_1}x_1$, while the relay transmits $\sqrt{P_2}x_2$ based on the prior received signals from the source. Both x_1 and x_2 are subject to power constraints $\mathbb{E}(|x_1|^2) = 1$ and $\mathbb{E}(|x_2|^2) = 1$. Mathematically, the received signals at the relay and destination can be expressed as

$$y_2 = \sqrt{P_1}\, h_{12}x_1 + z_2,$$
$$y_3 = \sqrt{P_1}\, h_{13}x_1 + \sqrt{P_2}\, h_{23}x_2 + z_3,$$

where z_2 and z_3 are additive white Gaussian noise (AWGN) at the relay and destination, respectively, each with mean zero and variance normalized to one. Each h_{ij} denotes the channel gain between nodes i and j, which takes account of the effects of path loss, shadowing and frequency nonselective fading. The h_{ij}'s are independent zero mean complex Gaussian random variables with variance σ_{ij}^2, i.e., $h_{ij} \sim \mathcal{CN}(0, \sigma_{ij}^2)$. Furthermore, we assume that the relay node works in full-duplex mode which usually offers higher spectrum efficiency than its half-duplex counterpart, perfect CSI is available at the corresponding receivers, P_1 and P_2 satisfy a total power constraint given as $P_1 + P_2 \leq P$, and perfect timing synchronization is available at all nodes.

3. ACHIEVABLE RATE

In this section, we introduce the definitions of the achievable rate when the fading process is ergodic, then obtain an analytical expression of the achievable rate of the DF relay system in an ergodic Rayleigh fading environment.

In the single-relay system described above, the highest achievable rate proved till now is [1, Theorem 1], [6]

$$R_{\rm df} = \max_{f(x_1,x_2)} \min\left\{\mathbb{E}[I(X_1;Y_2 \mid X_2)], \mathbb{E}[I(X_1,X_2;Y_3)]\right\}, \quad (1)$$

where $f(x_1,x_2)$ denotes the joint density function of X_1 and X_2, and the expectations $\mathbb{E}[\cdot]$ are with respect to the channels. Define ρ as the correlation coefficient of X_1 and X_2, i.e., $\rho = \mathbb{E}[X_1X_2^*]$, and ν as the power allocation ratio, i.e., $\nu = P_1/P$, which quantifies the spatial power allocation between the source and relay. For any realization of h_{ij}, $f(x_1,x_2)$ that maximizes the achievable rate is zero-mean jointly Gaussian [3, Proposition 2]. Thus, the maximization over all input distribution $f(x_1,x_2)$ in (1) reduces to the maximization only over the correlation coefficient ρ and the power allocation ratio ν. Accordingly, the highest achievable rate in (1) can be rewritten as [3, Eq. (106)], [5, Eq. (7)]

$$R_{\rm df} = \max_{\rho,\nu} \min\left\{\mathbb{E}[I_{\rm sr}], \mathbb{E}[I_{\rm mac}]\right\}, \quad (2)$$

where $I_{\rm sr}$ and $I_{\rm mac}$ are

$$I_{\rm sr} = \log_2\left(1 + (1-|\rho|^2)P_1|h_{12}|^2\right), \quad (3)$$
$$I_{\rm mac} = \log_2\left(1 + P_1|h_{13}|^2 + P_2|h_{23}|^2 + 2\sqrt{P_1P_2}\Re(\rho h_{13}h_{23}^*)\right).$$

Note that to make (2) non-trivial and meaningful, ν should be in the interval $(0,1)$. In the extreme case of $\nu = 1$, i.e., only the direct transmission takes place, the mutual information between the transmitted signals at source and the received signals at destination is $I_{\rm sd} = I(X_1,Y_3) = \log_2(1+P|h_{13}|^2)$, and the achievable rate is $R_{\rm sd} = \mathbb{E}[I_{\rm sd}]$ instead of $R_{\rm df}$ given in (2).

The quantity $\min\{\mathbb{E}[I_{\rm sr}], \mathbb{E}[I_{\rm mac}]\}$ is the achievable rate [6], which depends on the correlation coefficient ρ and the spatial power allocation ν. In order to determine the optimal ρ and ν, we need to express the achievable rate in terms of the parameters ρ and ν. To do so, we first express $I_{\rm sr}$ and $I_{\rm mac}$ in new forms, which facilitates us to obtain $\mathbb{E}[I_{\rm sr}]$ and $\mathbb{E}[I_{\rm mac}]$ for arbitrary ρ and ν.

Define $\mu = 1 - |\rho|^2$ and $\eta_{ij} (ij = 12, 13, 23)$ as follows

$$\eta_{12} = P_1\sigma_{12}^2 = \nu P\sigma_{12}^2, \qquad \eta_{13} = P_1\sigma_{13}^2 = \nu P\sigma_{13}^2,$$
$$\eta_{23} = P_2\sigma_{23}^2 = (1-\nu)P\sigma_{23}^2. \quad (4)$$

With the noise whitening and eigen-decomposition techniques [7], we can re-express $I_{\rm sr}$ and $I_{\rm mac}$ in (3) as

$$I_{\rm sr} = \log_2(1 + \mu\eta_{12}|\tilde{h}_{12}|^2),$$
$$I_{\rm mac} = \log_2(1 + \alpha|\tilde{h}_{13}|^2 + \beta|\tilde{h}_{23}|^2), \quad (5)$$

where $\tilde{h}_{12}$, $\tilde{h}_{13}$ and $\tilde{h}_{23}$ are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance, α and β are given as

$$\alpha = \frac{\eta_{13}+\eta_{23}}{2} + \sqrt{\left(\frac{\eta_{13}+\eta_{23}}{2}\right)^2 - \eta_{13}\eta_{23}\mu},$$
$$\beta = \frac{\eta_{13}+\eta_{23}}{2} - \sqrt{\left(\frac{\eta_{13}+\eta_{23}}{2}\right)^2 - \eta_{13}\eta_{23}\mu}. \quad (6)$$

With the expressions (5) of $I_{\rm sr}$ and $I_{\rm mac}$, we can easily obtain their mean values and the achievable rate, which are presented in the following theorem.

Theorem 1 *The achievable rate is a function of μ and ν, and it is given by*

$$R(\mu,\nu) = \min\{R_{\rm sr}(\mu,\nu),\ R_{\rm mac}(\mu,\nu)\}, \quad (7)$$

where $R_{\rm sr}(\mu,\nu)$ and $R_{\rm mac}(\mu,\nu)$ denote $\mathbb{E}[I_{\rm sr}]$ and $\mathbb{E}[I_{\rm mac}]$, respectively, and are given as

$$R_{\rm sr}(\mu,\nu) = \mathbb{E}[I_{\rm sr}] = \exp(\mu^{-1}\eta_{12}^{-1}){\rm E}_1(\mu^{-1}\eta_{12}^{-1})\log_2 e, \quad (8)$$
$$R_{\rm mac}(\mu,\nu) = \mathbb{E}[I_{\rm mac}] = \frac{\alpha\exp(\frac{1}{\alpha}){\rm E}_1(\frac{1}{\alpha}) - \beta\exp(\frac{1}{\beta}){\rm E}_1(\frac{1}{\beta})}{(\alpha-\beta)\ln 2},$$

with ${\rm E}_1(x)$ denoting the exponential integral function, i.e., ${\rm E}_1(x) = \int_x^\infty e^{-t}/t\, dt$ $(x > 0)$.

Remark: The achievable rate $R(\mu,\nu)$ does not depend the phase of ρ. This is due to the fact that in the Rayleigh fading case, the phase of ρ does not alter the distribution of the phase of $\rho h_{13}h_{23}^*$, since the phase of $h_{13}h_{23}^*$ is uniformly distributed.
Remark: For $\rho = 0$, $\mathbb{E}[I_{\rm mac}]$ was given in [6]; however, for ρ other than zero, $\mathbb{E}[I_{\rm mac}]$ does not seem available in the literature.

4. OPTIMAL POWER ALLOCATION

In this section, we will present our optimal spatial power allocation strategy that maximizes the achievable rate for the Rayleigh fading DF relay channel. As can been from (7), the achievable rate R depends on μ and ν. We first show that for any fixed power allocation ν, the optimal transmit signals are always independent, i.e., $\mu = 1$ is optimal for any fixed $\nu \in (0,1)$. This motivates us to solely focus on deriving the optimal spatial power allocation strategy for independent transmit signaling.

4.1. Optimal Transmit Signaling

For a fixed spatial power allocation $\nu \in (0,1)$, designing an optimal transmit signaling is equivalent to finding an optimal μ which maximizes the achievable rate. The optimal μ for a fixed $\nu \in (0,1)$ is denoted by $\mu_e(\nu)$, i.e.,

$$\mu_e(\nu) = \arg\max_{0\le\mu\le1} R(\mu,\nu), \qquad \nu \in (0,1). \quad (9)$$

It was concluded in [3, Theorem 8] that $\mu_e(\nu)$ is one, i.e.,$\mu_e(\nu) = 1$, for a more general case, which considers multiple relays and only assumes that the phases of the channel gains are uniformly distributed. However, the proof of the theorem involves using the Jensen's inequality to approximate $\mathbb{E}[\log(x)]$. Rigorously speaking, maximizing $\log(\mathbb{E}[x])$ is not necessarily equivalent to maximizing $\mathbb{E}[\log(x)]$. By using the expressions (5) of $I_{\rm sr}$ and $I_{\rm mac}$, the proof can be made mathematically rigorous in the Rayleigh fading single-relay case. Therefore, we introduce the following theorem that restates a part of the result [3, Theorem 8], but complements it with our finding on the monotonicity of $R(\mu,\nu)$, and a mathematically rigorous proof for the optimality of $\mu_e(\nu) = 1$ for any $\nu \in (0,1)$.

Theorem 2 *The achievable rate $R(\mu,\nu)$ is an increasing function of $\mu \in [0,1]$ (or, equivalently, a decreasing function of $|\rho| \in [0,1]$) for any fixed $\nu \in (0,1)$. Thus, $\mu_e(\nu)$ is one for any fixed $\nu \in (0,1)$. It means that the optimal transmit signals from source and relay that maximize the achievable rate R should be always independent. Furthermore, the maximum achievable rate $R_{\rm df}^I(\nu)$ for a fixed $\nu \in (0,1)$ is given by*

$$R_{\rm df}^I(\nu) = \max_{\mu} R(\mu,\nu) = R(1,\nu) = \min\{R_{\rm sr}^I(\nu), R_{\rm mac}^I(\nu)\}, \quad (10)$$

where $R^I_{\rm sr}(\nu) = R_{\rm sr}(1,\nu)$ *and* $R^I_{\rm mac}(\nu) = R_{\rm mac}(1,\nu)$*, i.e.,*

$$R^I_{\rm sr}(\nu) = \exp(\eta_{12}^{-1}){\rm E}_1(\eta_{12}^{-1})\log_2 e,$$
$$R^I_{\rm mac}(\nu) = \frac{\eta_{13}\exp(\eta_{13}^{-1}){\rm E}_1(\eta_{13}^{-1}) - \eta_{23}\exp(\eta_{23}^{-1}){\rm E}_1(\eta_{23}^{-1})}{(\eta_{13}-\eta_{23})\ln 2}.$$

Proof: Clearly, $R_{\rm sr}(\mu,\nu)$ is a monotonically increasing function of μ for any fixed $\nu \in (0,1)$, since $I_{\rm sr}$ is an increasing function of μ. However, the monotonicity of $R_{\rm mac}(\mu,\nu)$ is not obvious. To demonstrate the monotonicity of $R_{\rm mac}(\mu,\nu)$, we calculate

$$\frac{\partial R_{\rm mac}(\mu,\nu)}{\partial\mu} = -\frac{\eta_{13}\eta_{23}}{\alpha-\beta}\left[\frac{\partial R_{\rm mac}(\mu,\nu)}{\partial\alpha} - \frac{\partial R_{\rm mac}(\mu,\nu)}{\partial\beta}\right] = -\frac{\eta_{13}\eta_{23}}{\alpha-\beta}\,\mathbb{E}\left[\frac{w_1-w_2}{1+\alpha w_1+\beta w_2}\right], \quad (11)$$

where $w_1 = |\tilde{h}_{13}|^2$, $w_2 = |\tilde{h}_{23}|^2$, and the second equality follows from $R_{\rm mac}(\mu,\nu) = \mathbb{E}[\log_2(1+\alpha w_1+\beta w_2)]$. It is shown in [8, Theorem 3.1] that when w_1 and w_2 are i.i.d. and $\alpha \ge \beta$, we have

$$\mathbb{E}\left[\frac{w_1-w_2}{1+\alpha w_1+\beta w_2}\right] \le 0.$$

From (11), we can readily conclude that $\partial R_{\rm mac}(\mu,\nu)/\partial\mu \ge 0$. It implies that $R_{\rm mac}(\mu,\nu)$ is increasing in μ for any fixed $\nu \in (0,1)$. Thus, $R(\mu,\nu)$ is an increasing function of μ (or a decreasing function of $|\rho|$) for any fixed $\nu \in (0,1)$, and it is maximized at $\mu = 1$ (or $\rho = 0$). Lastly, we obtain $R^I_{\rm sr}(\nu)$ and $R^I_{\rm mac}(\nu)$ by evaluating $R_{\rm sr}(1,\nu)$ and $R_{\rm mac}(1,\nu)$, respectively. ■

4.2. Spatial Power Allocation

Since the optimal value of μ, $\mu_e(\nu)$, is one for any fixed $\nu \in (0,1)$ (Theorem 2), we now consider an optimal spatial power allocation for independent transmit signaling. In such a case, the optimal spatial power allocation strategy for the DF relay channel is $\hat{\nu}_e$ such that

$$\hat{\nu}_e = \arg\max_{0<\nu<1} R(1,\nu) = \arg\max_{0<\nu<1} R^I_{\rm df}(\nu). \quad (12)$$

Moreover, in the case of $\nu = 1$, as mentioned in Section 3, the mutual information is $I_{\rm sd}$, and the corresponding achievable rate is

$$R_{\rm sd} = \mathbb{E}[I_{\rm sd}] = \exp(\sigma_{13}^{-2}P^{-1}){\rm E}_1(\sigma_{13}^{-2}P^{-1})\log_2 e. \quad (13)$$

To determine whether to use direct transmission or DF relay transmission with optimal power allocation $\hat{\nu}_e$, we need compare $R_{\rm sd}$ with $R^I_{\rm df}(\hat{\nu}_e) = R_{\rm df}$. Thus, the optimal spatial power allocation ν_e is determined as

$$\nu_e = \begin{cases} \hat{\nu}_e, & R^I_{\rm df}(\hat{\nu}_e) > R_{\rm sd}, \\ 1, & R^I_{\rm df}(\hat{\nu}_e) \le R_{\rm sd}. \end{cases} \quad (14)$$

We denote the corresponding maximum achievable DF rate as $R^J_{\rm df} = \max\{R^I_{\rm df}(\hat{\nu}_e), R_{\rm sd}\} = \max\{R_{\rm df}, R_{\rm sd}\}$.

Theorem 3 *If* $\sigma_{12}^2 \le \sigma_{13}^2$*, the optimal transmission strategy is to use only direct transmission and allocate all the transmit power at the source, i.e.,* $\nu_e = 1$*, with* $R^J_{\rm df} = R_{\rm sd}$*.*

Proof: It follows from (10) that $R^I_{\rm df}(\nu) \le R^I_{\rm sr}(\nu)$. Since $R^I_{\rm sr}(\nu)$ is an increasing function and $\sigma_{12}^2 \le \sigma_{13}^2$, we have

$$R^I_{\rm df}(\nu) \le R^I_{\rm sr}(\nu) \le \lim_{\nu\to 1} R^I_{\rm sr}(\nu) \le R_{\rm sd}$$

for any $\nu \in (0,1)$. It implies that to achieve the highest rate, all the power should be allocated to the source. ■

However, in practice, the link from the source to relay is typically stronger than the direct link, i.e., $\sigma_{12}^2 > \sigma_{13}^2$. In such a case, we have

$$R^I_{\rm df}(\hat{\nu}_e) \ge \lim_{\nu\to 1} R^I_{\rm df}(\nu) = R_{\rm sd},$$

and thus $\nu_e = \hat{\nu}_e$. To determine $\hat{\nu}_e$, we need to know the monotonicity of $R^I_{\rm sr}(\nu)$ and $R^I_{\rm mac}(\nu)$. Clearly $R^I_{\rm sr}(\nu)$ is an increasing function, but the monotonicity of $R^I_{\rm mac}(\nu)$ is not obvious, which is stated in the following lemma.

Lemma 1 *The function* $R^I_{\rm mac}(\nu)$ *is increasing on the interval* $[0,\hat{\nu})$*, and decreasing on* $(\hat{\nu},1)$*, where* $\hat{\nu} = \arg\max_{0<\nu<1} R^I_{\rm mac}(\nu)$ *and is determined as follows:*

a) If $\sigma_{13}^2 = \sigma_{23}^2$*, then* $\hat{\nu}$ *equals* $1/2$*;*
b) If $\sigma_{13}^2 \ne \sigma_{23}^2$*, then* $\hat{\nu}$ *is the unique root of the equation*

$$\left[\frac{\sigma_{13}^2\sigma_{23}^2 P}{\eta_{13}-\eta_{23}} + \frac{\sigma_{13}^2}{\eta_{13}}\right]\exp(\eta_{13}^{-1}){\rm E}_1(\eta_{13}^{-1}) - \left[\frac{\sigma_{13}^2\sigma_{23}^2 P}{\eta_{13}-\eta_{23}} - \frac{\sigma_{23}^2}{\eta_{23}}\right]\exp(\eta_{23}^{-1}){\rm E}_1(\eta_{23}^{-1}) = \sigma_{13}^2 + \sigma_{23}^2. \quad (15)$$

Proof: The optimization problem $\max_\nu R^I_{\rm mac}(\nu)$ is a convex problem, because the second partial derivative of $R^I_{\rm mac}(\nu)$ is always non-positive, i.e.,

$$\frac{\partial^2 R^I_{\rm mac}(\nu)}{\partial\nu^2} = -\log_2 e \cdot \mathbb{E}\left[\frac{(P\sigma_{13}^2 w_1 - P\sigma_{23}^2 w_2)^2}{(1+\eta_{13}w_1+\eta_{23}w_2)^2}\right] \le 0.$$

Clearly $\hat{\nu}$ is the unique solution of $\partial R^I_{\rm mac}(\nu)/\partial\nu = 0$ as shown in (15). Since $\partial^2 R^I_{\rm mac}(\nu)/\partial\nu^2$ is non-positive, $\partial R^I_{\rm mac}(\nu)/\partial\nu$ is positive and $R^I_{\rm mac}(\nu)$ is increasing on the interval $[0,\hat{\nu})$; $\partial R^I_{\rm mac}(\nu)/\partial\nu$ is negative and $R^I_{\rm mac}(\nu)$ is decreasing on the interval $(\hat{\nu},1]$. In particular, when $\sigma_{13}^2 = \sigma_{23}^2$, we have $\hat{\nu} = 0.5$. ■

We next introduce the optimal spatial power allocation strategy for the case of $\sigma_{12}^2 > \sigma_{13}^2$.

Theorem 4 *If* $\sigma_{12}^2 > \sigma_{13}^2$*, the optimal spatial power allocation* $\nu_e = \hat{\nu}_e$ *is:*

a) If $R^I_{\rm sr}(\hat{\nu}) \ge R^I_{\rm mac}(\hat{\nu})$*, then* $\hat{\nu}_e = \hat{\nu}$ *and* $R^J_{\rm df} = R^I_{\rm mac}(\hat{\nu})$*.*
b) If $R^I_{\rm sr}(\hat{\nu}) < R^I_{\rm mac}(\hat{\nu})$*, then* $\hat{\nu}_e = \nu^* \in (\hat{\nu},1)$ *and* $R^J_{\rm df} = R^I_{\rm sr}(\nu^*)$*, where* ν^* *denotes the unique root of* $R^I_{\rm sr}(\nu) = R^I_{\rm mac}(\nu)$ *in* $(\hat{\nu},1)$*.*

Proof: a) If $R^I_{\rm sr}(\hat{\nu}) \ge R^I_{\rm mac}(\hat{\nu})$, the inequalities $R^I_{\rm df}(\nu) \le R^I_{\rm mac}(\nu) \le R^I_{\rm mac}(\hat{\nu}) \le R^I_{\rm sr}(\hat{\nu})$ hold for any $\nu \in (0,1)$. This fact indicates that $\hat{\nu}_e = \hat{\nu}$ and $R^J_{\rm df} = R^I_{\rm mac}(\hat{\nu})$.

b) Define $\psi(\nu) = R^I_{\rm sr}(\nu) - R^I_{\rm mac}(\nu)$. Since $R^I_{\rm sr}(\hat{\nu}) < R^I_{\rm mac}(\hat{\nu})$ and $\sigma_{12}^2 > \sigma_{13}^2$, we have

$$\psi(\hat{\nu}) < 0 \ \text{ and } \ \lim_{\nu\to 1}\psi(\nu) > 0,$$

which indicate that $\psi(\nu)$ has at least one root $\nu^* \in (\hat{\nu},1)$. Moreover, the facts that $R^I_{\rm mac}(\nu)$ is decreasing on the interval $(\hat{\nu},1)$ (Lemma 1), and $R^I_{\rm sr}(\nu)$ is increasing, imply that $\psi(\nu)$ is an increasing function of $\nu \in (\hat{\nu},1)$. Therefore, $\psi(\nu)$ has one and only one root $\nu^* \in (\hat{\nu},1)$. With $R^I_{\rm df}(\nu) \le R^I_{\rm sr}(\nu^*)$ for $\nu \in (0,\nu^*]$ and $R^I_{\rm df}(\nu) \le R^I_{\rm mac}(\nu^*)$ for $\nu \in (\nu^*,1)$, we can easily conclude that $\hat{\nu}_e = \nu^*$. ■

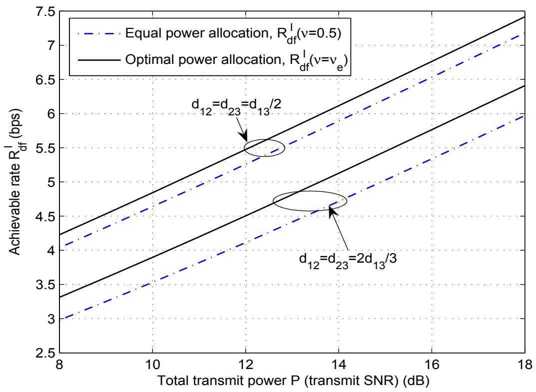

Fig. 2. Achievable rates $R^I_{\rm df}$ versus transmit SNR.

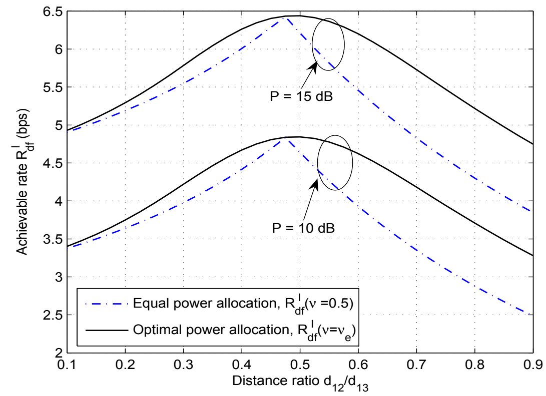

Fig. 3. Achievable rates versus d_{12}/d_{13}.

5. NUMERICAL RESULTS

In this section, we provide the following two examples to illustrate our findings. Assume that σ_{13}^2 is normalized to one. We adopt the well known path-loss model with path-loss exponent being 3, i.e., $\sigma_{ij}^2 \propto d_{ij}^{-3}$, where d_{ij} denotes the distance between nodes i and j.

Example 1: In Fig. 2, we show the achievable rate (10) versus the total transmit power P (or transmit SNR) for $d_{12} = d_{23} = d_{13}/2$ and $d_{12} = d_{23} = 2d_{13}/3$. We compare the achievable rates of the DF relay transmission with equal power allocation ($\nu = 0.5$), and with optimal power allocation (ν_e). From Fig. 2, we can see that the optimal power allocation strategy considerably outperforms the equal power allocation one.

Example 2: In this example, we assume that the source, relay and destination are collinear, i.e., $d_{12} + d_{23} = d_{13}$. In Fig. 3, we study the impacts of the location of the relay on the achievable rate, for both equal power allocation ($\nu = 0.5$) and optimal power allocation (ν_e). We observe that, when the relay is far from the source ($d_{12}/d_{13} > 0.5$), the optimal power allocation strategy can greatly improve the achievable rates. However, when the relay is close to the source ($d_{12}/d_{13} < 0.5$), the gain achieved by adopting optimal power allocation is negligible. We plot the corresponding optimal power allocation ν_e versus d_{12}/d_{13} in Fig. 4. It can be seen that the optimal power allocation ν_e becomes insensitive to the total transmit power P when d_{12}/d_{13} exceeds a certain value.

6. CONCLUSION

In this paper, we derived an analytical expression of the achievable rate of the DF relay system in an ergodic Rayleigh fading environment, proved in a mathematically rigorous manner that optimal transmit signals from source and relay are independent irrespective of the transmit power, and obtained the optimal spatial power allocation strategy that maximizes the achievable rate. We further disclosed that the optimal spatial power allocation depends on the variances of the channels and the total transmit power, and it can be obtained by solving certain transcendental equations numerically.

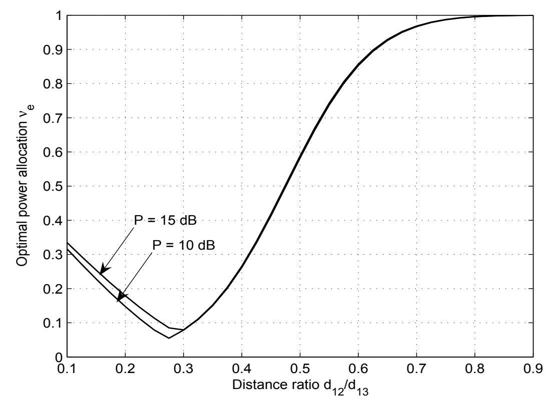

Fig. 4. Optimal power allocation ratio ν_e versus d_{12}/d_{13}.

7. REFERENCES

[1] T. M. Cover and A. A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 572–584, Sep. 1979.

[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: efficient protocols and outage behavior," *IEEE Trans. Inform. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.

[4] X. Cai, Y. Yao, and G. B. Giannakis, "Achievable rates in cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Commun.*, vol. 53, no. 1, pp. 184–194, Jan. 2005.

[5] A. Høst Madsen and J. Zhang, "Capacity bounds and power allocation for wireless relay channels," *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 2020–2040, Jun. 2005.

[6] B. Wang, J. Zhang, and A. Høst Madsen, "On the capacity of MIMO relay channels," *IEEE Trans. Inform. Theory*, vol. 51, no. 1, pp. 29–43, Jan. 2005.

[7] Y. Zhu, Y. Xin, and P. Y. Kam, "Outage-optimal transmission strategies for Rayleigh fading relay channels," in *Proc. of Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Oct. 29 - Nov. 1, 2006.

[8] H. Boche and E. A. Jorswieck, "On schur-convexity of expectation of weighted sum of random variables with applications," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, no. 2, 2004.