INFORMATION LOSSLESS SPACE-TIME CODING FOR MULTIPLE ACCESS SYSTEMS

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ABSTRACT

In this paper we consider multiple access systems where users and access point are equipped with multiple antennas. In order to exploit some of the MIMO potentials we have to resort to space-time coding. Unfortunately, such a processing may induce severe loss in terms of achievable information rates. In this paper we prove the necessary and sufficient condition ensuring that the space-time coding is information lossless, in the sense that it does not induce any modification in certain regions of achievable rates. In particular information lossless property is guaranteed if each user makes use of a Trace-Orthogonal Design (TOD), that is a linear space-time code whose encoding matrices are orthogonal with respect to the trace inner product. Noteworthy, users can also use the same set of encoding matrices.

Index Terms— MIMO Systems, Multiple Access Channel, Space-Time Coding, Information Lossless

1. INTRODUCTION

MIMO systems have attracted a lot of research in the recent years, since they make possible to increase spectral efficiency and diversity gain without sacrificing transmission power and/or bandwidth [1]. However, to exploit MIMO potentials, in particular diversity gain, we have to resort to space-time coding [2]. Unfortunately, this processing may induce severe loss in terms of capacity [3]. Nevertheless, spectral efficiency is one of the major motivations for using MIMO systems so, it is fundamental to discern which are the space-time coding properties essential to achieve the MIMO potentials without incurring capacity losses. In the single user scenario there are several studies in that direction [4], [5] and references therein. Despite this, it seems that no equivalent study has been carried out in the multiuser setting. To the best of authors' knowledge, it seems that the only contribution dealing with the problem of characterizing the space-time coding strategies allowing for lossless information transfer, in the case of multiple access systems, is [6], where the problem of invariance for the sum-rate has been addressed. In this work we consider multiple access systems and we extend the result of [6]. In particular we prove the necessary and sufficient condition, on the space-time coding strategy for each user, which guarantees that certain regions of achievable rates are not affected by the coding. The result is derived on a per-realization basis and thus it holds regardless of the statistics of the channels. Throughout the paper, we use the following convention: $\mathbb{C}^{n\times p}$ denotes the space of $n\times p$ matrices with complex entries; matrices are denoted by bold uppercase letters and vectors by bold lowercase ones; $\boldsymbol{I}_n$ denotes the $n\times n$ identity matrix; $\boldsymbol{I}$ and $\boldsymbol{0}$ denote respectively an identity matrix and a null vector with suitable dimensions; logarithms are in base e.

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2. SYSTEM MODEL

Consider a multiple access system composed of N users, each with n_T transmit antennas, and an access point (AP), with n_R receive antennas. Let us assume that the k-th user encodes its own n_s complex symbols $s_k(j), j = 1, \ldots, n_s$, through the following space-time linear encoder

$$\boldsymbol{X}_k = \sum_{j=1}^{n_s} \boldsymbol{A}_k(j) s_k(j) \tag{1}$$

where $\{\boldsymbol{A}_k(j), j = 1, \ldots, n_s\}$ is the set of $n_T \times Q$ complex matrices assigned to the k-th user.
A space-time encoder is a *Trace-Orthogonal Design* (TOD), if the corresponding matrices $\boldsymbol{A}_k(1), \cdots, \boldsymbol{A}_k(n_s)$ are orthonormal with respect the trace inner product, that is they satisfy

$$\operatorname{tr}\left(\boldsymbol{A}_k^H(j)\boldsymbol{A}_k(m)\right) = \delta_{jm}, \tag{2}$$

where δ_{jm} is the Kronecker delta.
Applying the $\operatorname{vec}(\cdot)$ operator to (1), we get

$$\boldsymbol{x}_k = \operatorname{vec}(\boldsymbol{X}_k) = \sum_{j=1}^{n_s} \operatorname{vec}\left(\boldsymbol{A}_k(j)\right) s_k(j) = \boldsymbol{F}_k \boldsymbol{s}_k, \tag{3}$$

where $\boldsymbol{F}_k$ is the $(Q\cdot n_T \times n_s)$ matrix whose j-th column is $\operatorname{vec}(\boldsymbol{A}_k(j))$ and $\boldsymbol{s}_k = [s_k(1)\cdots s_k(n_s)]^T$ is the vector of transmitted symbols for the k-th user.
To guarantee symbol recovery[1] for each user, matrices $\boldsymbol{F}_k$ must be full column rank, i.e. $\operatorname{rank}(\boldsymbol{F}_k) = n_s$. This means that the following inequality must be satisfied

$$n_s \leq Q \cdot n_T. \tag{4}$$

We will refer to codes for which $\boldsymbol{F}_k$ has full column rank, as *non-singular* codes.
As will be clear later, matrices $\boldsymbol{F}_k$ ($k = 1, \ldots, N$) play a fundamental role in characterizing the properties of the codes.

Now, let us consider the system. Denoting by $\boldsymbol{H}_k \in \mathbb{C}^{n_R\times n_T}$ the channel matrix characterizing the link between the k-th user and the AP, and by $\tilde{\boldsymbol{x}}_k$ the corresponding vector of transmitted symbols, the received vector is

$$\tilde{\boldsymbol{y}} = \sum_{k=1}^{N} \boldsymbol{H}_k \tilde{\boldsymbol{x}}_k + \tilde{\boldsymbol{v}}, \tag{5}$$

where $\tilde{\boldsymbol{v}}$ is the noise vector, assumed to be zero mean, circularly symmetric complex Gaussian, with covariance matrix $\sigma_v^2\boldsymbol{I}$. We will refer to (5) as the *uncoded system*.

[1] Symbol recovery is guaranteed if mapping in (3) is injective.

Consider now the system with space-time encoding, where the channels $\boldsymbol{H}_k$ are assumed to be constant over Q successive channel uses (block fading model). If each user transmits the matrix $\boldsymbol{X}_k$, built as in (1), the received matrix is

$$\boldsymbol{Y} = \sum_{k=1}^{N} \boldsymbol{H}_k \boldsymbol{X}_k + \boldsymbol{V}\,, \tag{6}$$

where $\boldsymbol{V}$ is the $(n_R \times Q)$ received noise matrix. Applying the $\mathrm{vec}(\cdot)$ operator[2] to (6) and using (1) and (3), we get

$$\boldsymbol{y} = \sum_{k=1}^{N} (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k) \boldsymbol{F}_k \boldsymbol{s}_k + \boldsymbol{v}, \tag{7}$$

where $\boldsymbol{v} = \mathrm{vec}(\boldsymbol{V})$. We will refer to (7) or (6) as the *coded system*.

We assume that P_k is the constraint on the transmit power for the k-th user, no channel state information (CSI) is available at the transmitters and the receiver at the AP has perfect knowledge of CSI. Under the hypothesis of no CSI at the transmitters a sensible choice is to use a uniform power allocation for each user. In this way, for any given realization of the channels, the multiple access system can be characterized by the instantaneous region of achievable rates, which denote the region of achievable rates for the specific realization of the channels, when uniform power allocation is assumed for all the users [2]. Introducing the notation

$$\boldsymbol{\mathcal{H}} = [\boldsymbol{H}_1 \cdots \boldsymbol{H}_N], \tag{8}$$

for the uncoded system in (5) it assumes the form

$$\mathcal{R}^{\text{unc}}(\boldsymbol{\mathcal{H}}) = \Big\{ (R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \sum_{k \in \mathcal{S}} R_k \leq C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S})\,, \forall \mathcal{S} \subseteq \{1, \ldots, N\} \Big\}, \tag{9}$$

where R_k is the rate for the k-th user, $\mathcal{S}$ is any subset of $\{1, \ldots, N\}$, and

$$C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) = \log \left| \boldsymbol{I} + \sum_{k \in \mathcal{S}} \gamma_k \boldsymbol{H}_k \boldsymbol{H}_k^H \right|, \tag{10}$$

where $\gamma_k = P_k / n_T \sigma_v^2$.
The corresponding expression for the coded system in (7), when the transmitted symbols for the k-th user are uncorrelated with variance P_k/n_T, is

$$\mathcal{R}^{\text{cod}}(\boldsymbol{\mathcal{H}}) = \Big\{ (R_1, \ldots, R_N) \in \mathbb{R}_+^N \Big| \sum_{k \in \mathcal{S}} R_k \leq C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S})\,, \forall \mathcal{S} \subseteq \{1, \ldots, N\} \Big\}, \tag{11}$$

where

$$C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) = \frac{1}{Q} \log \left| \boldsymbol{I} + \sum_{k \in \mathcal{S}} \gamma_k (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k) \boldsymbol{F}_k \boldsymbol{F}_k^H (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k^H) \right| \tag{12}$$

where the factor $1/Q$ accounts for the Q uses of the channels, and $\boldsymbol{F}_k$ is defined in (3).

[2] In deriving (7) we have used $\mathrm{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{B}^T \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{X})$.

3. INFORMATION LOSSLESS SPACE-TIME CODING

The objective of this section is to provide *necessary and sufficient* conditions on the encoding matrices for each user so that lossless information transfer is guaranteed. In general, the coded system (6) can experience rate reductions depending on the particular choice of the space-time encoder for each user. For example, in the single user scenario, it is well known [3] that orthogonal space-time block coding incurs severe loss in terms of capacity. In a multiuser scenario the rate loss is experienced in terms of modification and/or reduction of the corresponding region of achievable rates.
We are interested in space-time coding strategies that do not affect the instantaneous region of achievable rates, as defined in (9). The rationale behind this choice is that if such a coding scheme exists for all the users, whichever are the channel realizations, the property will hold regardless of the statistics of the channels. This motivates the introduction of the following definition

Definition 1. *A space-time*[3] *coding strategy is* information lossless *for a multiple access system, if the instantaneous regions of achievable rates $\mathcal{R}^{cod}(\boldsymbol{\mathcal{H}})$ and $\mathcal{R}^{unc}(\boldsymbol{\mathcal{H}})$, as given in* (9) *and* (11)*, coincide for all realizations of the channels.*

The main result of this section is Theorem 1, which gives a complete characterization of the information lossless coding strategy. Before proceeding, we need some preliminary results from matrix theory.

Lemma 1. *If $\boldsymbol{G} \in \mathbb{C}^{n \times n}$ is a Hermitian matrix, then*

$$\mathrm{tr}[\boldsymbol{G}^k] = n\,,\ k = 1, \ldots, n \iff \boldsymbol{G} = \boldsymbol{I}_n.$$

Proof. See Appendix. □

Lemma 2. *If $\boldsymbol{M} \in \mathbb{C}^{n \times n}$ is a Hermitian positive semidefinite matrix independent of t, then*

$$\left. \frac{\partial^k \log |\boldsymbol{I} + t\,\boldsymbol{M}|}{\partial t^k} \right|_{t=0} = (-1)^{k-1}(k-1)!\,\mathrm{tr}(\boldsymbol{M}^k), \quad k \in \mathbb{N}.$$

Proof. See Appendix. □

We are now ready to prove the following

Theorem 1. *In the setting of the previous section, if the receiver has perfect CSI, no CSI is available at the transmitters, and the k-th user transmits independent symbols with variance P_k/n_T, then*

$$\mathcal{R}^{cod}(\boldsymbol{\mathcal{H}}) = \mathcal{R}^{unc}(\boldsymbol{\mathcal{H}})\,, \qquad \forall \boldsymbol{\mathcal{H}} \in \mathbb{C}^{n_R \times N n_T}$$

that is, the instantaneous regions of achievable rates for the coded and the uncoded systems coincide for all realizations of the channels, if and only if $\boldsymbol{F}_k \boldsymbol{F}_k^H = \boldsymbol{I}$*, for $k = 1, \ldots, N$.*

Proof. Let us introduce the set $E = \{1, \cdots, N\}$. The instantaneous regions of achievable rates $\mathcal{R}^{\text{unc}}(\boldsymbol{\mathcal{H}})$ and $\mathcal{R}^{\text{cod}}(\boldsymbol{\mathcal{H}})$, as given in (9) and (11), are functions only of the boundaries expressed in terms of $C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S})$ and $C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S})$ respectively. Since we require the coincidence of $\mathcal{R}^{\text{unc}}(\boldsymbol{\mathcal{H}})$ and $\mathcal{R}^{\text{cod}}(\boldsymbol{\mathcal{H}})$ for *all* the realizations of the channels, it is easy to verify that this condition is equivalent to require

$$C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) = C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S}), \quad \forall \boldsymbol{\mathcal{H}} \in \mathbb{C}^{n_R \times N n_T}\,, \forall \mathcal{S} \subseteq E\,, \tag{13}$$

[3] We consider linear space-time codes.

that is, the coincidence of the corresponding functions $C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S})$ and $C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S})$ for *all* the realizations of the channels and for *all* the subsets $\mathcal{S}$ of E. Through the proof we will make expressly reference to this equivalent condition.

(*Sufficiency*) When $\boldsymbol{F}_k \boldsymbol{F}_k^H = \boldsymbol{I}$, for $k = 1, \ldots, N$, from (12), exploiting the properties of Kronecker product [7], we can draw

$$
\begin{aligned}
C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) &= \frac{1}{Q} \log \left| \boldsymbol{I} + \sum_{k \in \mathcal{S}} \gamma_k (\boldsymbol{I}_Q \otimes \boldsymbol{H}_k)(\boldsymbol{I}_Q \otimes \boldsymbol{H}_k^H) \right| \\
&= \frac{1}{Q} \log \left| \boldsymbol{I} + \sum_{k \in \mathcal{S}} \gamma_k \left(\boldsymbol{I}_Q \otimes \boldsymbol{H}_k \boldsymbol{H}_k^H \right) \right| \\
&= \frac{1}{Q} \log \left| \boldsymbol{I} + \left(\boldsymbol{I}_Q \otimes \sum_{k \in \mathcal{S}} \gamma_k \boldsymbol{H}_k \boldsymbol{H}_k^H \right) \right| \\
&= \frac{1}{Q} \log \left| \boldsymbol{I}_Q \otimes \left(\boldsymbol{I} + \sum_{k \in \mathcal{S}} \gamma_k \boldsymbol{H}_k \boldsymbol{H}_k^H \right) \right| = C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) \qquad (14)
\end{aligned}
$$

where in the last step we have used the identity $|\boldsymbol{I}_n \otimes \boldsymbol{M}| = |\boldsymbol{M}|^n$. Note that (14) holds true $\forall \boldsymbol{\mathcal{H}} \in \mathbb{C}^{n_R \times N n_T}$, and $\forall \mathcal{S} \subseteq E$. This coincides with (13) thus proving the sufficiency.

(*Necessity*). The following relation is assumed to be true

$$
C^{\text{cod}}(\boldsymbol{\mathcal{H}}, \mathcal{S}) = C^{\text{unc}}(\boldsymbol{\mathcal{H}}, \mathcal{S}), \quad \forall \boldsymbol{\mathcal{H}} \in \mathbb{C}^{n_R \times N n_T}, \forall \mathcal{S} \subseteq E\,. \qquad (15)
$$

Let us consider the generic subset of E having K elements $\mathcal{S} = \{i_1, \cdots, i_K\}$, where $1 \leq K \leq N$. So, denoting by

$$
\boldsymbol{H} = \left[\sqrt{\gamma_{i_1}} \boldsymbol{H}_{i_1} \cdots \sqrt{\gamma_{i_K}} \boldsymbol{H}_{i_K} \right], \qquad (16)
$$

and introducing the following block diagonal matrix

$$
\boldsymbol{F} = \operatorname{diag}\{\boldsymbol{F}_{i_1}, \cdots, \boldsymbol{F}_{i_K}\}\,, \qquad (17)
$$

identity (15), after some matrix algebra, can be recast as

$$
\frac{1}{Q} \log \left| \boldsymbol{I} + (\boldsymbol{I}_Q \otimes \boldsymbol{H}) \boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H (\boldsymbol{I}_Q \otimes \boldsymbol{H}^H) \right| = \log \left| \boldsymbol{I} + \boldsymbol{H} \boldsymbol{H}^H \right| \qquad (18)
$$

where $\boldsymbol{\Pi}$ is the following permutation matrix

$$
\boldsymbol{\Pi} = [\boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_1} \;\; \boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_2} \;\; \cdots \;\; \boldsymbol{I}_Q \otimes \boldsymbol{P}_{i_K}], \qquad (19)
$$

with $\boldsymbol{P}_k$ defined as

$$
\boldsymbol{P}_k = \boldsymbol{u}_k \otimes \boldsymbol{I}_{n_T}, \qquad (20)
$$

where $\boldsymbol{u}_k$ is the k-th unit vector[4] in $\mathbb{C}^K$, that is $\boldsymbol{u}_k(j) = \delta_{kj}$, for $j = 1, \ldots, K$. Note that $\boldsymbol{H}$, $\boldsymbol{F}$, and $\boldsymbol{\Pi}$ depend on the subset $\mathcal{S}$.

Resorting to the identity $|\boldsymbol{I} + \boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{I} + \boldsymbol{B}\boldsymbol{A}|$, (18) can be rewritten as

$$
\log \left| \boldsymbol{I} + (\boldsymbol{I}_Q \otimes \boldsymbol{H}^H \boldsymbol{H}) \boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H \right| = Q \log \left| \boldsymbol{I} + \boldsymbol{H}^H \boldsymbol{H} \right|. \qquad (21)
$$

Since (21) is assumed to hold for *any* channel realization, it is satisfied, in particular, for rank-one channel matrices such that $\boldsymbol{H}$ assumes the following structure[5]

$$
\hat{\boldsymbol{H}} = \sqrt{t}\,[\boldsymbol{h} \;\; \boldsymbol{0} \;\; \cdots \;\; \boldsymbol{0}]^H \qquad t \in \mathbb{R}^+,\; \boldsymbol{h} \in \mathcal{U}\,, \qquad (22)
$$

where $\mathbb{R}^+ = [0, +\infty)$, $\boldsymbol{0}$ is the Kn_T-size null vector, and $\mathcal{U} = \{\boldsymbol{h} \in \mathbb{C}^{Kn_T} : \|\boldsymbol{h}\| = 1\}$, that is $\mathcal{U}$ is the set of Kn_T-size column vectors with unit norm. With this particular choice, the product $\boldsymbol{H}^H \boldsymbol{H}$ becomes

$$
\hat{\boldsymbol{H}}^H \hat{\boldsymbol{H}} = t\, \boldsymbol{h} \boldsymbol{h}^H, \qquad (23)
$$

which, substituted in (21), leads to

$$
\log \left| \boldsymbol{I} + t\, (\boldsymbol{I}_Q \otimes \boldsymbol{h}^H) \boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H (\boldsymbol{I}_Q \otimes \boldsymbol{h}) \right| = Q \log(1 + t)\,. \qquad (24)
$$

Since (24) must hold true $\forall\, t \in \mathbb{R}^+$ and $\forall\, \boldsymbol{h} \in \mathcal{U}$, equation (24), for any $\boldsymbol{h}$, is an identity between infinitely differentiable functions of the variable $t \in \mathbb{R}^+$.

Taking the Q successive derivatives with respect to t of both sides of (24), evaluated at $t = 0$, and resorting to Lemma 2, we get the following identities valid for $k = 1, \ldots, Q$,

$$
\operatorname{tr} \left\{ \left[(\boldsymbol{I}_Q \otimes \boldsymbol{h}^H) \boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H (\boldsymbol{I}_Q \otimes \boldsymbol{h}) \right]^k \right\} = Q \quad \forall\, \boldsymbol{h} \in \mathcal{U}, \qquad (25)
$$

which, according to Lemma 1, hold true if and only if

$$
(\boldsymbol{I}_Q \otimes \boldsymbol{h}^H) \boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H (\boldsymbol{I}_Q \otimes \boldsymbol{h}) = \boldsymbol{I}_Q\,, \qquad \forall\, \boldsymbol{h} \in \mathcal{U}. \qquad (26)
$$

Now, let us rewrite the product $\boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H$ as a $Q \times Q$ block matrix where each block, denoted by $\boldsymbol{\Phi}_{ij}$, has dimensions $Kn_T \times Kn_T$

$$
\boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H = \begin{bmatrix} \boldsymbol{\Phi}_{11} & \cdots & \boldsymbol{\Phi}_{1Q} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\Phi}_{Q1} & \cdots & \boldsymbol{\Phi}_{QQ} \end{bmatrix} \qquad (27)
$$

With this position, identity (26) can be recast equivalently as

$$
\begin{bmatrix} \boldsymbol{h}^H \boldsymbol{\Phi}_{11} \boldsymbol{h} & \cdots & \boldsymbol{h}^H \boldsymbol{\Phi}_{1Q} \boldsymbol{h} \\ \vdots & \ddots & \vdots \\ \boldsymbol{h}^H \boldsymbol{\Phi}_{Q1} \boldsymbol{h} & \cdots & \boldsymbol{h}^H \boldsymbol{\Phi}_{QQ} \boldsymbol{h} \end{bmatrix} = \boldsymbol{I}_Q, \qquad \forall\, \boldsymbol{h} \in \mathcal{U}. \qquad (28)
$$

This holds true if and only if the following identities are satisfied

$$
\boldsymbol{h}^H \boldsymbol{\Phi}_{ij} \boldsymbol{h} = \delta_{ij} \quad \forall\, \boldsymbol{h} \in \mathcal{U}, \quad \text{for } i, j = 1, \ldots, Q\,. \qquad (29)
$$

Equivalently, (29) states that, for $i, j = 1, \ldots, Q$,

$$
\boldsymbol{h}^H (\boldsymbol{\Phi}_{ij} - \delta_{ij} \boldsymbol{I}_{Kn_T}) \boldsymbol{h} = 0, \quad \forall\, \boldsymbol{h} \in \mathcal{U}. \qquad (30)
$$

Since (30) must hold true $\forall\, \boldsymbol{h} \in \mathcal{U}$, this is possible, according to [7], if and only if

$$
\boldsymbol{\Phi}_{ij} = \delta_{ij} \boldsymbol{I}_{Kn_T}, \quad \text{for } i, j = 1, \ldots, Q, \qquad (31)
$$

that is, if and only if

$$
\boldsymbol{\Pi} \boldsymbol{F} \boldsymbol{F}^H \boldsymbol{\Pi}^H = \boldsymbol{I}_{QKn_T}, \qquad (32)
$$

which is equivalent to

$$
\boldsymbol{F} \boldsymbol{F}^H = \boldsymbol{I}_{QKn_T}, \qquad (33)
$$

since $\boldsymbol{\Pi}$ is a permutation matrix. Finally, taking into account the definition of $\boldsymbol{F}$ in (17) we conclude that (33) is equivalent to

$$
\boldsymbol{F}_{i_1} \boldsymbol{F}_{i_1}^H = \boldsymbol{F}_{i_2} \boldsymbol{F}_{i_2}^H = \cdots = \boldsymbol{F}_{i_K} \boldsymbol{F}_{i_K}^H = \boldsymbol{I}_{Qn_T}\,. \qquad (34)
$$

Since (34) holds true for any subset $\mathcal{S} = \{i_1, \cdots, i_K\}$, it implies the following final result

$$
\boldsymbol{F}_k \boldsymbol{F}_k^H = \boldsymbol{I}_{Qn_T}\,, \qquad (35)
$$

[4] It is a column vector.

[5] Note that the presence of $\gamma_1, \ldots, \gamma_K$ in (16) does not pose any problem in building (22), since those values are non null.

which holds true for $k = 1, \ldots, N$.

This proves the necessity for the subclass of rank-one channel matrices in (22). But since (35) is also the condition that guarantees (15) for *all* realizations of the channels, as follows from the proof of sufficiency, we conclude a *fortiori* [6] that (35) is also a necessary condition for the whole class of channel matrices. Stated differently, the proof of sufficiency demonstrates the existence of a solution to the problem of information invariance with respect to the channel realizations, namely (35). The proof of necessity guarantees that such a solution is unique showing that for a subclass of channel realizations, condition (35) must necessarily hold. The proof is thus complete. □

3.1. Consequences of Theorem 1

Condition (35) holds true only if $\boldsymbol{F}_k$ has full row rank. Since $\boldsymbol{F}_k$ is $Qn_T \times n_s$, it can occur only if

$$n_s \geq Qn_T \,. \tag{36}$$

Combining (36) with (4), we arrive at the following equality

$$n_s = Q \cdot n_T \,, \tag{37}$$

which is equivalent to say that the code rate, defined as $R = n_s/Q$, is equal to n_T, for all users. This means that all users must use a *full-rate* code.

Moreover condition (37) forces $\boldsymbol{F}_k$ to be square and this, together with (35), implies that $\boldsymbol{F}_k$ is *unitary*, i.e., it is also true that $\boldsymbol{F}_k^H \boldsymbol{F}_k = \boldsymbol{I}$. This is a strong result that allows us to fully characterize the encoding matrices $\boldsymbol{A}_k(j)$, for each user. In fact, taking into account the structure of $\boldsymbol{F}_k$ (see (3)), the generic element $\left\{\boldsymbol{F}_k^H \boldsymbol{F}_k\right\}_{i,j} = \{\boldsymbol{I}\}_{i,j}$ of the identity $\boldsymbol{F}_k^H \boldsymbol{F}_k = \boldsymbol{I}$ can be written as

$$\mathrm{vec}^H(\boldsymbol{A}_k(i))\,\mathrm{vec}(\boldsymbol{A}_k(j)) = \mathrm{tr}\left(\boldsymbol{A}_k^H(i)\boldsymbol{A}_k(j)\right) = \delta_{ij} \tag{38}$$

where $\mathrm{vec}^H(\boldsymbol{X})\,\mathrm{vec}(\boldsymbol{Y}) = \mathrm{tr}(\boldsymbol{X}^H\boldsymbol{Y})$ has been used.
The last equality in (38) holds true *if and only if* the the encoding matrices $\boldsymbol{A}_k(j)$ ($j = 1, \ldots, n_s$), for each user, constitute a Trace-Orthogonal Design. Thus, combining (37) and (38), the consequence of Theorem 1 can be summarized in the following

Corollary 1. *A space-time coding strategy for a multiple access system, based on nonsingular linear codes, is information lossless* if and only if *each user employs a full-rate Trace-Orthogonal Design.*

4. CONCLUSION

In this work we have considered multiple access systems where users and access point are equipped with multiple antennas. In order to exploit some of the MIMO potentials we have to resort to space-time coding. We have studied the way to carry out such a coding strategy in order to avoid information losses. In particular we have proved that for any realization of the channels we can guarantee that the instantaneous region of the achievable rates does not change if and only if each user encodes its symbol using a full-rate Trace-Orthogonal Design. The result holds true regardless of the statistics of the channels. Moreover no other constraints are imposed on the choice of the encoding matrices as far as they belong to a Trace-Orthogonal Design. So, as far as the invariance of the achievable rates region is concerned, all users can share the same set of encoding matrices.

[6] Note that without the proof of sufficiency, we would not have been able to extend the necessity condition to the whole class of channel matrices, since (35) could not be a feasible solution for such a class.

5. APPENDIX

In this Section, we collect the proofs of lemmas enunciated in the paper. Towards this end we will make use of the following result

Lemma 3. *If $\boldsymbol{G} \in \mathbb{C}^{n\times n}$ is a Hermitian matrix with all eigenvalues equal to* 1*, then $\boldsymbol{G} = \boldsymbol{I}_n$.*

Proof. Let us indicate with $\gamma_1 = \gamma_2 = \cdots = \gamma_n = 1$ the eigenvalues of $\boldsymbol{G}$. Since $\boldsymbol{G} = \boldsymbol{G}^H$, there exists a unitary matrix $\boldsymbol{U}$ such that the following chain of equalities holds

$$\boldsymbol{G} = \boldsymbol{U}\,\mathrm{diag}\{\gamma_1, \gamma_2, \cdots, \gamma_n\}\boldsymbol{U}^H = \boldsymbol{U}\boldsymbol{U}^H = \boldsymbol{I}_n.$$

That is, the identity matrix $\boldsymbol{I}_n$ is the only $n \times n$ Hermitian matrix with all the eigenvalues equal to 1. □

Proof of Lemma 1

Since $n = \mathrm{tr}[\boldsymbol{I}_n] = \mathrm{tr}[(\boldsymbol{I}_n)^k]$, it is known from [7] that

$$\mathrm{tr}[\boldsymbol{G}^k] = \mathrm{tr}[(\boldsymbol{I}_n)^k]\,,\ k = 1, \ldots, n$$

holds true if and only if $\boldsymbol{G}$ and $\boldsymbol{I}_n$ have the same eigenvalues, i.e. $\boldsymbol{G}$ has all its eigenvalues equal to 1. Since $\boldsymbol{G}$ is Hermitian, from Lemma 3, this is possible if and only if $\boldsymbol{G} = \boldsymbol{I}_n$. □

Proof of Lemma 2

Let us denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ the eigenvalues of $\boldsymbol{M}$. Since $\boldsymbol{M}$ is positive semidefinite, the function $\log|\boldsymbol{I} + t\,\boldsymbol{M}|$ is infinitely differentiable[7] in $t = 0$. Moreover, the following chain of equalities holds true for $k \in \mathbb{N}$

$$\left.\frac{\partial^k \log|\boldsymbol{I} + t\,\boldsymbol{M}|}{\partial t^k}\right|_{t=0} = \sum_{j=1}^{n} \left.\frac{\partial^k \log(1 + t\lambda_j)}{\partial t^k}\right|_{t=0} =$$

$$= (-1)^{k-1}(k-1)! \sum_{j=1}^{n} \left.\frac{\lambda_j^k}{(1 + t\lambda_j)^k}\right|_{t=0} = (-1)^{k-1}(k-1)!\,\mathrm{tr}(\boldsymbol{M}^k)\,,$$

where in the last equality, we have used the following identity [7] relating trace and eigenvalues $\sum_{j=1}^{n} \lambda_j^k = \mathrm{tr}(\boldsymbol{M}^k)$. □

6. REFERENCES

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[7] Note that $\log|\boldsymbol{I} + t\,\boldsymbol{M}|$ is defined for $t \in (-\lambda_1^{-1}, +\infty)$.