ENTROPY OF GENERAL GAUSSIAN DISTRIBUTIONS AND MIMO CHANNEL CAPACITY MAXIMIZING PRECODER AND DECODER

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ABSTRACT

Exploiting channel state information at the transmitter and receiver to design an optimal linear precoder and decoder for a multiple-input multiple-output (MIMO) communication system is an active research area. The design is often based on the information rate criterion, that is to design the precoder and decoder such that the system capacity is maximized, subject to the average transmit power constraint. Although such an optimization problem has been considered intensively and there have been numerous proposals so far, they are not rigorously correct. We propose a mathematically rigorous framework for solving this optimization problem. Our proposed solution is applicable to both MIMO flat fading and frequency selective fading channels. Simulations verify the theoretical analysis.

Index Terms— MIMO system, entropy, mutual information, precoder and decoder

1. INTRODUCTION

The problem of designing an optimal pair of linear precoder and decoder has been considered intensively in a number of publications [5, 8, 9, 10] based on different criteria such as minimum mean squared error (MMSE), bit error rate (BER), maximum channel capacity, and maximum signal to interference-plus-noise ratio (SINR). In the context, the channel capacity can be substantially improved by appropriately reshaping the correlation matrices of the transmitted and received blocks. In a multiple transmit and multiple receive antenna communication system, the linear precoder reshapes the correlation of the input signal before the transmission and the linear decoder reshapes the correlation of the output signal after reception [1, 9]. One of the arising issues is that the optimal correlation matrix may be singular and that there is a discrepancy in the definition of mutual information between input and output [1, 9]. The objective of this paper is twofold:

(**i**) in Subs 1.1 and 2.2 we argue that the formula for the mutual information between the input and the output given by [1], which was subsequently used in [9] and other related works, is not always correct and propose a corrected version.

(**ii**) Using the linear matrix inequality approach (very popular in control theory, see e.g. [3] and references therein), in Subs 2.3, 2.4 and the whole Section 3 we show how to obtain a pair of precoder and decoder that maximizes the mutual information between the input and output under very general conditions that are free from rank and dimension restrictions such as in [1, 9]. Thus this approach leads to also a new class of optimal solutions as well as a new mathematically rigorous derivation. Numerical examples in Section 4 show the advantages of our methods and verify our analytical results.

Notation: The superscript T denotes transpose, the superscript H denotes the Hermitian transpose, while $||.||$ denotes the Frobenius norm, $\mathbf{E}\{.\}$ the expectation, $\langle . \rangle$ the trace of a matrix, rank$\{.\}$ the rank of a matrix, $\jmath = \sqrt{-1}$ pure imaginary. I and 0 denote the identity and zero matrices respectively, however, sometimes their sizes are indicated to avoid confusion and I_N, $0_{N\times M}$ indicate the $N\times N$ identity and the $N\times M$ zero matrices respectively. $A>0$ ($A\geq 0$, resp.) stands for strictly positive Hermitian (positive, resp.) definite matrix A. In what follows, $\mathcal{M}_n$ denotes the set of all $n\times n$ unitary matrices, while $\mathcal{D}_n$ is the set of real $n\times n$ diagonal matrices. For a given matrix X we denote its pseudo-inverse matrix by $X^\dagger$ while x^+ means $\max\{0,x\}$ for a scalar x and $\mathsf{diag}\,[X \quad Y]$ means $\begin{bmatrix} X & 0\\ 0 & Y\end{bmatrix}$ for two square matrices X and Y.

2. PROBLEM DEFINITION

2.1. Pseudo-inverse and entropy of general Gaussian distribution

We start from a general definition, which is more popular in the statistical community, that allows singular covariance matrices:

Definition [7, p. 518]. *An n-dimensional random variable u is Gaussian if and only if every linear function $T^H u$ of u has a univariate normal distribution.*

• Such random variable u has a mean μ and covariance Σ.

• In accordance with this definition, if there are μ and Σ such that for every $T\in C^n$, $T^H u$ is a univariate Gaussian with mean $T^H\mu$ and covariance $T^H\Sigma T$ then u is an $n-$dimensional Gaussian with $\mathbf{E}(u)=\mu$ and $\mathbf{E}((u-\mu)(u-\mu)^H)=\Sigma$, and we denote it as $u\sim\mathcal{N}_n(\mu,\Sigma)$.

Theorem 1 *[7, p. 528] If $u\sim\mathcal{N}_n(\mu,\Sigma)$ then its density function is $p_u(v) = (2\pi)^{-k/2}(\prod_{i=1}^{k}\lambda_i)^{-1/2}e^{-\frac{1}{2}(v-\mu)^H\Sigma^\dagger(v-\mu)}$, where λ_i are nonzero eigenvalues of Σ. Consequently, the entropy of u is $H(u) = \frac{1}{2}\log[(2\pi e)^k\prod_{i=1}^{k}\lambda_i]$.*

In what follows we denote the above function $(2\pi e)^k\prod_{i=1}^{k}\lambda_i$ as $\mathsf{gdet}(2\pi e\Sigma)$. Note that $\mathsf{gdet}(2\pi e\Sigma)=\mathsf{det}(2\pi e\Sigma)$ if and only if $\Sigma>0$, and in general

$$\mathsf{gdet}(\Sigma\Psi)\neq\mathsf{gdet}(\Sigma).\mathsf{gdet}(\Psi). \tag{1}$$

For instance, for $\Sigma=\mathsf{diag}\,[2 \quad 1 \quad 0]$, $\Psi=\mathsf{diag}\,[0\quad 1\quad 2]$ then $\mathsf{gdet}(\Sigma\Psi)=1\neq 4=\mathsf{gdet}(\Sigma).\mathsf{gdet}(\Psi)$.

Now consider a multiple-input-multiple-output (MIMO) model

$$z=\mathcal{H}x+\nu \tag{2}$$

where the input $x \in C^n$ is complex, Gaussian zero mean with covariance R_{xx} (possibly singular), $\nu \in C^\nu$ is the Gaussian noise with covariance $R_{\nu\nu}$ and is independent of the input x and $\mathcal{H} \in C^{\nu\times n}$. It is known that the output $z \in C^\nu$ is also Gaussian, and its covariance as well as the input-output correlation R_{xz} are easily calculated

$$R_{zz} = \mathbf{E}(zz^H) = \mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu},\ R_{xz} = \mathbf{E}(xz^H) = R_{xx}\mathcal{H}^H. \quad (3)$$

The following result adapted from [7, p. 522] is important.

Theorem 2 *The conditional distribution $z|x$ of z given x is the Gaussian $\mathcal{N}_\nu(R_{xz}^H R_{xx}^\dagger x, R_{zz} - R_{xz}^H R_{xx}^\dagger R_{xz})$.*

Now, the mutual information between the output z and input x is defined as $I(x;z) = H(x) - H(x|z)$, where according to Theorem 1 $H(x) = \frac{1}{2}\log \mathsf{gdet}(2\pi e R_{xx})$ and $H(x|z) = \frac{1}{2}\log \mathsf{gdet}(2\pi e \mathcal{H}R_{xx})$. Thus, with the definition, $R_{x|z}^\perp = R_{xx} - R_{xx}\mathcal{H}^H(\mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu})^{-1}\mathcal{H}R_{xx}$, the mutual information between x and z is given by

$$I(x,z) = \frac{1}{2}\log\frac{\mathsf{gdet}(2\pi e R_{xx})}{\mathsf{gdet}(2\pi e R_{x|z}^\perp)} \quad (4)$$

Of course, the formula (4) is too complicated and needs simplification. The next section discusses several shortcomings of previous works and section 2.3 presents our remedy.

2.2. Discussion of previous results and remedy

In [1] the following steps have been taken to simplify the mathematical formula (4):

• Use the identity

$$\frac{\mathsf{gdet}(2\pi e R_{xx})}{\mathsf{gdet}(2e\pi e R_{x|z}^\perp)} = \mathsf{gdet}((R_{x|z}^\perp)^\dagger R_{xx}) \quad (5)$$

to rewrite (4) as

$$I(x,z) = \frac{1}{2}\log \mathsf{gdet}((R_{x|z}^\perp)^\dagger R_{xx}). \quad (6)$$

• Then use the matrix identity

$$(R_{x|z}^\perp)^\dagger = R_{xx}^\dagger + \mathcal{H}^H R_{\nu\nu}^{-1}\mathcal{H} \quad (7)$$

to simplify (6) to

$$I(x,z) = \frac{1}{2}\log \mathsf{gdet}((R_{xx}^\dagger + \mathcal{H}^H R_{\nu\nu}^{-1}\mathcal{H})R_{xx}). \quad (8)$$

The expression (8) has also been used, e.g. in [9], as the starting point. However, one observes that

• Although $1/\mathsf{gdet}(2\pi e R_{x|z}^\perp) = \mathsf{gdet}((2\pi e R_{x|z}^\perp)^\dagger)$ and consequently $\mathsf{gdet}(2\pi e R_{xx})/\mathsf{gdet}(2\pi e R_{x|z}^\perp) = \mathsf{gdet}(R_{xx})\mathsf{gdet}((R_{x|z}^\perp)^\dagger)$ (provided that R_{xx} and $R_{x|z}^\perp$ have the same number of nonzero eigenvalues) the identity (5) is still not always valid because

$$\mathsf{gdet}(R_{xx})\mathsf{gdet}((R_{x|z}^\perp)^\dagger) \neq \mathsf{gdet}(R_{xx}(R_{x|z}^\perp)^\dagger)$$

in general (see (1)). In other words, the expression (6) is valid only when R_{xx} and $R_{x|z}^\perp$ are both nonsingular and then the pseudo-inverse matrix $R_{xx}^\dagger$ in (6) is the usual inverse R_{xx}^{-1}.

• The identity (7) is not always valid. As a simple counter-example take $R_{xx} = \mathrm{diag}[1\ 1\ 0]$, $\mathcal{H} = R_{vv} = I_3$. Then it is easily seen that $R_{xx}^\dagger \mathrm{diag}[1\ 1\ 0]$ and $R_{x|z}^\perp = \mathrm{diag}[0.5\ 0.5\ 0]$ and then $(R_{x|z}^\perp)^\dagger = \mathrm{diag}[2\ 2\ 0] \neq \mathrm{diag}[2\ 2\ 1] R_{xx}^\dagger + \mathcal{H}^H R_{\nu\nu}^{-1}\mathcal{H}$. Therefore (8) and (6) are not equivalent.

• The approach of [1, 9], in the derivation of the optimal solution to the problem of maximizing the objective defined by (8), uses the following generalization of the Hadamard inequality $\mathsf{gdet}(\Sigma) \leq \prod_{\Sigma(i,i)\neq 0}\Sigma(i,i)\forall\Sigma \geq 0$. However, this generalization is again not always valid. For instance, take $\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then

$\mathsf{gdet}(\Sigma) = 2 > 1 = \Sigma(1,1)\Sigma(2,2)$.

In conclusion, the simplified formula (8) derived in [1] and used in [9] is always valid only under the condition that $R_{xx} > 0$ (and thus nonsingular). When R_{xx} is possibly singular, all main tools actually used for the derivation of the results of [1, 9] such as generalized determinant of matrix product, generalized matrix inverse formula, generalized Hadamard inequality fail to hold. An another approach for derivation of mutual information is needed as follows.

The formula (8) is not only invalid but also complex than necessarily. We now provide its corrected and simple version.

For, consider the following alternative expression $I(x;z) = H(z) - H(z|x)$ where $H(z) = \frac{1}{2}\log \mathsf{gdet}(2\pi e(\mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu}))$, $H(z|x) = \frac{1}{2}\log \mathsf{gdet}(2\pi e R_{\nu\nu})$.

Thus our result that corrects (8) can be summarized as follows

Theorem 3 *The mutual information between the input x and output z related by equation (2) is*

$$\frac{1}{2}\log\frac{\mathsf{gdet}(2\pi e(\mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu}))}{\mathsf{gdet}(2\pi e R_{\nu\nu})}. \quad (9)$$

When $R_{\nu\nu} > 0$ as assumed in [1] then $R_{zz} = \mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu} \geq R_{\nu\nu} > 0$, and (9) simplifies to

$$\frac{1}{2}\log \det((\mathcal{H}R_{xx}\mathcal{H}^H + R_{\nu\nu})R_{\nu\nu}^{-1}). \quad (10)$$

Observe that (10) is the same as (8) if and only if $R_{xx} > 0$ (i.e. R_{xx} is nonsingular).

2.3. Optimal precoder and decoder MIMO problem formulation and assumptions

A MIMO communication channel can be modeled by

$$y = HFx + n \quad (11)$$

where $x \in C^\ell$ is the transmitted signal, $y \in C^{n_R}$ is the received signal, $F \in C^{n_T\times\ell}$ is the prefiltering matrix (to be designed), $H \in C^{n_R\times n_T}$ is the channel state matrix (which is assumed known) and $n \in R^{n_R}$ is the noise. Thus n_T and n_R are numbers of transmit and receive antennas, respectively.

The following assumptions are made: $\mathbf{E}[xx^H] = R_\ell > 0, \mathbf{E}[nx^H] = 0$, $\mathbf{E}[nn^H] = R_{n_R} > 0$, $\ell \leq \min\{n_T, n_R\}$, in which case $n_T - \ell$ is the transmission redundancy. Unlike most works on precoder and decoder design [5, 8, 9, 10], we do not impose any restriction on the rank of the channel matrix H. For instance, the following restrictions have been implicitly assumed in [10]: $\ell = \min\{n_T, n_R\}$ and H is of full rank, so $\mathrm{rank}(H) = \min\{n_T, n_R\} = \ell$.

Our MIMO design problem is to design both precoder matrix F and decoder matrix $G \in C^{\ell\times n_R}$ such that the mutual information between the transmitted signal Fx and the transmitted symbol

$$z = Gy = GHFx + Gn \quad (12)$$

is maximal.

According to (9) this problem is formulated as

$$\max_{F,G,\langle FR_\ell F^H\rangle \leq P_T} f(F,G) \quad (13)$$

where $f(F,G) = \log\frac{\mathsf{gdet}(2\pi e((GHF)R_\ell(GHF)^H + GR_{n_R}G^H))}{\mathsf{gdet}(2\pi e G R_{n_R} G^H)}$, and the transmitted power P_T is given. We cannot simply simplify $\mathsf{gdet}(.)$

by $\det(.)$ in (13) because it is unknown beforehand whether the matrix $GR_{n_R}G^H$ is singular or not.
In the next sections we will derive the optimal decoder matrix G and precoder matrix F.

3. OPTIMAL DECODER FOR A FIXED PRECODER

To consider the optimization problem (13), first we fix F and consider the optimization problem

$$\max_{G\in C^{\ell\times n_R}} \log \frac{\mathsf{gdet}(2\pi e((GHF)R_\ell(GHF)^H+GR_{n_R}G^H))}{\mathsf{gdet}(2\pi eGR_{n_R}G^H)} \quad (14)$$

As it can be seen in the previous section, the function $\mathsf{gdet}(.)$ causes a lot of unexpected troubles that make the linear algebraic approach of [1, 8] hardly work. To address the problem (14), using the matrix inequality approach, which has been proved very powerfull in control theory [3] we are able to show the following result

Theorem 4 *For fixed $F\in C^{n_T\times\ell}$ and*

$$\mathcal{H}=R_{n_R}^{-1/2}HFR_\ell^{1/2}\in C^{n_R\times\ell} \quad (15)$$

the following equality holds

$$\max_{G\in C^{\ell\times n_R}} \frac{\mathsf{gdet}(2\pi e((GHF)R_\ell(GHF)^H+GR_{n_R}G^H))}{\mathsf{gdet}(2\pi eGR_{n_R}G^H)} = \det(I_\ell+\mathcal{H}^H\mathcal{H}). \quad (16)$$

Moreover, the maximum is attained at $G=\begin{bmatrix} G_\ell & O_{\ell\times(n_R-\ell)}\end{bmatrix} V_{n_R}R_{n_R}^{-1/2}G_\ell V_{\ell\times n_R}R_{n_R}^{-1/2}$, where G_ℓ is any nonsingular matrix of dimension $\ell\times\ell$ and V_{n_R} is a unitary matrix of the form $V_{n_R}=\begin{bmatrix} V_{\ell\times n_R}\\ V_{(n_R-\ell)\times n_R}\end{bmatrix}$ with $V_{(n_R-\ell)\times n_R}\mathcal{H}=0_{(n_R-\ell)\times\ell}$, i.e. $V_{(n_R-\ell)\times\ell}$ consists of $(n_R-\ell)$ orthonormal vectors from the left zero space of $\mathcal{H}$.

Proof. As $R_{n_R}>0$, using the variable change $G\leftarrow GR_{n_R}^{1/2}$ in (16), the proof then reduces to establishing the following

$$\max_{G\in C^{\ell\times n_R}} \frac{\mathsf{gdet}(2\pi e(G\mathcal{H}\mathcal{H}^HG^H+GG^H))}{\mathsf{gdet}(2\pi eGG^H)} = \det(I_\ell+\mathcal{H}^H\mathcal{H}) \quad (17)$$

First, we show that

$$\frac{\mathsf{gdet}(2\pi e(G\mathcal{H}\mathcal{H}^HG^H+GG^H))}{\mathsf{gdet}(2\pi eGG^H)} \le \det(I_\ell+\mathcal{H}^H\mathcal{H}) \quad \forall G\in C^{\ell\times n_R}. \quad (18)$$

When G is full rank, GG^H is nonsingular and it can be seen that

$$\frac{\mathsf{gdet}(2\pi e(G\mathcal{H}\mathcal{H}^HG^H+GG^H))}{\mathsf{gdet}(2\pi eGG^H)} = \det(I_\ell+\mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H})$$

One can see that $0<I_\ell+\mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H}\le I_\ell+\mathcal{H}^H\mathcal{H}$ and by

$$\det(I_\ell+\mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H})\le\det(I_\ell+\mathcal{H}^H\mathcal{H}) \quad (19)$$

hence we have (18). The proof of (18) when G is not full rank is more involved.
Apply the following SVD for G of rank $k<\ell$:
$G=U_\ell\begin{bmatrix}\Sigma_\ell & 0_{\ell\times(n_R-\ell)}\end{bmatrix}U_{n_R}, \Sigma_\ell=\mathsf{diag}\begin{bmatrix}\Sigma_k & 0_{\ell-k}\end{bmatrix}, U_\ell\in\mathcal{M}_\ell$, $U_{n_R}\in\mathcal{M}_{n_R}$.
Then

$$GG^H=U_\ell\Sigma_\ell U_\ell^H,\quad \mathsf{gdet}(2\pi eGG^H)=\det(2\pi e\Sigma_k^2), \quad (20)$$

$$G\mathcal{H}\mathcal{H}^HG^H+GG^H=U_\ell\mathsf{diag}\begin{bmatrix}\Sigma_kX\Sigma_k+\Sigma_k^2 & 0_{\ell-k}\end{bmatrix}U_\ell^H$$

with the partition $U_{n_R}\mathcal{H}\mathcal{H}^HU_{n_R}^H=\begin{bmatrix} X & *\\ * & *\end{bmatrix}, X\in C^{k\times k}$.
Hence

$$\mathsf{gdet}(2\pi eG\mathcal{H}\mathcal{H}^HG^H+GG^H)=\det(2\pi e(\Sigma_kX\Sigma_k+\Sigma_k^2)). \quad (21)$$

In view of (20) and (21) in can be shown that

$$\frac{\mathsf{gdet}(2\pi e(G\mathcal{H}\mathcal{H}^HG^H+GG^H))}{\mathsf{gdet}(2\pi eGG^H)}\le\det(\mathcal{H}^H\mathcal{H}+I_\ell)$$

Hence, (18) follows.
It remains to show that $G=G_\ell V_{\ell\times n_R}$ attains the right hand side of (18). It can be checked that

$$GG^H=G_\ell G_\ell^H,\qquad \mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H}=\mathcal{H}^H\mathcal{H}. \quad (22)$$

Thus, $I_\ell+\mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H}=I_\ell+\mathcal{H}^H\mathcal{H}$ and so $\det(I_\ell+\mathcal{H}^HG^H(GG^H)^{-1}G\mathcal{H})=\det(I_\ell+\mathcal{H}^H\mathcal{H})$, which together with (19) prove the statement of the Theorem.
Remark. For each precoder F, there are many choices for unitary V_{n_R}. For instance, $V_{n_R}=U_{n_R}^H$, where columns of U_{n_R} are orthonormal eigenvectors of $\mathcal{H}\mathcal{H}^H$ in the following SVD for $\mathcal{H}$: $\mathcal{H}=U_{n_R}\begin{bmatrix}\Sigma_{\ell\times\ell}\\ 0_{(n_R-\ell)\times\ell}\end{bmatrix}V_\ell=U_{n_R}\begin{bmatrix}\mathcal{H}_{\ell\times\ell}\\ 0_{(n_R-\ell)\times\ell}\end{bmatrix}$.

4. OPTIMAL PAIR OF PRECODER AND DECODER

Having solved the optimal decoder G in Section 3 for the optimization problem (13), it remains to derive its optimal precoder F, which is our aim in this section. We will explore more structures of the optimal precoder to derive the corresponding decoders to attain the maximal mutual information between input and output. Like [6] we will use the power of the matrix variational inequality machinery in tandem with genius matrix partition.
Using (16), the optimal problem (13) reduces to the following optimization problem in the precoder variable F:

$$\max_{F,\langle FR_\ell F^H\rangle\le P_T}\log\det(I_{n_T}+FR_\ell F^HH^HR_{n_R}^{-1}H). \quad (23)$$

We now derive the optimal solution of (23) in closed-form.
Define $h=\mathrm{rank}(H)$. Using SVD

$$H^HR_{n_R}^{-1}H=U_H^H\Sigma U_H \quad (24)$$

with $U_H\in\mathcal{M}_{n_T}$ and $\Sigma\in\mathcal{D}_{n_T}$ with diagonal elements in a decreasing order

$$\Sigma=\mathsf{diag}\begin{bmatrix}\Sigma_h & 0_{n_T-h}\end{bmatrix}\in R^{n_T\times n_T},\ 0<\Sigma_h\in\mathcal{D}_h, \quad (25)$$

problem (23) can be rewritten as

$$\max_{\langle FR_\ell F^H\rangle\le P_T}\log\det(I_{n_T}+\Sigma^{\frac12}FR_\ell F^H\Sigma^{\frac12}), \quad (26)$$

with $F\leftarrow U_HF$. Like [6], it can be shown that the optimal solution to problem (26) must have the structure $F=\begin{bmatrix}F_{h\ell}\\ 0_{(n_T-h)\times\ell}\end{bmatrix}$, i.e. according to the variable change in (26), the optimal solution of problem (23) must admit the form

$$F=U_H^H\begin{bmatrix}F_{h\ell}\\ 0_{(n_T-h)\times\ell}\end{bmatrix}=U_H^H[1:h,:]F_{h\ell}, \quad (27)$$

where $U_H(1:h,:)$ is the h first rows of U_H.
Hence, the problem (26) is in fact the following

$$\max_{\langle F_{h\ell}R_\ell F_{h\ell}^H\rangle\le P_T}\log\det(I_h+\Sigma_h^{1/2}F_{h\ell}R_\ell F_{h\ell}^H\Sigma_h^{1/2}). \quad (28)$$

For the case $\ell\le h$, using the following SVD [4, Th. 7.4.5, p. 414]:

$$\Sigma_hF_{h\ell}R_\ell F_{h\ell}^H\Sigma_h=U\mathsf{diag}\begin{bmatrix}D_x & 0_{h-\ell}\end{bmatrix}U^H, \quad (29)$$

with $U\in\mathcal{M}_h, 0\le D_x\in\mathcal{D}_\ell$, the problem (28) is rewriten as

$$\max_{(U,D_x)\in\mathcal{F}_{h,\ell}}\log\prod_{i=1}^{\ell}(1+D_x(i,i)) \quad (30)$$

where $\mathcal{F}_{h,\ell}=\{(U,D_x): U\in\mathcal{M}_h, D_x\in\mathcal{D}_\ell$, $\langle\Sigma_h^{-1}U\mathsf{diag}\begin{bmatrix}D_x & 0_{h-\ell}\end{bmatrix}U^H\rangle\le P_T\}$. A variational matrix result yields that the optimal solution (U,D_x) of (30) must satisfy

$$\langle\Sigma_h^{-1}U\mathsf{diag}\begin{bmatrix}D_x & 0_{h-\ell}\end{bmatrix}U^H\rangle=\min_{\tilde U\in\mathcal{M}_h}\langle\Sigma_h^{-1}\tilde U\mathsf{diag}\begin{bmatrix}D_x & 0_{h-\ell}\end{bmatrix}\tilde U^H\rangle.$$

Using [6, Lemma 1] we obtain

$$\min_{\tilde{U}\in\mathcal{M}_h} \langle \Sigma_h^{-1}\tilde{U}\mathsf{diag}\,[\,D_x \quad 0_{h-\ell}\,]\,\tilde{U}^H\rangle = \sum_{i=1}^{\ell}\Sigma_h^{-1}(i,i)D_x(\tau(i),\tau(i)), \tag{31}$$

where $\tau : \{1,2,...,\ell\} \to \{1,2,...,\ell\}$ is a one-to-one map that arranges $\{D_x(\tau(i),\tau(i))\}$ in deceasing order.
Thus problem (30) is reduced to

$$\max_{D_x(i,i)} \log\prod_{i=1}^{\ell}(1+D_x(i,i)) : \sum_{i=1}^{\ell}\Sigma_h^{-1}(i,i)D_x(i,i) \le P_T \tag{32}$$

because the water-filling structure of the optimal solution of the latter, i.e

$$D_x(i,i) = (\mu\Sigma_h(i,i)-1)^+, i = 1,2,...,\ell \tag{33}$$

is already in decreasing order.
In other words, $U = I_h$ in (29) and (30), while μ is chosen so that
$\sum_{i=1}^{\ell}\Sigma_h^{-1}(i,i)D_x(i,i) = P_T.$
Thus, in view of (29), (32), the optimal solution $F_{h\ell}$ of the optimization problem (28) in the case $\ell \le h$ is

$$\Sigma_h^{1/2}F_{h\ell}R_\ell^{1/2} = \begin{bmatrix} D_x \\ 0_{(h-\ell)\times\ell}\end{bmatrix}$$
$$\Leftrightarrow \quad F_{h\ell} = \begin{bmatrix}\Sigma_h^{-1/2}[1:\ell]\sqrt{D_x}R_\ell^{-1/2} \\ 0_{(h-\ell)\times\ell}\end{bmatrix}, \tag{34}$$

where $\Sigma_h^{-1/2}[1:\ell]$ is the diagonal matrix consisting of the ℓ first diagonal elements of $\Sigma_h^{-1/2}$. Then according to (27) the optimal solution F of the optimization problem (23) is

$$F = U_H^H(1:\ell,:)\Sigma_h^{-1}\sqrt{D_x}R_\ell^{-1/2}, \tag{35}$$

where $U_H(1:\ell,:)$ is the first ℓ rows of U_H.
Analogously, for the case $\ell \ge h$, the optimal solution F of the optimization problem (23) is

$$F = \left[\, U_H^H(1:h,:)\Sigma_h^{-1}\sqrt{D_x} \quad 0_{h\times(\ell-h)}\,\right] R_\ell^{-1/2} \tag{36}$$

Based on (35) and (36), we can state the following theorem to recap our results in this section.

Theorem 5 *With the SVD (24), (25) and the diagonal matrix D_x defined by (33), the optimal precoder F of the problem (23) is given by*

$$F = \left[\, U_H^H(1:L)\Sigma_h(1:L)^{-1}\sqrt{D_x} \quad O_{L\times(\ell-L)}\,\right] R_\ell^{-1/2}, \tag{37}$$

where $L = \min\{\ell,h\}$ and $U_H(1:L,:)$ is the first L rows of U_H while $\Sigma_h(1:L)$ is the first L rows/columns of Σ_h.

5. SIMULATION RESULTS

We now present simulation results to illustrate the performance of our solutions for frequency selective fading channels with various ranks of the channel matrix.
The channel taps are uncorrelated complex Gaussian random variables, i.e. Rayleigh fading channel. Each realization of the channel is assumed known at the transmitter and receiver. The linear precoder F and decoder G are optimized for each channel realization. The signal vectors x used in the simulations are drawn from the quadrature phase shift keying (QPSK) constellation $\{\pm 1 \pm \jmath\}$, correlated with covariance R_ℓ. The additive noise vectors n are correlated complex Gaussian random variables with covariance R_{n_R}. The total transmit power across all transmit antennas is normalized to unity, i.e. $\langle FR_\ell F^H\rangle = 1$. The signal to noise ratio (SNR) is defined as $SNR = \langle FR_\ell F^H\rangle/\langle R_{n_R}\rangle 1/\langle R_{n_R}\rangle$, which does not include possible gain/attenuation of the channel realization. In our simulations, the channel matrix H is normalized so that $\langle HH^H\rangle = 1$. A system of $n_T = 2$ transmit and $n_R = 2$ receive antennas is considered. The channel order is $L = 3$. Each block of input signal $x \in C^{10}$ is precoded as $z = Fx \in C^{14}$. To avoid interblock interference, we use the following transmission scheme. Let z_1 and z_2 denote the L-zero appended versions of the first half and the last half of z respectively, specifically: $z_1 = [z(1),\dots,z(7),0,\dots,0]^T \in C^{10}$ and $z_2[z(8),\dots,z(14),0,\dots,0]^T \in C^{10}$. The elements of z_1 are sent to the first antenna and the elements of z_2 to the second antenna. The noise-corrupted received vector $y = HFx + n$ is then used for decoding at the receiver. Note that the channel matrix $H \in C^{20\times 14}$ in this case is $H = \begin{bmatrix} H_{11}, H_{21} \\ H_{12}, H_{22}\end{bmatrix}$ where $H_{ij} \in C^{10\times 7}$ is channel matrix of the transmission link between the ith transmit antenna and the jth receive antenna. Two cases of the channel matrix rank: rank $= 14$ and rank $= 7$ are investigated. As for the case of $rank(H) = 7$, we let $h_{11} = h_{12} = h_{21} = h_{22}$. The same signal and noise covariance matrices as in the flat fading channel case are used. The information rate is given in Figures 1.

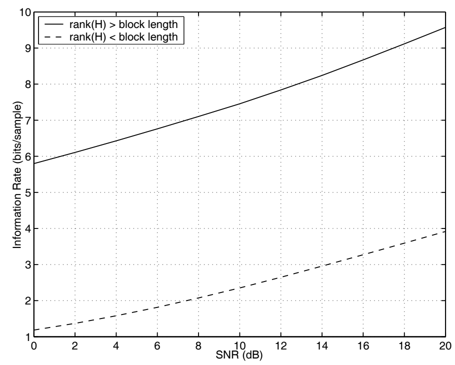

Fig. 1. Information rate for frequency-selective fading channel

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