ON THE TRACKING PERFORMANCE OF A FAMILY OF GENERALIZED CONSTANT MODULUS ALGORITHM

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ABSTRACT

Based on an energy conservation relation, the tracking performance of a family of generalized constant modulus algorithm (GCMA) is analyzed in this paper. Both theoretical analysis and simulation results validate the excellent tracking performance of the extended constant modulus algorithm (ECMA) in some situations.

***Index Terms*—** Adaptive filters, adaptive equalizers, tracking filters.

1. INTRODUCTION

Due to the growing demand for digital communications, blind adaptive algorithms have an important role in improving data transmission efficiency. Until now, the three best known and widely used blind equalization algorithms for two dimensional (2-D) modulation schemes, such as quadrature amplitude modulation (QAM) and carrierless amplitude and phase modulation (CAP), are the constant modulus algorithm (CMA) [1], the reduced constellation algorithm (RCA) [2] and the multimodulus algorithm (MMA) [3]. A fly in the ointment is that the CMA is blind to carrier phase. To recover the carrier phase, the CMA needs an additive rotator at the output of equalizer, but it increases the complexity of implementation of the receiver in steady-state operation. Both the RCA and the MMA are capable of recovering the carrier phase. However, since the equalization errors of the real parts and imaginary parts signals are assumed to be independent in the cost functions, the RCA and the MMA may erroneously cause a phase-splitting equalizer to converge to a degenerate diagonal solution [3], [5]. By combining the merits of RCA, MMA and CMA, Thaiupathump [5] proposed the square contour algorithm (SCA) and its generalization named the generalized SCA (GSCA) and sign SCA (SSCA). The SCA and its generalized algorithms can avoid the degenerate diagonal solution and recover the carrier phase simultaneously. Recently, by generalizing the definition of complex modulus, a family of generalized constant modulus algorithm (GCMA) is proposed in [6], which contains not only the well-known CMA, but also the sign Godard algorithm (SGA) [4], the SCA and its generalized algorithms as special examples. Naturally, the convergence and tracking analysis of GCMA is a problem of interest. This paper studies the tracking performance of GCMA in an environment with a small degree of nonstationarity. Both theoretical analysis and simulation results suggest that the extended constant modulus algorithm (ECMA) [6], a new constant modulus-type algorithm in the family of GCMA, can reach a lower minimum steady-state mean squared error than the CMA and the GSCA in some situations.

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2. BACKGROUND

We consider the tracking performance of GCMA implemented in fractionally-spaced form. Fig. 1 shows the channel-equalizer model that arises when $\frac{T}{2}$-fractionally spaced equalization (FSE) is used. The discussion here is also applicable to a more general $\frac{T}{P}$-FSE. We split the $2M$ length channel coefficients $\mathbf{c} = [c(0), c(1), \cdots, c(2M-1)]$ into even part and odd part as follows:

$$\mathbf{c}_e = [c(0), c(2), \cdots, c(2M-2)],$$

$$\mathbf{c}_o = [c(1), c(3), \cdots, c(2M-1)].$$

In the same way, two sub-equalizers of length N are defined as

$$\mathbf{w}_e = [w(0), w(2), \cdots, w(2N-2)]^T,$$

$$\mathbf{w}_o = [w(1), w(3), \cdots, w(2N-1)]^T$$

where ^{T} denotes the transpose operator of vectors. We further define the equalizer weight column vector $\mathbf{w} = [\mathbf{w}_e^T, \mathbf{w}_o^T]^T$ and the input row vector $\mathbf{u}_i = [\mathbf{u}_{o,i}, \mathbf{u}_{e,i}]$, where

$$\mathbf{u}_{o,i} = [u_o(i), u_o(i-1), \cdots, u_o(i-N+1)],$$

$$\mathbf{u}_{e,i} = [u_e(i), u_e(i-1), \cdots, u_e(i-N+1)].$$

Then the equalization output is given by $y(i) = \mathbf{u}_i\mathbf{w}$.

Let complex number $z = z_R + \mathrm{j}z_I$, where $\mathrm{j} = \sqrt{-1}$, and z_R and z_I are the real and imaginary part of z respectively. A family of generalized complex modulus of z can be defined as [6]

$$|z|_\ell = (|z_R|^\ell + |z_I|^\ell)^{1/\ell},\ \ell \geq 1 \quad (1)$$

where $|x|$ denotes the absolute value or modulus of real or complex number x. In [6], the GCMA minimizes cost function

$$J_{\ell,p,q}(\mathbf{w}) = E\{||y(i)|_\ell^p - R_{\ell,p,q}|^q\},\ \ell \geq 1, p \geq 1, q \geq 1 \quad (2)$$

via stochastic gradient descent algorithm, where $E\{\cdot\}$ is the expectation operator, ℓ, p and q are positive integers, and the dispersion constant $R_{\ell,p,q}$ is related with the statistics of source $s(i)$. Choosing different ℓ, p and q, different realization forms of GCMA, which includes the CMA [1], the SGA [4], and the GSCA, SCA and SSCA [5], can be obtained. The RCA and MMA are closely related with GCMA and their cost functions can be derived from the generalized cost function (2) by removing the statistical correlation terms of the real part component $y_R(t)$ and the imaginary part component $y_I(t)$ of the output $y(t)$ [6]. A disadvantage of such simplification is that the RCA and the MMA may erroneously cause a phase-splitting equalizer to converge to a degenerate diagonal solution [3], [5]. An important property of GCMA is its carrier phase recovery ability if $\ell \neq 2$ is selected [6]. In this paper, we focus on a special family of GCMA defined by choosing $p = q = 2$ and $\ell \geq 1$. This family includes most of the important algorithms such as the CMA ($\ell = 2$) in [1], the GSCA ($\ell = \infty$) in [5] and the ECMA ($\ell = 4$) in [6].

Considering the differential of $|z|_\ell$

$$\mathrm{d}|z|_\ell = |z|_\ell^{1-\ell}(z_R|z_R|^{\ell-2}\mathrm{d}z_R + z_I|z_I|^{\ell-2}\mathrm{d}z_I), \quad (3)$$

the conjugated gradient of $J_{\ell,2,2}(\mathbf{w})$ with respect to $\mathbf{w}$ can be shown to be

$$\begin{aligned}\frac{\partial J_\ell(\mathbf{w})}{\partial \mathrm{conj}(\mathbf{w})} &= 4E\{|y(i)|_\ell^{2-\ell}(|y(i)|_\ell^2 - R_\ell)[y_R(i)|y_R(i)|^{\ell-2} \\ &\quad + \mathrm{j}y_I(i)|y_I(i)|^{\ell-2}]\mathbf{u}_i^*\} \end{aligned} \quad (4)$$

where $J_\ell \equiv J_{\ell,2,2}$, $R_\ell \equiv R_{\ell,2,2}$, $\mathrm{conj}(\cdot)$ is the element-wise complex conjugation operator, and * denotes the conjugated transpose operator of vectors. The dispersion constant R_ℓ can be evaluated by assuming perfect equalization, i.e., $y(i) \to s(i)$. Thus, putting $\frac{\partial J_\ell(\mathbf{w})}{\partial \mathrm{conj}(\mathbf{w})} = \mathbf{0}$ leads to

$$R_\ell = \frac{E\{|s(i)|_\ell^4\}}{E\{|s(i)|_\ell^2\}}. \quad (5)$$

Specially, R_ℓ reduces to $R_2 = \frac{E\{|s(i)|^4\}}{E\{|s(i)|^2\}}$ when $\ell = 2$ is selected [1].

3. TRACKING ANALYSIS

We follow the way in [7] and [8] to analyze the tracking performance of GCMA. Descending along the instantaneous conjugated gradient descent direction (4) leads to the learning rule of GCMA with a constant step-size μ

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu\mathbf{u}_i^* f_e(i) \quad (6)$$

where

$$\begin{aligned} f_e(i) &= |y(i)|_\ell^{2-\ell}(R_\ell - |y(i)|_\ell^2)[y_R(i)|y_R|^{\ell-2}(i) \\ &\quad + jy_I(i)|y_I(i)|^{\ell-2}(i)]. \end{aligned} \quad (7)$$

In a nonstationary environment, the channel varies with time and therefore the zero forcing weight vector $\mathbf{w}^0$ itself may also assumed to vary with time, say $\mathbf{w}_i^0$. It is customary to assume that $\mathbf{w}_i^0$ follow the model [7]

$$\mathbf{w}_{i+1}^0 = \mathbf{w}_i^0 + \mathbf{q}_i. \quad (8)$$

In this model, $\mathbf{q}_i$ is an independently and identically distributed (i.i.d.) sequence with positive definite autocorrelation matrix $\mathbf{Q} = E[\mathbf{q}_i\mathbf{q}_i^*]$. We use the model (8) to study the tracking performance of GCMA. Subtracting both sides of (6) from the zero-forcing solution $\mathbf{w}_i^0$ gives the weight error equation

$$\bar{\mathbf{w}}_{i+1} = \bar{\mathbf{w}}_i - \mu\mathbf{u}_i^* f_e(i) + \mathbf{q}_i \quad (9)$$

where $\bar{\mathbf{w}}_i = \mathbf{w}_i^0 - \mathbf{w}_i$ and $\bar{\mathbf{w}}_{i+1}$ is defined in the same way. As we focus on the tracking performance of GCMA capable of carrier phase recovering, we assume that $y(i)$ differs from $s(i)$ only by an unknown delay D. Thus, we introduce respectively the *a-priori* and *a-posteriori* estimation errors

$$e_a(i) = s(i-D) - y(i) = \mathbf{u}_i\bar{\mathbf{w}}_i,$$

$$e_p(i) = \mathbf{u}_i(\bar{\mathbf{w}}_{i+1} - \mathbf{q}_i).$$

Then we can rewrite (9) as

$$\bar{\mathbf{w}}_{i+1} = \bar{\mathbf{w}}_i - \bar{\mu}(i)\mathbf{u}_i^*[e_a(i) - e_p(i)] + \mathbf{q}_i \quad (10)$$

where $\bar{\mu}(i) = 1/\|\mathbf{u}_i\|^2$. Assuming the sequences $\mathbf{u}_i$ and $\mathbf{q}_i$ are mutually statistically independent, we have the following important equality in the steady-state [7]

$$E\{\bar{\mu}(i)|e_a(i)|^2\} = \mathrm{tr}(\mathbf{Q}) + E\left\{\bar{\mu}(i)\left|e_a(i) - \frac{\mu}{\bar{\mu}(i)}f_e(i)\right|^2\right\}. \quad (11)$$

The steady-state mean squared error (MSE) is given by $\mathrm{MSE} = \lim_{i\to\infty} E\{|e_a(i)|^2\}$.

To write the following derivation more compactly, we define

$$e_a \equiv e_a(i),\ \bar{\mu} \equiv \bar{\mu}(i),\ y \equiv y(i),\ \mathbf{u} \equiv \mathbf{u}_i,\ s \equiv s(i-D)$$

for $i \to \infty$. To simplify the discussion, let us further assume [7], [8]: (1) the transmitted signal $s(i-D)$ and the estimation error $e_a(i)$ are independent in steady-state so that $E[s^*(i-D)e_a(i)] = 0$, since $s(i-D)$ is assumed zero mean; (2)

the scaled regressor energy $\mu^2\|\mathbf{u}_i\|^2$ is independent of $y(i)$ in steady-state. Then, expanding (11) leads to

$$E[e_a^* f_e]+E[e_a f_e^*] = \mathrm{tr}(\mathbf{Q})/\mu+\mu E\{\|\mathbf{u}\|^2\}E\{|f_e|^2\}. \quad (12)$$

To approximate both sides of (12), we write f_e as a function of y explicitly, say $f_e \equiv f_e(y)$[1]. For $2 \leq \ell \ll \infty$, $|y|_\ell$ is a smooth function of y and so that $f_e(y)$. In this case, we can approximate $f_e(y)$ by utilizing its first order derivative. As $y = s - e_a$, for small error e_a, it is reasonable to approximate $f_e(y)$ as

$$f_e(y) \approx f_e(s) + \mathrm{d}f_e(s) \quad (13)$$

where

$$\mathrm{d}f_e(s) = -\frac{\mathrm{d}f_e(s)}{\mathrm{d}s_R}e_{a,R} - \frac{\mathrm{d}f_e(s)}{\mathrm{d}s_I}e_{a,I}.$$

For $\ell = 1$ or $\ell = \infty$, as $|y|_\ell$ and $f_e(y)$ are not smooth functions of y, approximation (13) no longer holds. We will consider this case separately later. Using (13), for small step-size μ, expression $\mu E\{\|\mathbf{u}\|^2\}E\{|f_e|^2\}$ can be approximated as

$$\mu E\{\|\mathbf{u}\|^2\}E\{|f_e|^2\} \approx \mu E\{\|\mathbf{u}\|^2\}E\{|f_e(s)|^2\}. \quad (14)$$

Supposing the signal constellation is symmetrical, we can write $E\{|f_e(s)|^2\}$ as

$$E\{|f_e(s)|^2\} = 2E\{|s|_\ell^{4-2\ell}|s_R|^{2\ell-2}(R_\ell - |s|_\ell^2)^2\}. \quad (15)$$

On the other hand, expression $E[e_a^* f_e]+E[e_a f_e^*]$ can be evaluated as

$$\begin{aligned} E[e_a^* f_e] + E[e_a f_e^*] &= 2\mathrm{Re}\{E[e_a^* f_e(y)]\} \\ &\approx 2\mathrm{Re}\{E[e_a^* f_e(s)]\} \\ &\quad +2\mathrm{Re}\{E[e_a^* \mathrm{d}f_e(s)]\} \\ &= 2\mathrm{Re}\{E[e_a^* \mathrm{d}f_e(s)]\}. \end{aligned} \quad (16)$$

Supposing the signal constellation satisfies the so-called circularity condition $E[s^2] = 0$ and considering the approximation

$$\begin{aligned} \mathrm{d}|y|_\ell &= |y|_\ell^{1-\ell}(y_R|y_R|^{\ell-2}\mathrm{d}y_R + y_I|y_I|^{\ell-2}\mathrm{d}y_I) \\ &\approx -|s|_\ell^{1-\ell}(s_R|s_R|^{\ell-2}e_{a,R} + s_I|s_I|^{\ell-2}e_{a,I}), \end{aligned}$$

we can evaluate $E[e_a^* f_e]+E[e_a f_e^*]$ after neglecting the higher order terms of e_a through straightforward algebraic calculations as

$$E[e_a^* f_e] + E[e_a f_e^*] \approx A_\ell E\{|e_a|^2\} \quad (17)$$

where

$$\begin{aligned} A_\ell = E\{&4|s|_\ell^{4-2\ell}|s_R|^{2\ell-2} \\ &+(2\ell-4)|s|_\ell^{2-2\ell}|s_R|^{2\ell-2}(R_\ell - |s|_\ell^2) \\ &-(2\ell-2)|s|_\ell^{2-\ell}|s_R|^{\ell-2}(R_\ell - |s|_\ell^2)\}. \end{aligned} \quad (18)$$

[1]We no longer use the notation $f_e(i)$ in the following, where i is the discrete time index.

Note that quantity A_ℓ only depends on the statistics of source $s(i)$. To simplify the notation, we introduce another quantity B_ℓ that also only depends on the statistics of source as

$$B_\ell = E\{|f_e(s)|^2\} = 2E\{|s|_\ell^{4-2\ell}|s_R|^{2\ell-2}(R_\ell - |s|_\ell^2)^2\}. \quad (19)$$

Now, considering (12), (14), (17) and the definition of B_ℓ, we get the MSE of GCMA at steady-state as

$$\mathrm{MSE}_{\mathrm{GCMA},\ell} = E\{|e_a|^2\} = \frac{\mathrm{tr}(\mathbf{Q})/\mu + \mu B_\ell E\{\|\mathbf{u}\|^2\}}{A_\ell}. \quad (20)$$

Choosing $\ell = 2$ in (20), we get the MSE formula of CMA derived in [7].

In the case of $\ell = 1$ or $\ell = \infty$, the formula of A_ℓ and B_ℓ in (18) and (19) should be modified because the basic approximation (13) no longer holds. Noting $|z\mathrm{e}^{\frac{\mathrm{j}\pi}{4}}|_1 = \sqrt{2}|z|_\infty$, GCMAs with generalized complex moduli defined by choosing $\ell = 1$ and $\ell = \infty$ are equivalent in the sense of an immaterial fixed rotation angle ($\mathrm{e}^{\mathrm{j}\frac{\pi}{4}}$). So, we only present the expression of A_∞ and B_∞ here. Noting $J_\infty(\mathbf{w}) = \frac{1}{16}J_{GSCA}(\mathbf{w})$, we slightly modify the results in [5] to get the expressions of A_∞ and B_∞ as

$$A_\infty = 0.5E\{(3s_R^2 - R_\infty)X'\}, \quad (21)$$

$$B_\infty = 2E\{s_R^2(s_R^2 - R_\infty)^2X'\} \quad (22)$$

where $X' = \frac{\mathrm{sgn}(s_R)}{2}[\mathrm{sgn}(s_R + s_I) + \mathrm{sgn}(s_R - s_I)]$.

By observing (20), one can find that in the noiseless case and for non-constant modulus signal, there exists a finite optimal step size that minimizes the steady-state MSE, which gives

$$\mu_\ell^{[\mathrm{opt}]} = \sqrt{\frac{\mathrm{tr}(\mathbf{Q})}{B_\ell E\{\|\mathbf{u}\|^2\}}}, \quad (23)$$

$$\mathrm{MSE}_\ell^{[\mathrm{opt}]} = \frac{2\sqrt{B_\ell \mathrm{tr}(\mathbf{Q})E\{\|\mathbf{u}\|^2\}}}{A_\ell}. \quad (24)$$

We use the quantity

$$\eta_\ell = 10\log_{10}\frac{\mathrm{MSE}_{\mathrm{CMA}}^{[\mathrm{opt}]}}{\mathrm{MSE}_{\mathrm{GCMA},\ell}^{[\mathrm{opt}]}} \quad (25)$$

defined as the ratio of the minimum MSE of CMA at steady-state to the one of GCMA to evaluate the tracking performance of GCMA with different ℓ. Table I summarizes the results for some commonly used signal constellations. From Table I, one observes that neither the CMA nor the GSCA provides the best tracking performance.

4. SIMULATION RESULTS

In this section, we provide the simulation results to compare the experiment performance with the one predicted by the derived expressions. We consider the channel

$$\mathbf{c} = [0.1, 0.3, 1, -0.1, 0.5, 0.2]$$

and an FIR filter with 4 taps as a $\frac{T}{2}$-fractionally spaced equalizer. $\mathbf{q}_i = \mathbf{q}_{i,R} + \mathrm{j}\mathbf{q}_{i,I}$ are modeled as a complex i.i.d. sequence with all the elements of $\mathbf{q}_{i,R}$ and $\mathbf{q}_{i,I}$ drawn from the normal distribution $\mathcal{N}(0, 10^{-3})$ independently. In this way, the autocorrelation matrix $\mathbf{Q} = 2 \times 10^{-6}\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. QAM-16 constellation is considered. Fig. 2 shows the analysis and simulation MSE of ECMA and GSCA with varying step sizes. The value of simulation MSE is obtained through averaging over 100 independent experiments and it is close to the one predicted by (20). The minimum achievable simulation MSE of ECMA is about 1.8dB lower than the one of GSCA.

5. CONCLUSION

This paper analyzes the tracking performance of a family of GCMA capable of carrier phase recovering through a recently proposed energy conservation relation. Simulation results with QAM-16 source validate the theoretical analysis and better tracking performance of ECMA compared with GSCA.

6. REFERENCES

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Table 1. Tracking Performance Index η_ℓ of GCMA

	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = \infty$
QAM-16	1.1dB	1.8dB	2.0dB	1.9dB	-0.7dB
QAM-32	0.2dB	0.0dB	-0.3dB	-0.7dB	-4.7dB
QAM-64	0.9dB	1.4dB	1.5dB	1.5dB	-2.7dB
QAM-128	0.2dB	0.2dB	0.1dB	-0.1dB	-4.7dB

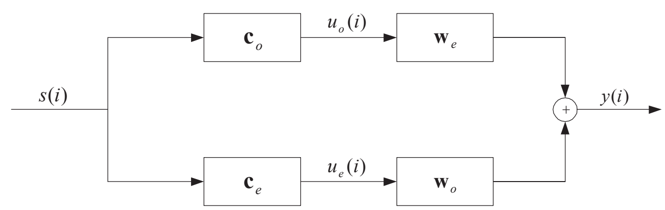

Fig. 1. A multichannel model for $\frac{T}{2}$-fractionally spaced equalization, where i is the discrete time index.

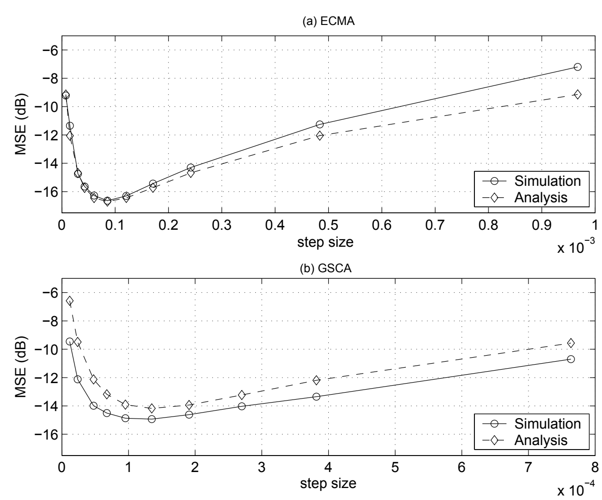

Fig. 2. Analysis and simulation MSE of ECMA and GSCA averaging over 100 experiments. QAM-16 constellation is considered.