THE STOCHASTIC SINUSOIDAL MODEL FOR RAYLEIGH FADING CHANNEL SIMULATION

J. Grolleau, D. Labarre, E. Grivel and M. Najim

Equipe Signal et Image, ENSEIRB, UMR CNRS 5131 LAPS, Universite Bordeaux I
1, Avenue du Dr. Albert Schweitzer, BP 99 F-33402 Talence Cedex, France
julie.grolleau@etu.u-bordeaux1.fr, david.labarre@laps.u-bordeaux1.fr, egrivel@u-bordeaux1.fr, mnajim@u-bordeaux1.fr

ABSTRACT

In this paper, we propose a new Rayleigh channel simulator. Modeling the channel by an AR process leads to numerical problems due to the bandlimitation of the theoretical Power Density Spectrum (PSD) of a Rayleigh channel. Therefore, we suggest modeling the channel by a low-pass filtered version of the so-called stochastic sinusoidal process. It consists of sinusoids in quadrature with random magnitudes modeled as AR processes. To estimate the AR parameters of the amplitudes, we take advantage of the asymptotic behavior of the first-kind zero-order Bessel function. We show that unlike an AR channel modeling, this simulator has the advantage of exhibiting the PSD peaks at the maximum Doppler frequency, for any AR process order.

***Index Terms*—** Rayleigh channels, autoregressive processes, Bessel functions.

1. INTRODUCTION

In the framework of wireless systems, the transmitted signal suffers from the effects of the relative velocity of the receiver and of the propagation environment. Thus, to test the receiver performance of a communication system, a great deal of attention has been paid to channel simulators.

The Rayleigh channel process $h(n)$ is usually generated in accordance with Clarke's model. In an environment with no direct line-of-sight between transmitter and receiver, the marginal distributions $P_\phi(n)$ of the phase $\phi(n)$ and $P_r(n)$ of the amplitude $r(n)$ of the channel process are respectively uniform and Rayleigh. In that case, the channel is modeled as a zero-mean Wide-Sense Stationnary (WSS) Gaussian complex process. The propagation path is assumed to be a two-dimensional isotropic scattering with a vertical monopole antenna at the receiver [1]. Then, the theoretical PSD of the real and imaginary parts of the fading samples are defined as follows:

$$S_h^{theo}(f) = \begin{cases} \frac{1}{\pi f_d}\sqrt{\frac{1}{1-(\frac{f}{f_d})^2}}, & |f| < f_d, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

where f_d is the maximum Doppler frequency.
The corresponding theoretical normalized discrete-time autocorrelation function is therefore given by:

$$R_h^{theo}[k] = J_0\left(2\pi f_{d_n}\,|k|\right)\ \forall\ k\ \in Z \quad (2)$$

where $J_0(.)$ denotes the zero-order Bessel function of the first kind and f_{d_n} is the normalised maximum Doppler frequency.

Based on these assumptions, various studies have been carried out to simulate channel samples. Among them, three main families of approaches have been considered:

- the Sum-Of-Sinusoids (SOS) based methods [1]-[4],
- the Inverse Discrete Fourier Transform (IDFT) based algorithms [5] [6],
- the white noise filtering methods [8]-[10].

Thus, Jakes' fading model [1] is one of the pioneering deterministic approach to simulate time-correlated waveforms. However, simulating uncorrelated channels with a method based on a sum of sinusoids is not an easy task. To remove correlation between channels, Dent et al. [2] propose to weight the sinusoids by orthogonal random codes. Nevertheless, Pop et al. [3] have shown that the autocorrelations of the real and the imaginary parts and the cross-correlations between these two parts do not match the desired correlation properties even if the number of sinusoids tends to infinity. Then, Zheng et al. [4] have proposed an improved SOS simulator whose second order statistics correspond to the desired ones for any number of sinusoids. In addition, the marginal distributions of the phase and the amplitude can be also well approximated when using a small number of sinusoids. It should be noted that this simulation model cannot be used to estimate and/or predict a channel.

As an alternative to this simulator, Smith [5] propose to design a channel generator that combines a filtering step and an IDFT. However, the storage requirement makes Smith's approach less attractive when the number of channel samples to be simulated becomes large. Although Young et al. [6] manage to reduce the computational cost and the memory storage, the use of IDFT leads to an off-line simulation, and hence still results in a high storage requirement.

The third approach consists in filtering white Gaussian noises. Thus, Wu et al. [8] and Mamfoumbi Ocloo et al. [7] model the Rayleigh channel by a 1^{st} or 2^{nd}-order AutoRegressive (AR) process. However, the estimation procedure proposed in [8] leads to twin peaks at $\pm\frac{f_{d_n}}{\sqrt{2}}$, instead of $\pm f_{d_n}$. More generally, this approach may be questionable. Due to the bandlimitation of the fading process, only an infinite-order AR process can lead in theory to the U-shaped PSD [9] [10]. In practice, Baddour et al. [9] suggest considering a model order at least higher than 50 and estimating the parameters based on the Yule-Walker equations by using $R_h^{theo}[k]$ (2). Nevertheless, this still does not guarantee the frequency peaks to be at the maximum Doppler frequency. Moreover, the channel autocorrelation matrix used in the Yule-Walker equations may become ill-conditioned. To overcome this difficulty, in [9], a small constant is added to the diagonal of the channel autocorrelation matrix, yielding a lower matrix condition number. So, this solution consists in simulating a Rayleigh fading channel disturbed by a white Gaussian noise.

In this paper, we propose to consider a low-pass filtered version of a sum of two sinusoids in quadrature at the maximum Doppler frequency and whose amplitudes are AR processes. This combination has the advantage of providing a process whose PSD is bandlimited and has two peaks at the maximum Doppler frequency for any AR order. However, as the autocorrelation function of the AR amplitudes is not available, we study two ways to estimate the autocorrelation function and the AR parameters, pointing out their advantages and drawbacks.

The paper is organized as follows: In Section 2, a statistical analysis of the stochastic sinusoidal modeling for fading channel simulation is provided. A comparative study with the AR-model based simulators [9] is carried out in Section 3.

2. THE STOCHASTIC SINUSOIDAL MODEL

2.1. The Stochastic Sinusoidal Model for channel modeling

Let us consider the following stochastic sinusoidal process:

$$z(n) = a(n)cos(2\pi f_{d_n} n) + b(n)sin(2\pi f_{d_n} n) \tag{3}$$

where $a(n)$ and $b(n)$ are independant p^{th} order AR processes:

$$a(n) = -\sum_{i=1}^{p} c_i a(n-i) + u_a(n) \tag{4}$$

$$b(n) = -\sum_{i=1}^{p} c_i b(n-i) + u_b(n) \tag{5}$$

where $\{c_i\}_{i=1,\ldots,p}$ are the real AR parameters. $u_a(n)$ and $u_b(n)$ are independant zero-mean complex white Gaussian noise processes whose real and imaginary parts have the same variance σ_p^2.

In addition, to make $z(n)$ WSS, the real and imaginary parts of $a(n)$ and $b(n)$ are assumed to have the same autocorrelation function $R_a[k]$. Therefore, it can be easily shown that the modulus and the phase of the stochastic sinusoidal process $z(n)$ are respectively Rayleigh and uniformly distributed.

However, the PSD of $z(n)$ is not bandlimited, unlike $S_h^{theo}(f)$. For this reason, we propose to filter the process $z(n)$ by a low-pass filter with cut-off frequency f_{d_n}.

At that stage, the autoregressive parameters $\{c_i\}_{i=1,\ldots,p}$ have to be estimated by using the Yule-Walker equations. This hence requires the estimation of the AR process autocorrelation function $R_a[k]$.

2.2. Estimation of $R_a[k]$: First method and resulting numerical problems

On the one hand, due to (3), (4) and (5), the relationship between $R_a[k]$ and the autocorrelation function $R_z[k]$ of the stochastic sinusoidal process $z(n)$ is given by:

$$R_z[k] = R_a[k]cos(2\pi f_{d_n} k). \tag{6}$$

On the other hand, according to section 2.1, the PSD $S_h(f)$ of the simulated channel $h(n)$ satisfies:

$$S_h(f) = S_z(f).\Pi_{2f_{d_n}}(f) \tag{7}$$

where $\Pi_{2f_{d_n}}(f)$ is the frequency response of an ideal low-pass filter with cut-off frequency f_{d_n}.

Combining (6) and the inverse Fourier transform of $\Pi_{2f_{d_n}}(f)$ leads to:

$$R_h[k] = (R_a[k]cos(2\pi f_{d_n} k)) * g(k) \;\; \text{for any } k \tag{8}$$

where the real even sequence $g(k)$ is the inverse Fourier transform of $\Pi_{2f_{d_n}}(f)$. Nevertheless, for the subsequent estimation of $R_a[k]$, a truncated version of the impulse response $g(k)$ has to be considered (i.e. $k \in [-L, L]$ with L high enough to weaken Gibb's oscillations at discontinuities in the frequency domain).

Thus, the equation (8) leads to the following relationship:

$$\mathbf{GCr}_a = \mathbf{r}_h \tag{9}$$

where:

- $\mathbf{G}$ is a $2L+1 \times 2L+1$ matrix whose element $G_{i,j}$ of the i^{th} row and j^{th} column of $\mathbf{G}$ is given by:
  for $i = 1$:
  $$G_{1,j} = g(L-j+1) \tag{10}$$
  for $1 < i < L+1$:
  $$G_{i,j} = \begin{cases} g(L-i+1), \text{ for } j = 1, \\ g(L-i+j) + g(L-i-j+2), \text{ for } j \in [2, i], \\ g(L-i-j+2), \text{ for } j \in [i+1, 2L+2-i], \\ 0, \text{ otherwise,} \end{cases} \tag{11}$$
  for $i = L+1$:
  $$G_{i,j} = \begin{cases} g(0), \text{ for } j = 1, \\ g(j-1) + g(1-j), \text{ for } j \in [2, L+1], \\ 0, \text{ otherwise,} \end{cases} \tag{12}$$
  for $2L+1 \geq i > L+1$, the element can be deduced thanks to the following relation:
  $$G_{i,j} = G_{2L+2-i,j}. \tag{13}$$
- $\mathbf{C}$ is a diagonal matrix whose i^{th} element is:
  $$C_{i,i} = cos(2\pi f_{d_n}(i-1)) \tag{14}$$
- $\mathbf{r}_a$ is the autocorrelation vector to be retrieved given by:
  $$\mathbf{r}_a = [R_a[0] \; R_a[1] \; \ldots \; R_a[2L]]^T \tag{15}$$
- and $\mathbf{r}_h$ is the channel autocorrelation vector defined as:
  $$\mathbf{r}_h = \left[R_h^{theo}[-L] \; \ldots \; R_h^{theo}[0] \; \ldots \; R_h^{theo}[L]\right]^T \tag{16}$$

As $R_h^{theo}[k]$ is available, the autocorrelation $R_a[k]$ can be estimated by solving the set (9) of the $2L+1$ equations . However, two numerical problems may appear:

- Firstly, **GC** may not be inversible. Indeed, for $f_{d_n} \leq 0.5$ such as $2\pi k f_{d_n} = (2l+1)\frac{\pi}{2}$ where $l \in N$, $cos(2\pi k f_{d_n}) = 0$. Therefore, one has:
$$det(\mathbf{C}) = det(\mathbf{GC}) = 0 \quad (17)$$
- Secondly, for other values of f_{d_n}, **GC** may be ill-conditionned. See the proof in appendix.

Therefore, an alternative estimation method has to be found.

2.3. Estimation of $R_a[k]$: Second method using the asymptotic behavior of the Bessel function

In this subsection, our purpose is to express the Bessel function as in equation (8). For this purpose, we take advantage of its asymptotic behavior.

Indeed, when $|k| \geq \frac{1}{2\pi f_{d_n}}$, the Bessel function $J_0(2\pi f_{d_n} k)$ can be approximated [11] by :

$$J_0(2\pi f_{d_n} k) \approx \sqrt{\frac{1}{\pi^2 f_{d_n} |k|}} cos(2\pi f_{d_n} |k| - \frac{\pi}{4}) \quad (18)$$

Therefore, when $f_{d_n} \geq \frac{1}{2\pi}$, the relation (18) can be rewritten as follows:

$$J_0(2\pi f_{d_n} k) \approx f[k] \left(cos\left(2\pi f_{d_n} k\right) + sin\left(2\pi f_{d_n} |k|\right) \right) \quad (19)$$

where:

$$f[k] = \sqrt{\frac{1}{2\pi^2 f_{d_n} |k|}} \text{ for } k \neq 0 \quad (20)$$

It should be noted that relation (19) still holds for $k = 0$ providing $f[0] = 1$.

At that stage, let us consider $R_{appr}(k)$ defined as follows:

$$R_{appr}(k) = 2f[k] cos\left(2\pi f_{d_n} k\right) \text{ for } k \neq 0 \quad (21)$$

where $R_{appr}(0)$ is the solution of:

$$TF(R_{appr}(k))_{f=0} = TF(J_0(2\pi f_{d_n} k))_{f=0} \quad (22)$$

with TF(.) the Fourier transform. Combining (1), (20) and (21) leads to:

$$\sum_{k=-\infty}^{+\infty} 2f[k] cos(2\pi f_{d_n} k) - 2f[0] + R_{appr}(0) = \frac{1}{\pi f_{d_n}} \quad (23)$$

which can be rewritten as follows:

$$R_{appr}(0) = \frac{1}{\pi f_{d_n}} - TF\left(2f[k] cos\left(2\pi f_{d_n} k\right)\right)_{f=0} + 2f[0] \quad (24)$$

Therefore, due to (18) and (19), the difference $D(k) = J_0(2\pi f_{d_n} k) - R_{appr}(k)$ is approximated by:

$$D(k) \approx \begin{cases} -f[k] cos\left(2\pi f_{d_n} k + \frac{\pi}{4}\right), & k > 0, \\ 1 - R_{appr}(0), & k = 0, \end{cases} \quad (25)$$

As shown in figure 1, the Fourier Transform of $D(k)$ is close to zero between $-f_{d_n}$ and f_{d_n}. The Fourier Transform of $R_{appr}(k)$ and $J_0(2\pi f_{d_n} k)$ are therefore very close to each other in this frequency band. Low-pass filtering at the frequency f_{d_n} hence leads to the following approximation of the Bessel function:

$$J_0(2\pi f_{d_n} k) \approx R_{appr}(k) * g(k) \quad (26)$$

So, according to the equations (8), (21) and (26), $R_a[k]$ can be approximated by:

$$R_a[k] \approx \begin{cases} \sqrt{\frac{2}{\pi^2 f_{d_n} |k|}} & \text{for } k \neq 0 \\ R_{appr}(0) & \text{for } k = 0 \end{cases} \quad (27)$$

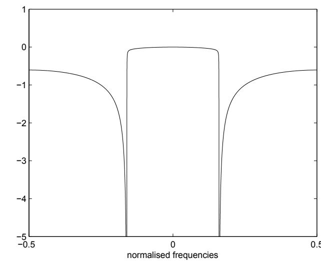


Fig. 1. Fourier transform of $D(k)$ with f_{d_n}=0.16

2.4. Summary of the channel simulation

1. The AR parameters $\{c_i\}_{i=1,\ldots,p}$ are estimated from the Yule-Walker equations using the estimation (27) of the autocorrelation function.
2. Two complex AR processes, whose AR coefficients are $\{c_i\}_{i=1,\ldots,p}$ are simulated, are modulated by a cosine and a sine and summed in quadrature.
3. The resulting process is filtered by a low-pass filter whose L'-length impulse response $p(n)$ is defined as follows:
$$p(n) = 2f_{d_n} \frac{sin(2\pi(n - L'/2)f_{d_n})}{2\pi(n - L'/2)f_{d_n}} \quad (28)$$
where L' is high enough to weaken Gibb's oscillations at discontinuities in the frequency domain.

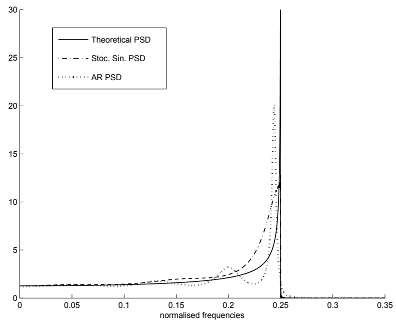
3. PERFORMANCE COMPARISON TO THE AR BASED SIMULATOR

In this section, we study the relevance of the low-pass filtered version of the stochastic sinusoidal process based simulator to generate bandlimited Rayleigh samples. For this purpose, we carry out a comparative study between the proposed method and the AR-model based simulator. In the following, we present the results for the spectra of the real parts of the simulators. Similar results are obtained for their imaginary parts. The orders of both the AR models in the proposed method and the AR-model based simulator vary from 10 to 100 while the normalised maximum Doppler frequency varies between 0.16 to 0.35. 10000 channel samples are generated. The length L' of the impulse response defined in (28) is assigned to 700.

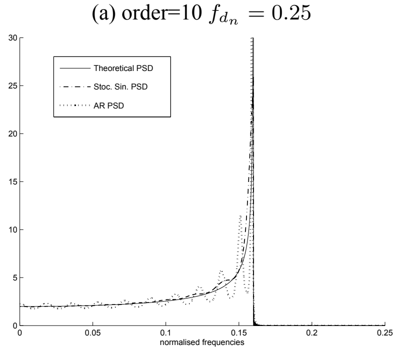
The simulations illustrated by Fig.2 confirm that the spectra of the generated channel with the AR-model based approach does not exhibit peaks close to the maximum Doppler frequency when the model order is less than 50 [9]. The proposed method always enables the twin peaks of the PSD to be located at $\pm f_d$, for any AR process order. Furthermore, the stochastic sinusoidal-model based method provides a PSD closer to the true one than the standard AR-model based method.

4. CONCLUSION

In this paper, we have investigated the relevance of the filtered stochastic sinusoidal model for the simulation of Rayleigh fading channels. As the straightforward way to estimate the AR parameters leads to numerical problems, we have proposed a new estimation method based on the asymptotic



(a) order=10 $f_{d_n} = 0.25$



(b) order=50 $f_{d_n} = 0.16$

Fig. 2. PSD of various simulators.

behavior of the zero-order Bessel function. Thus, the simulated process has the advantage of exhibiting twin peaks at the Doppler frequency in its PSD, with a reduced number of AR parameters. The resulting process better fits the Clarke's model statistical properties than the AR process.

APPENDIX

In the following , let $\mathbf{M} = \mathbf{GC}$ for the sake of simplicity. We aim at finding a lower bound for the 2-norm condition number of $\mathbf{M}$ defined as follows:

$$cond(M) = \|\mathbf{M}\|_2 \left\|\mathbf{M}^{-1}\right\|_2 \tag{29}$$

For this purpose, let us find lower bounds of $\|\mathbf{M}\|_2$ and $\left\|\mathbf{M}^{-1}\right\|_2$.
Step 1: Let x be the $(2L+1)$-length column vector with 1 in the first row and 0 elsewhere. As $\mathbf{M}x = [g(L) \dots g(0) \dots g(L)]^T$, the 2-norm of $\mathbf{M}$ satisfies:

$$\|\mathbf{M}\|_2 \geq 2f_{d_n}. \tag{30}$$

Step 2: Let $k \notin \{1, 2L+1\}$ and x be the $(2L+1)$-length column vector with 1 in the $(k+1)^{th}$ row and 0 elsewhere. One has:

$$\mathbf{C}x = \left[\overbrace{0 \dots 0}^{k} \; cos(2\pi k f_{d_n}) \; \overbrace{0 \dots 0}^{2L-k} \right] \tag{31}$$

So, $y = \mathbf{M}x$ is a $(2L+1)$-length vector whose element on the i^{th} row satisfies:

$$y_i = cos(2\pi(k) f_{d_n}).G_{i,k}. \tag{32}$$

where $G_{i,k}$ is defined by equations (11)-(13).
As $|g(i)| \leq g(0) \leq 2f_{d_n}$ for all i, one has:

$$\|\mathbf{M}x\|_2 \leq 4f_{d_n}\sqrt{L+1}\,|cos(2\pi k f_{d_n})|\,. \tag{33}$$

At that stage, let us introduce:

$$y' = \frac{1}{4f_{d_n}\sqrt{2L+1}\,|cos(2\pi k f_{d_n})|}\mathbf{M}x. \tag{34}$$

The 2-norm of $\mathbf{M}^{-1}y$ is equal to:

$$\left\|\mathbf{M}^{-1}y\right\|_2 = \frac{1}{4f_{d_n}\sqrt{2L+1}\,|cos(2\pi k f_{d_n})|} \tag{35}$$

As $\|\mathbf{y'}\|_2 \leq 1$, a lower bound of the 2-norm of $\mathbf{M}^{-1}$ can be given by:

$$\left\|\mathbf{M}^{-1}\right\|_2 \geq \frac{1}{4f_{d_n}\sqrt{2L+1}\,|cos(2\pi k f_{d_n})|} \tag{36}$$

So, using (30) and (36), the condition number of $\mathbf{GC}$ satisfies:

$$cond = \|\mathbf{M}\|_2 \left\|\mathbf{M}^{-1}\right\|_2 \geq \frac{1}{2\sqrt{2L+1}\,|cos(2\pi k f_{d_n})|} \tag{37}$$

For IEEE floating point precision of 16 decimal digits, a condition number superior to 10^{16} causes significant errors in the computation of the inverse of $\mathbf{M}$. So, given (37), when L is equal to a few hundreds, if $\exists\ k \leq 2L$ such as:

$$|cos(2\pi k f_{d_n})| \leq 10^{-17} \tag{38}$$

then the condition number is too large. Thus, for instance, $f_{d_n} = 0.475$ makes the cosinus equal to zero for $k = 10$, so for $0.475 - 10^{-17} \leq f_{d_n} \leq 0.475 + 10^{-17}$, $\mathbf{M}$ will be ill-conditioned.

AKNOWLEDGEMENT

We would like to thank Flavius Turcus for the fruitful discussion we have with him about the condition number issue.

5. REFERENCES

[1] W. C. Jakes, "Microwave mobile communications", *IEEE Press Eds. Wiley, 1974.*

[2] P. Dent, G. E. Bottomley and T. Croft, "Jakes' fading model revisited", *Electron. Lett. vol.29, no.3, Jun. 1993.*

[3] M. F. Pop and N. C. Beaulieu, "Limitations of sum-of-sinusoids fading channel simulators", *IEEE Trans. on Commun. vol.49, no.4, Apr. 2001.*

[4] Y. R. Zheng and C. Xiao, "Simulation models with correct statistical properties for Rayleigh fading channels", *IEEE Trans. on Commun. vol. 51, no. 6, Jun. 2003.*

[5] J. I. Smith, "A computer generated multipath fading simulation for mobile radio", *IEEE Trans. on Veh. Technol. vol.VT-24, Aug. 1975.*

[6] D. J. Young and N.C. Beaulieu, "The generation of correlated Rayleigh random variates by inverse Fourier transform", *IEEE Trans. on Commun. vol.48, no.7, Jul. 2000.*

[7] J. M. Mamfoumbi Ocloo and F. Alberge, "OFDM Channel Estimation by a Linear EM-MAP Algorithm", *IEEE ICASSP, May 2006.*

[8] H. Wu and A. Duel-Hallen, "Multiuser detectors with disjoint Kalman channel estimators for synchronous CDMA mobile radio channels", *IEEE Trans. on Commun. vol.48, no.5, May 2000.*

[9] K. E. Baddour and N. C. Beaulieu, "Autoregressive modeling for fading channel simulation", *IEEE Trans. on Commun. vol.4, no.4, Jul. 2005.*

[10] P. Sadeghi, P. Rapajic, R. Kenedy and T. Adhayapala, "Autoregressive Time-Varying Flat-Fading Channels: Model Order and Information Rate Bounds", *IEEE ISIT, July 2006.*

[11] G. N. Watson, "A treatise on the theory of Bessel function", *Cambridge at the university press, 1944.*