WHAT DETERMINES THE DIVERSITY ORDER OF LINEAR EQUALIZERS?

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ABSTRACT

Diversity techniques are important in combating the deleterious effects of channel fading. To enjoy diversity, both transmitter and receiver have to be designed appropriately. Many existing designs have exploited different flavors of diversity by employing maximum likelihood (ML) or near-ML detectors at the receiver, which is of high complexity. On the other hand, empirical results have shown that linear equalizers (LEs) offer inferior performance but come with low complexity. Unfortunately, the diversity collected by general LEs has not been quantified, which reduces the attraction of LEs in theoretical and practical aspects. In this paper, we reveal a fundamental yet simple condition that determines whether LEs can achieve the same diversity order as an ML equalizer does. The condition is based on the orthogonal deficiency (od) of channel matrix. Based on the distribution of the channel od, we propose a framework of hybrid equalizers which guarantees maximum diversity while trading off complexity for coding gains. The theoretical findings are verified by simulation results.

Index Terms— Diversity, fading channels, linear systems, linear equalizers, orthogonality deficiency

1. INTRODUCTION

A major challenge in designing wireless systems is to mitigate the fading propagation effects of the wireless channels given the prescribed power, bandwidth, and complexity constraints. To cope with the deleterious fading effects on the system performance, diversity-enriched transmitters and receivers have well-appreciated merits [9]. The higher the collected diversity is, the smaller the error probability is at high signal-to-noise ratio (SNR). Diversity itself is an inherent property from fading channels. Different channels provide different diversity flavors, e.g., frequency-selective channels provide multipath diversity [4], multi-antenna channels provide spatial diversity [7].

To enjoy the diversity from fading channels, two conditions are necessary. One condition is that we have to design the transmitter properly so that the diversity is enabled. This has been well studied in the literature. For example, the performance of uncoded OFDM does not have any diversity, but precoded OFDM enables multipath diversity [4]. Another condition is that the receiver has to be able to collect the diversity. Most of the existing diversity-enabled schemes require maximum-likelihood equalizers (MLEs) or near-ML decoders at the receiver to collect diversity [4]. Recently, lattice-reduction aided LEs have been proposed which can also collect full diversity [6, 11].

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Decoding complexity is another major figure of merit to evaluate a receiver. Although MLE enjoys the maximum diversity, its exponential decoding complexity makes itself infeasible for certain practical systems. Some near-ML schemes (e.g., sphere-decoding [3]) have been used to reduce the decoding complexity. However, at low SNR or when large decoding blocks and/or high signal constellations are employed, the complexity of near-ML schemes is still high. To further reduce the complexity, when the system model is linear, one may apply linear equalizers (LEs). Although it is well-known that LEs have inferior performance relative to that of (near-) MLE, unlike MLE, the diversity order of LEs has not been quantified in general. For specific cases, it has been shown that LEs cannot collect maximum diversity for MIMO V-BLAST systems [11]. It has also been shown that LEs collect full diversity for precoded OFDM systems in [8] and orthogonal space-time block coding schemes in [7]. Therefore, a fundamental question is what determines the diversity order of LEs if they exist.

In this paper, by introducing a metric, orthogonality deficiency (od) of the channel matrix, we reveal the fundamental condition with which LEs collect the same diversity as MLE does. Based on our analysis on the distribution of the channel od, we propose a framework to design hybrid equalizers with maximum diversity while also enabling coding gain and complexity trade-off.

2. SYSTEM MODEL AND PROBLEM STATEMENT

Consider general linear block transmissions

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{w}, \tag{1}$$

where $\boldsymbol{H}$ is an $M \times N$ complex random matrix with zero mean, the $N \times 1$ vector $\boldsymbol{s}$ consists of the information symbols, $\boldsymbol{y}$ is the $M \times 1$ received vector, and $\boldsymbol{w}$ is complex additive white Gaussian noise with variance σ_w^2. We assume a quasi-static environment that the channel coefficients are time-invariant during a certain block which is greater than a symbol period and change independently from block to block, and they are known to the receiver. Note that the channel matrix $\boldsymbol{H}$ is general enough to represent a number of cases, e.g., multi-antenna MIMO [7], precoded OFDM [4], and single-carrier Toeplitz channels [8]. Given the model in (1), there are various ways to decode $\boldsymbol{s}$ from the observation $\boldsymbol{y}$. Here, we briefly summarize two conventional LEs.

The zero-forcing (ZF) detector for the model in (1) is given as

$$\boldsymbol{x}_{zf} = \boldsymbol{H}^{\dagger}\boldsymbol{y} = \boldsymbol{s} + \boldsymbol{H}^{\dagger}\boldsymbol{w} = \boldsymbol{s} + \boldsymbol{\eta}, \tag{2}$$

where $\boldsymbol{H}^{\dagger} = (\boldsymbol{H}^{\mathcal{H}}\boldsymbol{H})^{-1}\boldsymbol{H}^{\mathcal{H}}$ denotes the Moore-Penrose pseudo-inverse of $\boldsymbol{H}$ and $\boldsymbol{\eta} := \boldsymbol{H}^{\dagger}\boldsymbol{w}$. A quantization step is then used to map each entry of $\boldsymbol{x}_{zf}$ to the symbol alphabet $\mathcal{S}$.

Another often used LE is the linear minimum mean-square error (MMSE) detector, which is given as

$$\boldsymbol{x}_{mmse} = \left(\boldsymbol{H}^{\mathcal{H}}\boldsymbol{H} + \sigma_w^2\boldsymbol{I}_N\right)^{-1}\boldsymbol{H}^{\mathcal{H}}\boldsymbol{y}. \tag{3}$$

Here, we notice that, with the definition of an extended system: $\bar{\boldsymbol{H}} = [\boldsymbol{H}^T \ \sigma_w\boldsymbol{I}_N]^T$, $\bar{\boldsymbol{y}} = [\boldsymbol{y}^T \ \boldsymbol{0}_{1\times N}]^T$; the MMSE equalizer in

(3) can be rewritten as $\boldsymbol{x}_{mmse} = \bar{\boldsymbol{H}}^{\dagger}\bar{\boldsymbol{y}}$. Therefore, the MMSE equalizer has the same form as the ZF equalizer in (2) with respect to this extended system. The analysis in the following sections is based on the ZF equalizer in (2), and thus can be extended to MMSE equalizer accordingly.

It is well-known that LEs come with low decoding complexity relative to those ML or near-MLEs (e.g. [3]) which use LEs as their initial or preprocessing steps. In practice, when complexity is of high priority, LEs are preferred. However, one major reason that hinders LEs to get more attention in theory and practice is that their performance loss is not quantified. In the following we remove this obstacle by revealing the diversity of LEs.

3. THE DIVERSITY OF LINEAR EQUALIZERS

Let us first define the orthogonality deficiency (od) of an $M \times N$ matrix $\boldsymbol{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_N]$ as in [6]:

$$od(\boldsymbol{H}) = 1 - \frac{\det(\boldsymbol{H}^{\mathcal{H}}\boldsymbol{H})}{\prod_{n=1}^{N} \|\boldsymbol{h}_n\|^2}, \tag{4}$$

where $\|\boldsymbol{h}_n\|, 1 \le n \le N$ is the norm of the nth column of $\boldsymbol{H}$. Note that $0 \le od(\boldsymbol{H}) \le 1, \forall \boldsymbol{H}$. If $\boldsymbol{H}$ is singular, $od(\boldsymbol{H}) = 1$, and if the columns of $\boldsymbol{H}$ are orthogonal, $od(\boldsymbol{H}) = 0$. The smaller $od(\boldsymbol{H})$ is, the more orthogonal $\boldsymbol{H}$ is. Given the model in (1), if $od(\boldsymbol{H}) = 0$, i.e., $\boldsymbol{H}$ is orthogonal, then LEs have the same performance as that of MLE. Interestingly, we find that $od(\boldsymbol{H})$ is directly related to the performance of LEs.

With the general system model in (1) and the definition of od in (4), we establish the following result regarding the diversity order of the ZF equalizers.

Theorem 1 *Consider a linear system in* (1)*, where the entries of the channel matrix are complex Gaussian distributed with zero mean, and the information symbols are drawn from integer lattice (Gaussian integer ring). The linear ZF equalizer in* (2) *collects the same diversity as MLE does if there exists a constant $\epsilon \in (0,1)$ such that $\forall \boldsymbol{H}$, $od(\boldsymbol{H}) \le \epsilon$.*

Proof: Let us define $\boldsymbol{H}^{\dagger} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_N]^T$, where $\boldsymbol{a}_n^T, n \in [1, N]$ is the ith row of $\boldsymbol{H}^{\dagger}$. Hence, if $|\boldsymbol{a}_n^T\boldsymbol{w}|$ is less than $\frac{1}{2}$, that means both the real and imaginary parts of η_n are less than $\frac{1}{2}$, we will definitely decode the nth symbol of $\boldsymbol{s}$ correctly, since all the entries of $\boldsymbol{s}$ belong to the Gaussian integer ring. Thus, the error probability of the nth symbol for a given $\boldsymbol{H}$, $P_{e,n|H}$ is upper-bounded by

$$P_{e,n|H} \le P\left(|\boldsymbol{a}_n^T\boldsymbol{w}| \ge \tfrac{1}{2} \big| \boldsymbol{H}\right). \tag{5}$$

From [6, Lemma 1], we obtain the following inequality:

$$|\boldsymbol{a}_n^T\boldsymbol{w}| \le \|\boldsymbol{a}_n^T\| \cdot \|\boldsymbol{w}\| \le \frac{1}{\sqrt{1 - od(\boldsymbol{H})} \cdot \|\boldsymbol{h}_n\|}\|\boldsymbol{w}\|, \tag{6}$$

where $\boldsymbol{h}_n, n \in [1, N]$ represents the nth column of $\boldsymbol{H}$. Furthermore, as we have assumed, if there exists $\epsilon \in (0,1)$ such that $\forall \boldsymbol{H}$, $od(\boldsymbol{H}) \le \epsilon$, we obtain the following inequality:

$$P_{e,n|H} \le P\left(\|\boldsymbol{w}\| \ge \tfrac{\sqrt{1-\epsilon}}{2}\|\boldsymbol{h}_n\| \Big| \boldsymbol{H}\right). \tag{7}$$

Here, we notice that $\epsilon \in (0,1)$ is a constant independent from $\boldsymbol{H}$. By averaging (7) with respect to the random matrix $\boldsymbol{H}$, the average error probability can be further simplified as

$$\begin{aligned} P_{e,n} = E_H\left[P_{e,n|H}\right] &\le E_H\left[P\left(\|\boldsymbol{w}\|^2 \ge \frac{1-\epsilon}{4}\|\boldsymbol{h}_n\|^2 \Big| \boldsymbol{H}\right)\right] \\ &= E_w\left[P\left(\|\boldsymbol{h}_n\|^2 \le \frac{4\|\boldsymbol{w}\|^2}{1-\epsilon} \Big| \boldsymbol{w}\right)\right], \end{aligned} \tag{8}$$

where $E[\cdot]$ denotes the expectation.

Suppose that the rank of the covariance matrix $\boldsymbol{R} = E[\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}]$ is D_n and $D_n \le M$. Using the eigenvalue decomposition, we have $\boldsymbol{R} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathcal{H}}$, where $\boldsymbol{U}$ is an $M \times D_n$ unitary matrix and $\boldsymbol{\Lambda}$ is a $D_n \times D_n$ diagonal matrix. Define $\tilde{\boldsymbol{h}}_n$ as a $D_n \times 1$ vector, whose entries are independent Gaussian random variables with zero mean and variance $\sigma_{n,d}^2$. Since $\boldsymbol{h}_n$ has identical distribution with $\boldsymbol{U}\tilde{\boldsymbol{h}}_n$, for any $\beta > 0$ we have:

$$P\left(\|\boldsymbol{h}_n\|^2 \le \beta\right) = P\left(\|\tilde{\boldsymbol{h}}_n\|^2 \le \beta\right) \le \prod_{d=1}^{D_n} P(|\tilde{h}_{n,d}|^2 \le \beta), \tag{9}$$

where $\tilde{h}_{n,d}$ is the dth entry of $\tilde{\boldsymbol{h}}_n$. Because $\tilde{h}_{n,d}$ is Gaussian distributed, we have

$$P\left(\frac{2|\tilde{h}_{n,d}|^2}{\sigma_{n,d}^2} \le \frac{2\beta}{\sigma_{n,d}^2}\right) = e^{-\gamma_{n,d}} \sum_{k=1}^{\infty} \frac{(\gamma_{n,d})^{k-1}}{k!}\gamma_{n,d} \le \frac{\beta}{\sigma_{n,d}^2}, \tag{10}$$

where $\gamma_{n,d} = \frac{\beta}{\sigma_{n,d}^2}$. Plugging (10) into (9), we obtain

$$P\left(\|\boldsymbol{h}_n\|^2 \le \beta\right) \le c_n\beta^{D_n}, \tag{11}$$

where $c_n = \prod_{d=1}^{D_n} \frac{1}{\sigma_{n,d}^2}$.

Consequently, the error probability in (8) is bounded as

$$\begin{aligned} P_{e,n} &\le E_w\left[P\left(\|\boldsymbol{h}_n\|^2 \le \frac{4\|\boldsymbol{w}\|^2}{1-\epsilon} \Big| \boldsymbol{w}\right)\right] \\ &= c_n\left(\frac{4}{1-\epsilon}\right)^{D_n} \frac{(2D_n-1)!}{(D_n-1)!}\left(\frac{1}{\sigma_w^2}\right)^{-D_n}. \end{aligned} \tag{12}$$

Therefore, the diversity order of the ZF equalizer for the nth symbol is greater than or equal to $D_n = \text{rank}(E[\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}])$, if there exists $\epsilon \in (0,1)$ such that $\forall \boldsymbol{H}$, $od(\boldsymbol{H}) \le \epsilon$. In general, for the system in (1), the diversity order of the ZF equalizer is greater than or equal to $\min_n\{\text{rank}(E[\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}])\}$.

Now, let us revisit the diversity order of MLE. According to [9, p. 66], we know that for MLE, the pairwise error probability for an error pattern $\boldsymbol{e} = \boldsymbol{s} - \boldsymbol{s}'$ ($\boldsymbol{s} \ne \boldsymbol{s}'$) is bounded as

$$P(\boldsymbol{s} \to \boldsymbol{s}' \mid \boldsymbol{H}) \le \exp\left(-\frac{\|\boldsymbol{H}\boldsymbol{e}\|^2}{4\sigma_w^2}\right) = \exp\left(-\frac{\|\boldsymbol{h}_e\|^2}{4\sigma_w^2}\right), \tag{13}$$

where $\boldsymbol{h}_e = \boldsymbol{H}\boldsymbol{e}$, which is a linear combination of $\boldsymbol{h}_n$'s, $n \in [1, N]$, with coefficients drawn from Gaussian integer ring. Furthermore, by averaging (13) with respect to $\boldsymbol{H}$, we obtain the error probability as

$$P(\boldsymbol{s} \to \boldsymbol{s}') \le C_e\left(\frac{1}{\sigma_w^2}\right)^{-G_{d,e}}, \tag{14}$$

where C_e is a finite constant and $G_{d,e} = \text{rank}(E[\boldsymbol{h}_e\boldsymbol{h}_e^{\mathcal{H}}])$. Thus, the diversity order that MLE can collect is

$$G_d^{ML} = \min_{e \ne 0} G_{d,e} = \min_{e \ne 0} \text{rank}(E[\boldsymbol{h}_e\boldsymbol{h}_e^{\mathcal{H}}]). \tag{15}$$

Compared with the diversity of the ZF equalizer, it is ready to obtain

$$G_d^{ML} = \min_{e \ne 0} \text{rank}(E[\boldsymbol{h}_e\boldsymbol{h}_e^{\mathcal{H}}]) \le \min_n \text{rank}(E[\boldsymbol{h}_n\boldsymbol{h}_n^{\mathcal{H}}]) \le G_d^{ZF}. \tag{16}$$

Thus, we conclude that, if $od(\boldsymbol{H}) < \epsilon$ and $\epsilon < 1$, the ZF equalizer collects the same diversity as that exploited by MLE. ■

The same claim can be made for the MMSE equalizer. Furthermore, we can extend Theorem 1 for any channel distribution as:

Table 1. Distributions of the od of i.i.d. channel matrix $\boldsymbol{H}$

n	Exact Distribution $f(1-x)$	Approximate $\beta(a,b)$
2	1	(1, 1)
3	$4(1-x)+4x\ln(x)$	(3.1818, 0.9091)
4	$13.5+216x-229.5x^2+(108x+135x^2)\ln x-27x^2\ln^2 x$	(7.4529, 0.7710)
5	$\frac{128}{3}+6912x-58752x^2+\frac{155392}{3}x^3+1728x\ln x-20736x^2\ln x-25792x^3\ln x$ $-6912x^2\ln^2 x+4992x^3\ln^2 x-384x^3\ln^3 x$	(16.1011, 0.6430)

Corollary 1 *Given a linear model as in* (1)*, if* $od(\boldsymbol{H}) < \epsilon$*, and* $\epsilon \in (0,1)$*,* $\forall \boldsymbol{H}$*, then LEs collect the same diversity as MLE does.*

On the other hand, if the supreme of $od(\boldsymbol{H})$ is 1 (i.e., $\sup(od(\boldsymbol{H})) = 1$), i.e., there is no $\epsilon < 1$ such that $od(\boldsymbol{H}) \leq \epsilon$, then in general, LEs lose diversity relative to MLE. Intuitively, this can be observed from (12). When ϵ approaches 1, the upper bound of $P_{e,n}$ becomes 1. Thus, LEs lose diversity. In the following, we give some examples to show how to apply our results in practical systems.

4. APPLICATIONS

First, we use two examples to verify Theorem 1. Consider the channel matrix in [7], which is orthogonal, then $od(\boldsymbol{H}) = 0$ for any non-zero realization of $\boldsymbol{H}$. In this case, the ZF detector has the same performance as MLE does. Surely, they have the same diversity. Another example is the Toeplitz matrix $\boldsymbol{H}$ in [8] generated by the first column $[h_1 ... h_L \ 0...0]^T$. In this case,

$$od(\boldsymbol{H}) \leq 1 - \left(\frac{\max_\ell |h_\ell|^2}{\sum_{\ell=1}^{L} |h_\ell|^2}\right)^N \leq 1 - \frac{1}{L^N} < 1. \qquad (17)$$

Based on Theorem 1, LEs collect the same diversity as MLE does, which is consistent with the claim in [8].

However, for many transmission systems, $od(\boldsymbol{H})$ does not have an upper bound less than 1, e.g., MIMO V-BLAST channels which have independent entries. As shown in [11], for MIMO V-BLAST systems, the ZF equalizer loses diversity relative to MLE. Theorem 1 also hints that if one can reduce $od(\boldsymbol{H})$ lower than a certain bound, then the diversity of LEs can be restored. Lattice reduction (LR) is one of these approaches, which guarantees the upper bound of $od(\boldsymbol{H})$ is strictly less than 1 and thus LR-aided LEs collect maximum diversity [6, 11].

Here, we propose an alternative way to guarantee maximum diversity.

Proposition 1 *Consider the model in* (1) *and a constant* $\epsilon \in (0,1)$*. A hybrid equalizer is designed as: if* $od(\boldsymbol{H}) \leq \epsilon$*, LEs are employed; otherwise, MLE (or any other decoder with maximum diversity) is adopted. This equalizer collects the same diversity order as pure MLE does, but requires much lower complexity.*

Based on Theorem 1, it is ready to prove Proposition 1. We notice that by selecting ϵ, we can trade-off the performance with the complexity. That means, when ϵ is small, more realizations of $\boldsymbol{H}$ are dealt with using MLE, then the performance of this hybrid equalizer is better with larger coding gain while the complexity is also higher. Later our simulation results verify this trade-off. To further quantify the trade-off (e.g., control the percentage of the usage of MLE), we have to learn more about the statistical properties of $od(\boldsymbol{H})$, so that we can choose the trigger ϵ wisely. Thus, in the following we derive the distribution of $od(\boldsymbol{H})$.

5. THE DISTRIBUTION OF $od\ (\boldsymbol{H})$

Since the channel matrix $\boldsymbol{H}$ is random, $od(\boldsymbol{H})$ is also random. When $od(\boldsymbol{H})$ has an upper bound which is strictly less than one, LEs guarantee diversity. Therefore, in this section, we focus on general Gaussian random channel $\boldsymbol{H}$ where $\sup(od(\boldsymbol{H})) = 1$ (no guarantee on diversity).

First, we consider i.i.d. channels, i.e., the channel coefficients are i.i.d. complex Gaussian random variables with zero mean and unit variance. By applying the QR decomposition of $\boldsymbol{H}$, $\boldsymbol{H} = \boldsymbol{QR}$, we can rewrite $od(\boldsymbol{H})$ in (4) as

$$1 - od(\boldsymbol{H}) = \prod_{n=1}^{N} \frac{R_{n,n}^2}{R_{n,n}^2 + \sum_{m=1}^{n-1} |R_{m,n}|^2}, \qquad (18)$$

where $R_{m,n}$ is the (m,n)th entry of the upper triangular matrix $\boldsymbol{R}$. According to [10, Lemma 2.1], $2R_{n,n}^2$ is Chi-square distributed with $2(M-n+1)$ degrees of freedom, while the off diagonal entries, $R_{m,n}$ for $m < n$, are complex Gaussian random variables with zero mean and unit variance, and they are independent. Based on [5, p. 188], $\frac{R_{n,n}^2}{R_{n,n}^2 + \sum_{m=1}^{n-1} |R_{m,n}|^2}$ is a Beta random variable with parameters $(M-n+1, n-1)$. Hence, $1 - od(\boldsymbol{H})$ in (18) is a product of N independent Beta random variables.

The distribution of the product of N independent Beta random variables can be computed by induction from the distribution of the product of $N-1$ Beta variables. The exact distribution of (18) can be found by following the approach in [2, p. 58]. However, the number of parameters increases as N increases. Fortunately, it has been shown by [1] that the product in (18) can be well approximated by a Beta variable which only needs two parameters. Here, we list some expressions of exact and approximate distributions of $od(\boldsymbol{H})$ in Table 1 where $M = N = n$ and $\boldsymbol{H}$ has i.i.d. entries.

If the channels are not i.i.d. but still Gaussian distributed, $od(\boldsymbol{H})$ can also be approximated as a Beta random variable, because $1 - od(\boldsymbol{H})$ is still a product of N Beta random variables. Though these Beta random variables may be correlated with each other, the product can still be approximated by a Beta random variable. Thus, in practice, we may treat $od(\boldsymbol{H})$ as Beta distributed. We summarize the results regarding the distribution of $od(\boldsymbol{H})$ as follows.

Proposition 2 *Suppose that* $\boldsymbol{H}$ *is complex Gaussian distributed with zero mean and* $\sup(od(\boldsymbol{H})) = 1$*. Then* $od(\boldsymbol{H})$ *is approximately a Beta random variable with only two parameters.*

With the distribution of $od(\boldsymbol{H})$ by estimating the two parameters of Beta distribution, one has more control on the system design, e.g., by choosing ϵ according to the distribution of $od(\boldsymbol{H})$, one can control the percentage of the usage of LEs, and thus achieve the trade-off between the complexity that we can afford and the optimal coding gain. Also the distribution of $od(\boldsymbol{H})$ will be helpful for further analyzing the performance and capacity of LEs.

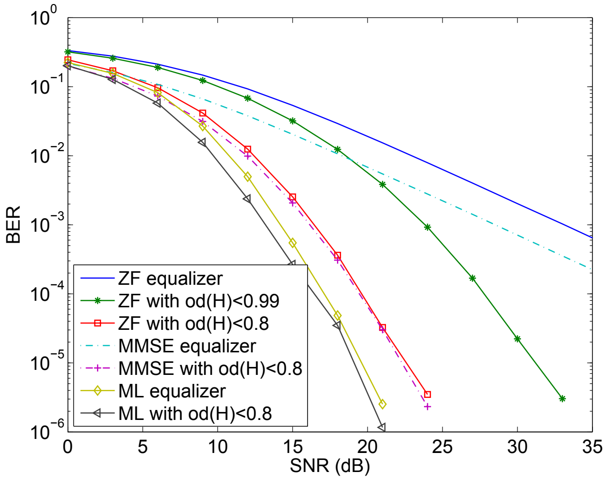

Fig. 1. Performance of LEs for i.i.d. channels

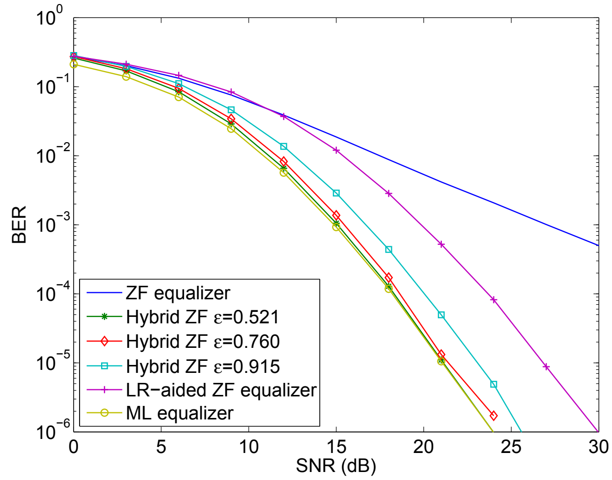

Fig. 2. Performance of hybrid LEs for precoded OFDM

6. SIMULATION RESULTS

In the following examples, QPSK modulation is used and SNR is defined as symbol energy versus noise power, i.e., $E[|s|^2]/\sigma_w^2$.

Example 1 (Performance of LEs with bounded $od(\boldsymbol{H})$): In this example, we generate the channel coefficients as i.i.d. complex Gaussian variables and discard those realizations that $od(\boldsymbol{H}) > \epsilon$. Thus, we upper bound $od(\boldsymbol{H})$ by ϵ. In Fig. 1, we plot the BERs of the ZF and MMSE equalizers with $\epsilon = 0.8, 0.99$ respectively, where $M = N = 4$. The BERs of the ZF and MMSE with unconstrained $od(\boldsymbol{H})$ (without discarding those $od(\boldsymbol{H}) > \epsilon$ cases), and MLEs are also plotted as references. From Fig. 1, we observe that, if $od(\boldsymbol{H}) \leq \epsilon < 1$, LEs collect the same diversity order as MLE does, which is 4 here, while the LEs without od bound only collect diversity 1. On the contrary, the performance of MLEs is not highly influenced by $od(\boldsymbol{H})$. We also notice that when ϵ is smaller, the gap between LEs and MLE is smaller. This is consistent with the observation that when ϵ is smaller, the upper bound of the BER performance in (12) also becomes smaller, which shows that in general for LEs, a smaller $od(\boldsymbol{H})$ indicates higher coding gain while the diversity is the same as that collected by MLE.

Example 2 (Hybrid equalizer for precoded OFDM systems): In this example, we apply our proposed hybrid equalizer to the precoded OFDM systems in [4], in which the equivalent system model can also be expressed as in (1). We fix the channel order $L = 3$ (4 taps), so the maximum diversity is $L + 1 = 4$. In Fig. 2, we plot the BER curves of the hybrid ZF-ML equalizer with $\epsilon = 0.521, 0.760, 0.915$, for which the percentages of channel realizations that are decoded by the ZF equalizer are $25\%, 50\%, 75\%$ respectively according to the distribution of $od(\boldsymbol{H})$. The BERs of the pure ZF equalizer and the pure MLE are also plotted as references. From the figure, we notice that the diversity order of the hybrid equalizers is 4, which is the same as that of MLE. Surprisingly, even a hybrid equalizer with 75% ZF and 25% MLE has performance close to that of pure MLE, and outperforms an alternative low-complexity decoder in [6] (LR-aided ZF equalizer), and provides more flexibility to trade-off complexity and performance.

7. CONCLUSIONS

In this paper, we shed the light of the fundamental link between the orthogonality deficiency of channel matrix and the diversity collected by LEs. We show that when $od(\boldsymbol{H})$ has a bound strictly less than 1, LEs collect the same diversity as MLE does. Furthermore, we derive the pdf of $od(\boldsymbol{H})$. Based on that, we can switch the equalizer among the existing ones according to the affordable complexity at the receiver without sacrificing diversity.[1]

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