OFDM GUARD INTERVAL: ANALYSIS AND OBSERVATIONS

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ABSTRACT

Orthogonal Frequency Division Multiplexing (OFDM) is a modulation technique that is used in many modern digital communication systems and has been explored in numerous publications. Specifically, the tradeoff between throughput and ICI/ISI (Inter Carrier Interference/Inter Symbol Interference), which affects the length of the guard interval (GI) is well known. Many equalization methods that decrease the interference for a given (short) GI have been proposed. However, it seems that the exact expressions for the ICI and ISI have never been derived. In this paper these expressions are derived for a general (Per tone equalization) setup. Often, the matched filter bound (MFB) for single carrier systems is used for OFDM systems [1]. We make two new observations. First, we prove that under mild conditions, met in common scenarios, the ICI power and the ISI power are approximately equal. Second, we show by example that the MFB does not necessarily hold for OFDM. Both cyclic prefix (CP) and zeros prefix (ZP) are examined.

Index Terms— OFDM, ISI, ICI, Per tone equalization, MFB

1. INTRODUCTION

In Section 2 we briefly recall the proof of per tone TEQ and multi tap FEQ equivalence [2]. In Section 3 we derive the exact expressions for ICI and ISI. We show that in common scenarios the ICI power and the ISI power are approximately equal. In section 4 we construct an example that violates the MFB. In [3] an analysis of the ICI/ISI for the CP is provided. The analysis here is different. We also provide an analysis for the ZP GI.

2. SETUP FORMULATION AND DEFINITIONS

2.1. General description

Our framework is a discrete time, deterministic, linear time invariant (LTI), channel with additive *proper* ([4], or *circularly symmetric* [5]) complex stationary Gaussian noise $y_n = z_n + \sum_{m=-\infty}^{\infty} h_{n-m} x_m$, where y_n, z_n, h_n, x_n are the time samples of the received signal (at the receiver input), the noise, the channel and the transmitted signal (at the transmitter output) respectively. The noise has zero mean, an autocorrelation function $R_z[m] = E\{z_n^* z_{n+m}\}$ and spectrum $S_z(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_z[m] e^{-j\omega m}$. The channel is stable in the Bounded In Bounded Out (BIBO) sense. A similar restriction holds for the noise autocorrelation function $\sum_{m=-\infty}^{\infty} |R_z[m]| < \infty$. Denote the channel DTFT $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_n e^{-j\omega n}$.

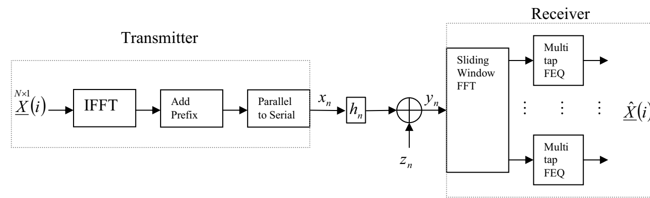

Fig. 1. OFDM transmitter, receiver and channel model.

Figure 1 depicts the OFDM transmitter, receiver and channel. The size of the OFDM symbol $\underline{X}(i)$ is N. A Prefix of length υ is appended to each symbol (either a CP or a ZP). The index i denotes time, *i.e.* the symbol $\underline{X}(i)$ determines (IFFT+prefix) the samples $\{x_n\}_{n=-\upsilon+i\cdot(N+\upsilon)}^{N-1+i\cdot(N+\upsilon)}$. At the receiver a "sliding FFT" (a scheme proposed in [2]) is applied to the received samples. This process may discard samples if a CP is being used. Then each frequency (bin) undergoes a multi tap Frequency Equalizer (FEQ) to produce the estimate $\hat{\underline{X}}(i)$. Van Acker et al [2] proved the equivalence of a Time Equalizer (TEQ) to this sliding FFT multi tap FEQ scheme. We repeat this proof and explain the equivalence. This multi tap FEQ scheme (compared to the standard TEQ scheme) permits a more flexible equalization (with the same number of multiplications per symbol!) due to the fact that each bin is equalized with a different equalizer. In [2] it is also shown how the sliding FFT actually costs only "one FFT" (We show it here as well).

2.2. Transmitter

Let $\underline{X}(i) = \begin{bmatrix} X_0(i) & \cdots & X_{N-1}(i) \end{bmatrix}^T$ be the OFDM symbol in frequency domain (i is the symbol index). Some of it's components may not be used and the rest carry the data each with power P. The bins are uncorrelated (across frequency and time) and their expectation is zero. $\underline{X}(i)$ is *circularly symmetric* ([5]). In summary, $E\{\underline{X}(i)\} = \underline{0}$, $E\{\underline{X}(i)\underline{X}(j)^*\} = K_{\underline{X}}\delta_{i-j}$ and $K_{\underline{X}}$ is diagonal. Some of the diagonal components are P and some are zero.
The Transmitter's output is:

$$x_n = \sum_{i=-\infty}^{\infty} x_n(i) \cdot \begin{cases} \sqrt{\frac{N+\upsilon}{N}}\,\mathrm{I}(0 \le n-(N+\upsilon)i < N) & \text{if } ZP \\ \mathrm{I}(-\upsilon \le n-(N+\upsilon)i < N) & \text{if } CP \end{cases} \tag{1}$$

where $\mathrm{I}(\bullet)$ is the Indicator function:

$$\mathrm{I}(term) = \begin{cases} 0 \text{ if } term = \text{false} \\ 1 \text{ if } term = \text{true} \end{cases}$$

and

$$x_n(i) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k(i)\, e^{j\frac{2\pi}{N}k(n-(N+\upsilon)i)} \tag{2}$$

The "fairness" factor $\sqrt{\frac{N+\upsilon}{N}}$ normalizes the power so that the average transmitter output power is equal in both the ZP and CP cases.

2.3. Receiver

Define for brevity $y_n(i) \triangleq y_{n+d+(N+\upsilon)i}$, where d is some channel dependent delay (FFT placement). The sliding FFT produces the following output:

$$\underset{k=0\ldots N-1,l=0\ldots L}{Y_{k,l}(i)} = \begin{cases} \frac{1}{\sqrt{N}} \sum_{n=0}^{N+\upsilon-1} y_{n-l}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } ZP \\ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_{n-l}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } CP \end{cases} \tag{3}$$

Note that for the ZP case, this operation is actually done by appending the last υ samples to the first υ samples and then executing an ordinary FFT. Also, this sliding FFT cost (in terms of number of multiplications) is almost equal to the cost of executing one FFT since we can use the recursive formula:

$$Y_{k,l-1}(i) = e^{j\frac{2\pi}{N}k} \cdot \begin{cases} Y_{k,l}(i) + \frac{1}{\sqrt{N}}\left[y_{N+\upsilon-l}(i)\, e^{-j\frac{2\pi}{N}k\upsilon} - y_{-l}(i)\right] & \text{if } ZP \\ Y_{k,l}(i) + \frac{1}{\sqrt{N}}\left[y_{N-l}(i) - y_{-l}(i)\right] & \text{if } CP \end{cases} \tag{4}$$

Let $\underline{Y}_k(i) \triangleq \begin{bmatrix} Y_{k,0}(i) & \cdots & Y_{k,L}(i) \end{bmatrix}$. The FEQ operation is:

$$\hat{X}_k(i) = \underline{Y}_k(i) \cdot \underline{w}_k \qquad k = 0 \ldots N-1 \tag{5}$$

where $\underline{w}_k = \begin{bmatrix} w_{k,0} & \cdots & w_{k,L} \end{bmatrix}^T$ is the FEQ.

2.4. Per bin TEQ, multi tap FEQ equivalence

We now want to show that the above receiver (multi tap FEQ) is equivalent to using a different TEQ for every bin. Let us first define this alternative scheme. The output of this bin dependent TEQ:

$$f_{k,n} \triangleq y_n * w_{k,n} = \sum_{l=0}^{L} w_{k,l} y_{n-l} \tag{6}$$

that is, the receiver takes a single stream of samples, insert them into different filters to create a stream for each bin. Then each stream is transformed to the frequency it belongs to. As we defined $y_n(i)$, we similarly define $f_{k,n}(i) \triangleq f_{k,n+d+(N+\upsilon)i}$ which means $f_{k,n}(i) = \sum_{l=0}^{L} w_{k,l} y_{n-l}(i)$. Now we transform each stream to the appropriate frequency:

$$F_k(i) = \begin{cases} \frac{1}{\sqrt{N}} \sum_{n=0}^{N+\upsilon-1} f_{k,n}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } ZP \\ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_{k,n}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } CP \end{cases} \tag{7}$$

We need to prove that $F_k(i) = \underline{Y}_k(i) \cdot \underline{w}_k$, which means that the receivers' output is identical (i.e. the schemes are equivalent). Proof:

$$\begin{aligned} F_k(i) &= \begin{cases} \frac{1}{\sqrt{N}} \sum_{n=0}^{N+\upsilon-1} f_{k,n}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } ZP \\ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_{k,n}(i)\, e^{-j\frac{2\pi}{N}kn} & \text{if } CP \end{cases} = \\ &= \begin{cases} \frac{1}{\sqrt{N}} \sum_{n=0}^{N+\upsilon-1} \sum_{l=0}^{L} w_{k,l} y_{n-l}(i) e^{-j\frac{2\pi}{N}kn} & \text{if } ZP \\ \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{l=0}^{L} w_{k,l} y_{n-l}(i) e^{-j\frac{2\pi}{N}kn} & \text{if } CP \end{cases} = \\ &= \begin{cases} \sum_{l=0}^{L} w_{k,l} \frac{1}{\sqrt{N}} \sum_{n=0}^{N+\upsilon-1} y_{n-l}(i) e^{-j\frac{2\pi}{N}kn} & \text{if } ZP \\ \sum_{l=0}^{L} w_{k,l} \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} y_{n-l}(i) e^{-j\frac{2\pi}{N}kn} & \text{if } CP \end{cases} = \\ &= \sum_{l=0}^{L} w_{k,l} Y_{k,l}(i) = \underline{Y}_k(i) \cdot \underline{w}_k \end{aligned}$$

Note the equivalence is independent of what is transmitted (OFDM or other modulation); samples that go through both schemes would give exactly the same output. Equalizing in the frequency domain (in comparison to per tone TEQ) is more efficient, because it takes advantage of the FFT implementation efficiency.

3. ICI AND ISI

3.1. Receiver's output for a single transmitted symbol

In this subsection we write (the derivations were left out for brevity) the receiver's output for a single transmitted symbol, that is, a single symbol is transmitted and in the rest of the time the transmitter is silent. Obviously, $\underline{Y}_k(i)$ is a linear combination of $\underline{X}(i)$ and of the additive noise samples which we ignore (as if they were equal to zero) in this section. It turns out that:

$$\underline{Y}_k(i) = \left[X_k(i)\, \underline{H}_k^{signal} + \sum_{\substack{m=0 \\ m \neq k}}^{N-1} X_m(i)\, \underline{H}_{k,m} \right] \cdot \begin{cases} \sqrt{\frac{N+\upsilon}{N}} & \text{if ZP} \\ 1 & \text{if CP} \end{cases} \tag{8}$$

where:

$$\begin{aligned} \underset{m\neq k}{\underline{H}_{k,m}} &= c(k-m) \begin{pmatrix} \underline{H}_k^{before} - \underline{H}_m^{before} - \\ -\left(\underline{H}_k^{after} - \underline{H}_m^{after}\right) \begin{cases} e^{-j\frac{2\pi}{N}k\upsilon} & \text{if ZP} \\ e^{-j\frac{2\pi}{N}m\upsilon} & \text{if CP} \end{cases} \end{pmatrix} \\ \underline{H}_k^{before} &= \begin{bmatrix} H_k^{(before,0)} & \cdots & H_k^{(before,L)} \end{bmatrix} \\ \underline{H}_k^{signal} &= \begin{bmatrix} H_k^{(signal,0)} & \cdots & H_k^{(signal,L)} \end{bmatrix} \\ \underline{H}_k^{after} &= \begin{bmatrix} H_k^{(after,0)} & \cdots & H_k^{(after,L)} \end{bmatrix} \\ c(m) &= \frac{je^{j\frac{\pi}{N}m}}{2N\sin\left(\frac{\pi}{N}m\right)} \end{aligned} \tag{9}$$

and:

$$\begin{aligned} H_k^{(signal,l)} &= \sum_{n=0}^{\upsilon} h_{n+d-l} e^{-j\frac{2\pi}{N}kn} + \\ &+ \sum_{n=1}^{N-1} \left(h_{n-N+d-l} \tfrac{n}{N} + h_{n+\upsilon+d-l} \tfrac{N-n}{N} e^{-j\frac{2\pi}{N}k\upsilon} \right) e^{-j\frac{2\pi}{N}kn} \\ H_k^{(before,l)} &= \sum_{n=1}^{N-1} h_{n-N+d-l} e^{-j\frac{2\pi}{N}kn} \\ H_k^{(after,l)} &= \sum_{n=1}^{N-1} h_{n+\upsilon+d-l} e^{-j\frac{2\pi}{N}kn} \end{aligned} \tag{10}$$

Note that the difference between ZP and CP is minor (equation 9). Also, the term $|c(k-m)|$ indicates that the closer the bin, the stronger it's interference. In fact, this "distance" is a "cyclic distance", explained via an example: for $N=256, k=20, m_1=49, m_2=248$, we have $|c(k-m_2)| > |c(k-m_1)|$. For $N=256$, figure 2 depicts $|c(k-m)|$.

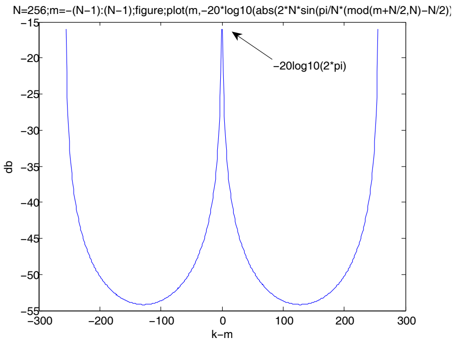

Fig. 2. $|c(k-m)|$. The closer the bin, the stronger it interferes.

We are now interested in the effect of $\underline{X}(i)$ on $\underline{Y}_k(i+r)$ for some integer r. This will represent ISI (or inter block interference). We can reuse equation (8) by simply replacing d with $d+(N+v)r$ (when calculating 10):

$$\underline{Y}_k(i+r) = \begin{bmatrix} X_k(i)\,\underline{H}_k^{signal}\Big|_{d\leftarrow d+(N+v)r} + \\ \sum_{\substack{m=0\\ m\neq k}}^{N-1} X_m(i)\,\underline{H}_{k,m}\Big|_{d\leftarrow d+(N+v)r} \end{bmatrix} \begin{cases} \sqrt{\frac{N+v}{N}} & \text{if ZP} \\ 1 & \text{if CP} \end{cases} \tag{11}$$

It can now be easily observed that taking $L=0$ (regular FEQ), if h_n exists only for $n=d...d+v$, there is zero ICI and zero ISI.

3.2. ISI from adjacent symbols

Up until now, we haven't assumed any restriction on the length (and causality) of the channel. In this subsection and the subsequent subsection we assume that the channel h_n exists only at time samples $-(N+v)+d \le n \le N+2v+d-L$. This assumption is reasonable for realistic OFDM systems. This guarantees that each OFDM symbol interferes only with it's adjacent symbols:

$$\underline{Y}_k(i) = \begin{bmatrix} X_k(i)\,\underline{H}_k^{signal} + \\ \underline{ISI}_{-1}^{k\rightarrow k} + \underline{ISI}_{1}^{k\rightarrow k} + \\ \underline{ICI}^{k} + \underline{ISI}_{1}^{others\rightarrow k} + \\ \underline{ISI}_{-1}^{others\rightarrow k} \end{bmatrix} \begin{cases} \sqrt{\frac{N+v}{N}} & \text{if ZP} \\ 1 & \text{if CP} \end{cases} \tag{12}$$

where:

$$\begin{aligned} \underline{ISI}_{-1}^{k\rightarrow k} &\triangleq X_k(i-1)\,\underline{H}_k^{signal}\Big|_{d\leftarrow d+(N+v)} \\ \underline{ISI}_{1}^{k\rightarrow k} &\triangleq X_k(i+1)\,\underline{H}_k^{signal}\Big|_{d\leftarrow d-(N+v)} \\ \underline{ISI}_{-1}^{others\rightarrow k} &\triangleq \sum_{\substack{m=0\\ m\neq k}}^{N-1} X_m(i-1)\,\underline{H}_{k,m}\Big|_{d\leftarrow d+(N+v)} \\ \underline{ISI}_{1}^{others\rightarrow k} &\triangleq \sum_{\substack{m=0\\ m\neq k}}^{N-1} X_m(i+1)\,\underline{H}_{k,m}\Big|_{d\leftarrow d-(N+v)} \\ \underline{ICI}^{k} &\triangleq \sum_{\substack{m=0\\ m\neq k}}^{N-1} X_m(i)\,\underline{H}_{k,m} \end{aligned} \tag{13}$$

Observe (equations 9 and 10) that:

$$\underline{H}_k^{after}\Big|_{d\leftarrow d+(N+v)r} = \underline{H}_k^{before}\Big|_{d\leftarrow d+(N+v)(r+1)} \tag{14}$$

Also, due to our time restriction:

$$\underline{H}_k^{after}\Big|_{d\leftarrow d+(N+v)} = \underline{H}_k^{before}\Big|_{d\leftarrow d-(N+v)} = \underline{0} \tag{15}$$

Using equations (9,14,15) we derive:

$$\begin{aligned} \underline{H}_{\substack{k,m\\ m\neq k}}\Big|_{d\leftarrow d+(N+v)} &= c(k-m)\cdot\left(\underline{H}_k^{after} - \underline{H}_m^{after}\right) \\ \underline{H}_{\substack{k,m\\ m\neq k}}\Big|_{d\leftarrow d-(N+v)} &= -c(k-m)\cdot\left(\underline{H}_k^{before} - \underline{H}_m^{before}\right)\cdot \\ &\quad \cdot\begin{cases} e^{-j\frac{2\pi}{N}kv} & \text{if ZP} \\ e^{-j\frac{2\pi}{N}mv} & \text{if CP} \end{cases} \end{aligned} \tag{16}$$

So the transmission factors of $\underline{ICI}^k$ are actually the sum of the respective transmission factors of $\underline{ISI}_1^{others\rightarrow k}$ and $\underline{ISI}_{-1}^{others\rightarrow k}$:

$$\begin{aligned} \underline{H}_{\substack{k,m\\ m\neq k}} = &- \underline{H}_{\substack{k,m\\ m\neq k}}\Big|_{d\leftarrow d-(N+v)} \cdot \begin{cases} e^{j\frac{2\pi}{N}kv} & \text{if ZP} \\ e^{j\frac{2\pi}{N}mv} & \text{if CP} \end{cases} \\ &- \underline{H}_{\substack{k,m\\ m\neq k}}\Big|_{d\leftarrow d+(N+v)} \cdot \begin{cases} e^{-j\frac{2\pi}{N}kv} & \text{if ZP} \\ e^{-j\frac{2\pi}{N}mv} & \text{if CP} \end{cases} \end{aligned} \tag{17}$$

3.3. ISI and ICI have approximately the same power in "most" scenarios

The main idea in OFDM is the "orthogonality" of bins, that is, when equalized properly, the OFDM scheme provides (almost) independent virtual channels (each bin is a channel), i.e. realistic OFDM systems work with channels for which most of the energy (if not before then after the equalizer) is within the GI. If this isn't the case, one should consider extending the GI or using a different modulation scheme. Bottom line: for a proper OFDM system, the ICI/ISI is at least 20-30db below the signal. Further, in most cases the energy outside the GI is concentrated after the GI rather than before it, because of decaying poles. For example, figure 3 depicts an impulse response (provided by G.Arslan, Texas university). For a GI of 32 samples, the samples inside the GI (32+1 samples) contain 0.9384 of the energy, the samples after the GI contain 0.0615 of the energy and the samples before the GI contain less than 1e-4 of the energy. For these cases the first term of equation (16) is much "stronger" than the second term and we can approximate (17):

$$\underline{H}_{\substack{k,m\\ m\neq k}} \approx - \underline{H}_{\substack{k,m\\ m\neq k}}\Big|_{d\leftarrow d+(N+v)} \cdot \begin{cases} e^{-j\frac{2\pi}{N}kv} & \text{if ZP} \\ e^{-j\frac{2\pi}{N}mv} & \text{if CP} \end{cases} \tag{18}$$

Consequently:

$$E\left\{\left|\underline{ICI}^k\cdot\underline{w}_k\right|^2\right\} \approx E\left\{\left|\underline{ISI}_{-1}^{others\rightarrow k}\cdot\underline{w}_k\right|^2\right\} \tag{19}$$

We now wish to make a few comments about the other ISI terms in (12). We have already implicitly indicated that $E\left\{\left|\underline{ISI}_1^{others\rightarrow k}\cdot\underline{w}_k\right|^2\right\}$

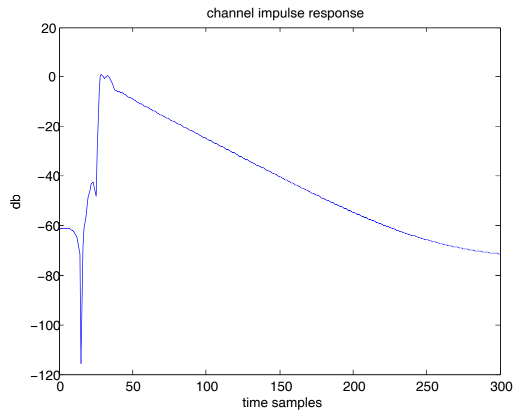

Fig. 3. channel impulse response. the ratio between the energy of samples after the GI and the energy of samples before it is 28db.

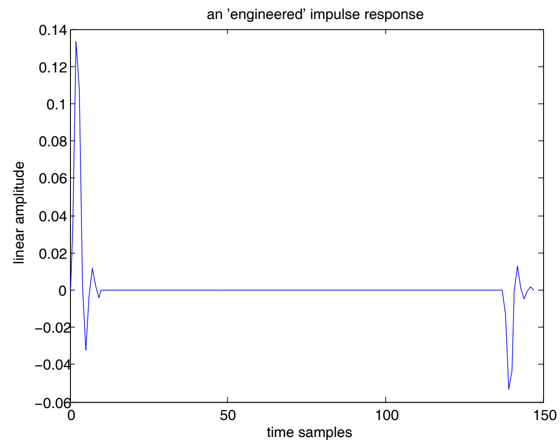

Fig. 4. A channel that causes only the interference $ISI_{-1}^{k\to k}$

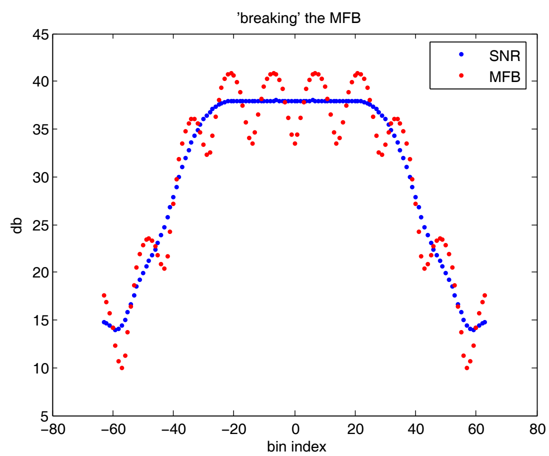

Fig. 5. MFB and SNR

is negligible compared with the terms of (19) and $E\left\{\left|\underline{ISI}_{1}^{k\to k}\cdot \underline{w}_k\right|^2\right\}$ is even more negligible (because it constitutes one interferer as opposed to many interferers). As for $E\left\{\left|\underline{ISI}_{-1}^{k\to k}\cdot \underline{w}_k\right|^2\right\}$ we can propose a receiver scheme (almost without increasing the complexity) that utilizes per tone decision feedback which completely cancels this term (assuming zero error propagation).

4. VIOLATING THE MFB

The MFB adapted to OFDM is an upper bound on per bin SNR. The rational behind it is that the "best" scenario for OFDM is when the channel is completely confined to the GI. In that case there is no ICI and no ISI and the bound is achieved. It is defined as:

$$MFB_k = \frac{P\left|H_k\right|^2}{S_k} \tag{20}$$

where:

$$\begin{aligned} S_k &= S_z\left(e^{j\frac{2\pi}{N}k}\right) \\ H_k &= H\left(e^{j\frac{2\pi}{N}k}\right) \end{aligned} \tag{21}$$

According to the insights obtained in the previous section, we can identify channels that violate the MFB for some bins. Specifically, consider a channel that causes only the interference $ISI_{-1}^{k\to k}$. We use the per bin decision feedback scheme proposed above to completely cancel this interference at the receiver. Figure 4 depicts this impulse response (system parameters: $N = 128, \upsilon = 9, d = 0, L = 0$). Figure 5 depicts this channel MFB and achieved SNR for white noise (both the ZP and CP cases have equal SNR for this scenario). We are not claiming that the MFB can be easily violated in realistic scenarios, nor do we challenge it's importance as a good tool for assessing realistic systems' performance. We only claim that the MFB was analytically proven to be a bound for single carrier modulation and not for OFDM. One may claim that the decision feedback scheme is not within the common scope of OFDM, but the MFB can be violated even without decision feedback.

5. CONCLUSIONS

We have provided exact expressions for ISI and ICI in OFDM. We used these expressions to show that for common scenarios the ICI power and the ISI power are approximately equal. We identified an example that violates the MFB for OFDM.

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